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NOTE ON ROBINSON'S FUNCTIONAL EQUATION

Abstract. The purpose of this note is to give a new proof of the fact, that the only entire solutions of the Robinson's functional equation are given by $f(z) = Az$ or $f(z) = A\sin az$, where A, a are complex constants and a is real or purely imaginary.

Robinson in [3] solved the following functional equation

$$|f(s+it)| = |f(s) + f(it)|, \tag{1}$$

where f is entire function of a complex variable, and s, t are real variables. Hille [2] found all entire solutions of the functional equation

$$|f(s+it)|^2 = |f(s)|^2 + |f(it)|^2. \tag{2}$$

Haruki [1] showed that equations (1) and (2) are equivalent.

Robinson's result from the paper [3] may be presented in the following form:

THEOREM. *The only entire solutions of* (1) *are*

$$f(z) = Az \quad \textit{and} \quad f(z) = A\sin az,$$

where A is an arbitrary complex constant and a is an arbitrary real or purely imaginary constant.

This paper gives a new proof of the above theorem - different to the proof we can find in [3].

First we will prove the following

LEMMA 1. *The only entire solutions of the functional equation*

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2, \qquad x, y \in \mathbb{C} \tag{3}$$

are

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$$f(z) = Az \quad \textit{and} \quad f(z) = A\sin az,$$

where A, a are complex constants.

Proof. Let an entire function f satisfy (3). We may assume that $f \neq 0$. Putting in (3) $y = x = 0$ we have

$$f(0) = 0. \tag{4}$$

Differentiating both sides of (3) with respect to x and y we obtain

$$f''(x+y)f(x-y) - f(x+y)f''(x-y) = 0$$

whence

$$f''(x)f(y) = f(x)f''(y) \tag{5}$$

for all $x, y \in \mathbb{C}$. Since $f \neq 0$, there exists a non-empty domain D, where $f(z) \neq 0$. By (5)

$$\frac{f''(z)}{f(z)} = c^2 \quad \text{for all} \quad z \in D,$$

where c is a complex number. From the Identity Theorem we obtain

$$f''(z) = c^2 f(z) \quad \text{for all} \quad z \in \mathbb{C}.$$

If $c = 0$, then

$$f(z) = Az \tag{6}$$

with respect to (4), where A is a complex constant. If $c \neq 0$, than

$$f(z) = A\frac{e^{cz} - e^{-cz}}{2i} = A\sin az, \tag{7}$$

where A is a constant and $ai = c$. Simple calculations show, that (6) and (7) with arbitrary complex constants a and A satisfy (3). ■

Let f be an entire function. It is easy to see that the function defined as

$$g(z) = \overline{f(\overline{z})} \tag{8}$$

is also entire.

The following result can be found in [1].

LEMMA 2. *If an entire function f satisfies equation* (1), *then*

$$f(x+y)g(x-y) = (f(x) + f(y))(g(x) - g(y)) \tag{9}$$

for all complex x and y, where g is given by (8) and

$$|f(z)| = |g(z)| \tag{10}$$

for all complex z.

In the proof of Theorem, Robinson determined coefficients of power series expansion of an entire solutions of (1). We will apply the above lemmas.

Proof of Theorem. Let an entire function f satisfy equation (1). We may assume that $f \neq 0$. By (10) and by the Maximum Modulus Principle we obtain that

$$g(z) = cf(z) \quad \text{for } z \in \mathbb{C} \tag{11}$$

where c is a complex constant of modulus 1. By (11) we can rewrite (9) in the form

$$f(x+y)f(x-y) = f(x)^2 - f(y)^2.$$

Hence and by Lemma 1 we obtain that

$$f(z) = Az \quad \text{or} \quad f(z) = A\sin az,$$

where A,a are complex constants. To end the proof it is enough to show that $a^2 \in \mathbb{R}$. Write

$$f(z) = A\sin az = A\sum_{n=o}^{\infty}(-1)^n\frac{(az)^{2n+1}}{(2n+1)!}.$$

From (8) and (11) we obtain

$$\overline{A}\sum_{n=0}^{\infty}(-1)^n\frac{(\overline{a}z)^{2n+1}}{(2n+1)!} = cA\sum_{n=0}^{\infty}(-1)^n\frac{(az)^{2n+1}}{(2n+1)!}.$$

Comparing coefficients of z and z^3 yields

$$\overline{A}\overline{a} = cAa \quad \text{and} \quad \overline{A}\overline{a}^3 = cAa^3.$$

Since $acA \neq 0$, we get $\overline{a}^2 = a^2$, hence a^2 is real. ∎

References

[1] H. Haruki, *On the equivalent of Hille's and Robinson's functional equations*, Ann. Pol. Math. 28 (1973), 261–264.

[2] E. Hille, *A Pythagoren functionl euqation*, Ann. Math. 24 (1923), 175–180.

[3] R. M. Robinson, *A curious trigonometric identity*, Amer. Math. Monthly 64 (1957), 83–85.

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