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THE GROWTH OF HARMONIC FUNCTIONS IN HYPERSPHERES

Abstract. The present paper deals with the growth of solutions (harmonic in n-dimensional Euclidean space R^n) of Laplace's differential equation.

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_n^2}\right)H=0.$$

Characterizations of q-order and q-type of harmonic functions $H(x)$ have been obtained explicitly in terms of the m-th gradients $|\nabla_m H(o)|$. The $H(x)$ are taken to be regular in a finite hyperball $B_R=\{x:|x|<R\}$, where $|x|=(x_1^2+x_2^2+\ldots+x_n^2)^{1/2}$.

1. Introduction

A real valued function $H(x)$, $x=(x_1,x_2,\ldots,x_n)$ is said to be harmonic in n-dimensional Euclidean space R^n if it has continuous partial derivatives of the first and second order and satisfies the Laplace equation

$$\frac{\partial^2 H}{\partial x_1^2}+\frac{\partial^2 H}{\partial x_2^2}+\ldots+\frac{\partial^2 H}{\partial x_n^2}=0,$$

throughout a neighbourhood of the origin in R^n. Such functions have spherical harmonic expansions

$$H(x)=\sum_{m=0}^{\infty} H_m(x), \tag{1.1}$$

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where $H_m(x)$ are harmonic, homogeneous polynomials of degree m in $x_1, x_2, \ldots, x_n$, [2. p. 45]. It has been shown [4] that the series (1.1) converges absolutely and uniformly on compact subsets of the open ball $B_R = \{x : |x| < R\}$, where $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$.

$$R^{-1} = \limsup_{m \to \infty} (\|\nabla_m H(o)\|/m!)^{1/m}, \tag{1.2}$$

and the norm of the m-th gradient of $H(x)$ is defined as follows:

For each n-tuple $a = (a_1, a_2, \ldots, a_n)$ of nonnegative integers, let $|a| = a_1 + a_2 + \ldots + a_n, a! = a_1!a_2! \ldots a_n!$, and

$$D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \ldots \partial x_n^{a_n}}.$$

Then

$$\|\nabla_m H(o)\| = \left(\frac{m!}{2^m} \sum_{|a|=m} (D^a H(o))^2/a! \right)^{1/2}. \tag{1.3}$$

Further, such convergence of the series (1.1) does not hold on a larger ball centered at the origin.

The concepts of index q, the q-order $\varrho(q)$ and q-type $T(q)$ are introduced by Bajpai et al. [1] in order to obtain a measure of growth of the maximum modulus, when it is rapidly increasing. Thus, let $M(r, H) \to \infty$ as $r \to R$ and let for $q = 2, 3, \ldots$, set

$$\varrho_q(H, R) = \limsup_{r \to R} \frac{\log^{(q)} M(r, H)}{\log(R/R - r)}, \tag{1.4}$$

where $\log^{(o)} M(r, H) = M(r, H)$ and $\log^{(q)} M(r, H) = \log \log^{(q-1)} M(r, H)$.

For an analytic function f with Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $R^{-1} = \limsup_{n \to \infty} |a_n|^{1/n}$, it has been shown [6] that $\varrho q(R)$ as defined by (1.4) with $M(r) = \max_{|z|=r} |f(z)|$ is given by the expression.

$$\varrho_q(R) + A(q) = \limsup_{n \to \infty} \frac{\log^{(q-1)} n}{\log n - \log^+ \log^+ |a_n| R^n}, \tag{1.5}$$

where $A(q) = 1$ if $q = 2$, $A(q) = 0$ if $q \geq 3$ and for $x > 0$, we put $\log^+ x = \max(\log x, 0)$. Now we need the definition of q-type of the function $H(x)$. Suppose that $H(x)$ is of finite nonzero q-order $\varrho q(H, R)$ in the ball B_R. We define the q-type $T_q(H, R)$ of $H(x)$ just like the q-type of a function of a complex variable analytic in the disc $|z| < R$, $0 < R < \infty$, [5]. The q-type

$T_q(H,R)$ of $H(x)$ is defined by the relation

$$(1.6)\qquad T_q(H,R) = \limsup_{r\to R} \frac{\log^{(q-1)} M(r,H)}{(R/(R-r))^{\varrho_q(H,R)}}.$$

If the analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of finite nonzero q-order $\varrho q(R)$ in the disc $|z| < R$, then the explicit characterization of its q-type $T_q(R)$ in terms of the Taylor coefficients a_n is given by [5] by the relation

$$(1.7)\qquad T_q(R)B(q) = V(q),$$

where

$$V(q) = \limsup_{n\to\infty}(\log^{(q-2)} n)(\frac{\log^+ |a_n|R^n}{n})^{\varrho_q(R)+A(q)},$$

$B(q) = (\varrho_q(R)+1)^{(\varrho_q(R)+1)}/(\varrho_q(R))^{\varrho_q(R)}$ for $q = 2$ and $B(q) = 1$ if $q \geq 3$. Here $M(r,H) = \max|H(x)|_{|x|=r}$. The $H(x)$ is said to have the index q if $\varrho_q(H,R) < \infty$, and $\varrho_{q-1}(H,R) = \infty$. If q is the index of H then $\varrho_q(H,R)$ and $T_q(H,R)$ are called the q-order and q-type of $H = H(x)$ respectively.

The notions of index and q-order play a significant role in classifying the rapidly increasing functions analytic in B_R. However, these concepts fail to compare the rates of growth of any two functions analytic in B_R that have same q-order. This paper gives the distinct parameters for the rates of growth of such functions. Our results extend and improve the results due to Fryant and Shankar [3].

The text has been divided into two parts. Section 1, consists of introductory exposition of the topic and a lemma which is used in proving the main theorems. In Section 2, we obtain expressions for the q-growth order and type of H, explicitly in terms of the m-th gradients $|\nabla_m H(o)|$, interior to the ball B_R.

The following lemma [3] gives upper and lower bounds on the maximum value attained by a harmonic function on the sphere $S(r)$ of radius r centered at the origin, in terms of analytic functions of r.

LEMMA A. *Let $H = H(x)$ be a harmonic in a neighbourhood of the origin in R^n. Then for all $r < R$.*

$$(1.8)\qquad M_2(r,H) \leq M(r,H) \leq M_1(r,H),$$

where

$$M_2(r,H) = \left[\frac{1}{C_n r^{n-1}} \int_{S(r)} H^2(x)d\sigma(x)\right]^{1/2}$$

$$= \left[\Gamma(n/2)\sum_{m=0}^{\infty} \frac{|\nabla H_m H(o)|^2}{m!\Gamma(m+(n/2))} r^{2m}\right]^{1/2}$$

and

$$M_1(r,H)=\sqrt{\Gamma(n/2)}\sum_{m=0}^{\infty}\sqrt{dm}(|\nabla_m H(o)|\sqrt{m!\Gamma(m+(n/2))})r^m,$$

$d\sigma$ is the element of the surface area on the sphere $S(1)$, and C_n is the surface area of $S(1)$.

2. Main results

THEOREM 1. *Let $H(x)$, $x=(x_1,x_2,\dots,x_n)$ be harmonic in a neighbourhood of origin in R^n, having index q, and suppose the radius R of harmonicity of $H(x)$ is given by (1.2). Then the q-order of $H(x)$ is given by the expression.*

$$\varrho_q(H,R)+A(q)=\limsup_{m\to\infty}\frac{\log^{(q-1)} m}{\log m-\log^+\log^+(|\nabla_m H(o)|/m!)R^m}.$$

Proof. Let $M_2(r,H)$, $M(r,H)$ and $M_1(r,H)$ be as defined in Lemma A, we observe that $[M_2(r,H)]^2$ is an analytic function of r, and thus can be continued analytically to complex variables, where the radius of convergence R of

$$[M_2(z,H)]^2=\Gamma(n/2)\sum_{m=0}^{\infty}\frac{|\nabla_m H(o)|^2}{m!\Gamma(m+(n/2))}z^{2m}$$

is given by the expression (1.2). Further $|M_2(z,H)|^2\le[M_2(r,H)]^2$, where $r=|z|$, and

$$\limsup_{r\to R}\frac{\log^{(q)}[M_2(r,H)]^2}{\log(R/R-r)}=\limsup_{r\to R}\frac{\log^{(q)} M_2(r,H)}{\log(R/R-r)}$$
$$\le\limsup_{r\to R}\frac{\log^{(q)} M_2(r,H)}{\log(R/R-r)}\equiv\varrho_q(H,R)$$

Thus the growth of the function $[M_2(z)]^2$ is less than or equal to the q-growth order of the harmonic function $H(x)$. Further, by the classical function theoretic result [6] the q-growth order of the function $[M_2(z)]^2$ can be expressed in terms of its Taylor Coefficients as follows :

$$\varrho_q(M_2^2,R)+A(q)=\limsup_{m\to\infty}\frac{\log^{(q-1)} 2m}{\log 2m-\log^+\log^+\frac{\Gamma(n/2)|\nabla_m H(o)|^2}{m!\Gamma(m+\frac{n}{2})}R^{2m}}.$$

Since $\Gamma(m+\frac{n}{2})/m!\sim m^{n/2-1}$, so we get

$$\varrho_q(M_2^2,R)+A(q)=\limsup_{m\to\infty}\frac{\log^{(q-1)} m}{\log m-\log^+\log^+\frac{|\nabla_m H(o)|}{m!}R^m}.$$

Now we observe that $M_1(r,H)$ is also an analytic function of r, and thus can be continued to complex variables, where the radius of convergence of the power series

$$M_1(z,H) = \sqrt{\Gamma(n/2)} \sum_{m=0}^{\infty} \sqrt{dm}\left(|\nabla_m H(o)| / \sqrt{m!\Gamma\left(m+\left(\frac{n}{2}\right)\right)}\right) z^m$$

is also R given by the expression (1.2). Further applying the result of Lemma A, we obtain $M(r,H) \le M_1(r,H) \le \max_{|z|=r} M_1(z,H)$, and thus the q-growth order of $H(x)$ is less than or equal to the q-growth order $\varrho_q(M_1,R)$ of the function $M_1(z,H)$. But in view of the result (1.5), the q-order $\varrho_q(M_1,R)$ is given by the expression

$$\begin{aligned}\varrho_q(M_1,R) + A(q) &= \limsup_{m\to\infty} \frac{\log^{(q-1)} m}{\log m - \log^+ \log^+ \sqrt{\frac{\Gamma(n/2)}{dm}} \frac{|\nabla_m H(o)| R^m}{\sqrt{m!\Gamma(m+(n/2))}}} \\ &= \limsup_{m\to\infty} \frac{\log^{(q-1)} m}{\log m - \log^+ \log^+ \frac{|\nabla_m H(o)|}{\sqrt{m!\Gamma(m+(n/2))}} R^m}\end{aligned}$$

or

$$\begin{aligned}\limsup_{m\to\infty} \frac{\log^{(q-1)} m}{\log m - \log^+ \log^+(|\nabla_m H(o)|/m!) R^m} \\ &= \varrho_q(M_2,R) + A(q) \le \varrho_q(H,R) + A(q) \\ &\le \varrho(M_1,R) + A(q) \\ &= \limsup_{m\to\infty} \frac{\log^{(q-1)} m}{\log m - \log^+ \log^+ \frac{|\nabla_m H(o)|}{m!} R^m}.\end{aligned}$$

Hence the theorem is proved.

REMARK 1. For $q = 2$, this theorem includes Theorem 1 by Fryant and Shankar [3].

THEOREM 2. *Let $H(x)$, $x = (x_1, x_2, \ldots, x_n)$ be harmonic in a neighbourhood of origin in R^n, having index q and suppose the radius R of harmonicity of $H(x)$ is given by* (1.2). *Then the q-type of $H(x)$ is given by the expression*

$$T_q(H,R) = \frac{1}{B(q,H)} \limsup_{m\to\infty} (\log^{(q-2)} m) \left[\frac{\log^+ \frac{|\nabla_m H(o)|}{m!} R^m}{m}\right]^{\varrho_q(H,R)+A(q)},$$

where $B(q,H) = (\varrho_q(H,R)+1)^{(\varrho_q(H,R)+1)} / \varrho_q(H,R)^{\varrho_q(H,R)}$ *for* $q = 2$ *and* $B(q,H) = 1$ *if* $q \ge 3$, *and* $\varrho_q(H,R) > 0$, $q = 2, 3, \ldots$

Proof. In the proof of Theorem 1 it has been shown that the q-order of $[M_2(z,H)]^2$, $H(x)$ and $M_1(z,H)$ are all equal. Thus the q-order of $[M_2(z,H)]^2$ is equal to $\varrho_q(H,R)$, and so the q-type $T_q(M_2^2,R)$, according to (1.7), is given by

$$T_q(M_2^2,R) = \frac{1}{B(q,H)} \limsup_{m\to\infty}(\log^{(q-2)} 2m) \times \left[\frac{\log^+ \frac{\Gamma(n/2)|\nabla_m H(o)|^2}{m!\Gamma(m+(n/2))} R^{2m}}{2m}\right]^{\varrho_q(H,R)+A(q)}$$

$$= \frac{1}{B(q,H)} \limsup_{m\to\infty}(\log^{(q-2)} 2m)\left[\frac{\log^+(\frac{|\nabla_m H(o)|}{m!} R^m)}{m}\right]^{\varrho_q(H,R)+A(q)}$$

$$= \frac{1}{B(q,H)} \limsup_{m\to\infty}(\log^{(q-2)} 2m)\left[\frac{\log^+(\frac{|\nabla_m H(o)|}{m!} R^m)}{m}\right]^{\varrho_q(H,R)+A(q)}$$

Since $[M_2(z,H)]^2$ has a Taylor series expansion with real, nonnegative coefficients,

$$\max_{|z|=r} |M_2(z,H)|^2 = [M_2(r,H)]^2.$$

Thus,

$$\text{(2.1)}\qquad \frac{1}{B(q,H)} \limsup_{m\to\infty}(\log^{(q-2)} 2m)\left[\frac{\log^+ \frac{|\nabla_m H(o)|}{m!} R^m}{m}\right]\varrho_q(H,R) + A(q) = \limsup_{r\to R} \frac{\log^{(q-2)}[M_2(r,H)]^2}{(R/R-r)^{\varrho_q(H,R)}}.$$

For $q=2$, it follows from (2.1) that

$$\{2\varrho_2(H,R)^{\varrho_2(H,R)}/(\varrho_2(HR)+1)^{\varrho_2(H,R)+1}\} \times \limsup_{m\to\infty}\left\{[\log^+ \frac{|\nabla_m H(o)|}{m!} R^m]^{\varrho_2(H,R)+1}/m^{\varrho_2(H,R)}\right\} = 2\limsup_{r\to R}\frac{\log^+ M_2(r,H)}{(R/R-r)^{\varrho_2(H,R)}}.$$

Using Lemma A, we get

$$\leq 2\limsup_{r\to R}\frac{\log^+ M(r,H)}{(R/R-r)^{\varrho_2(H,R)}}.$$

For $q \geq 3$, (2.1) gives

$$\frac{1}{B(q,H)} \limsup_{m\to\infty} (\log^{(q-2)} 2m) \left[\frac{\log^+ \frac{|\nabla_m H(o)|}{m!} R^m}{m}\right]^{\varrho_q(H,R)} = \limsup_{r\to R} \frac{\log^{(q-1)} M_2(r,H)}{(R/R-r)^{\varrho_q(H,R)}}.$$

Again using Lemma A, we have

$$\leq \limsup_{r\to R} \frac{\log^{(q-1)} M(r,H)}{(R/R-r)^{\varrho_q(H,R)}}.$$

Next, the q-order of $M_1(z,H)$ is also $\varrho_q(H,R)$, and thus by the relation (1.7) the q-type $T_q(M_1,R)$ is given by

$$T_q(M_1,R) = \frac{1}{B(q,H)} \limsup_{m\to\infty} (\log^{(q-2)} m) \times \left[\frac{\log^+ \sqrt{\frac{\Gamma(n/2)}{dm}} \frac{|\nabla_m H(o)|}{\sqrt{m!\Gamma(m+(n/2))}} R^m}{m}\right]^{\varrho_q(H,R)+A(q)}$$

$$= \frac{1}{B(q,H)} \limsup_{m\to\infty} (\log^{(q-2)} m) \left[\frac{\log^+ \frac{|\nabla_m H(o)|}{m!} R^m}{m}\right]^{\varrho_q(H,R)+A(q)}$$

We thus have

$$T_q(M_2,R) \leq T_q(H,R) \leq T_q(M_1,R) = \frac{1}{B(q,H)} \limsup_{m\to\infty} (\log^{(q-2)} m) \times \left[\frac{\log^+ \frac{|\nabla_m H(o)|}{m!} R^m}{m}\right]^{\varrho_q(H,R)+A(q)}$$

Hence the theorem is proved.

REMARK 2. For $q = 2$, this theorem includes Theorem 2 by Fryant and Shankar [3].

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