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THE HADAMARD INEQUALITIES FOR s-CONVEX FUNCTIONS IN THE SECOND SENSE

Abstract. We derive some inequalities of Hadamard's type for s-convex functions in the second sense and give some applications connected with special means.

1. Introduction

In the paper [11] the following class of functions was considered.

DEFINITION 1.1. Let $s \in (0,1]$. A real valued function on an interval $I \subset [0,\infty)$ is *s-convex in the second sense* provided

$$f(\alpha u + \beta v) \leq \alpha^s f(u) + \beta^s f(v) \tag{1.1}$$

for all $u, v \in I$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. This is denoted by $f \in K_s^2$.

This definition of s-convexity was considered by Breckner [1], where the problem of whether rationally s-convex functions are s-convex was explored.

We record here some of the results from [11] about s-convex functions.

THEOREM 1.2. *Let $0 < s < 1$. If $f \in K_s^2$ then f is nonnegative.*

Indeed one can put $u = v = x$ and $\alpha = \beta = 1/2$ in (1.1) to get $f(x) \geq 0$.

Recall that a function $f : [0,\infty) \to [0,\infty)$ is a *ϕ-function* if $f(0) = 0$ and f is nondecresing and continuous.

THEOREM 1.3 [11, Corollary 2]. *If Φ is a convex ϕ-function and f is a ϕ-function in K_s^2 then the composition $\Phi \circ f$ belongs to K_s^2. In particular $\Phi^s \in K_s^2$.*

There are, however, ϕ-functions in K_s^2 which are neither of the form $\Phi(u^s)$ nor Φ^s for any convex ϕ-function Φ ([11], Example 3]).

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For a convex function in an interval I, *Hadamard's inequalities* are

$$f((a+b)/2) \leq \int_a^b f(x)\,dx/(b-a) \leq (f(a)+f(b))/2$$

for $a, b \in I$ with $a < b$.

Some generalizations and applications of these inequalities are given in the papers [2–10] and the book [12], which gives further references.

In this paper we prove some inequalities like Hadamard's for s-convex functions in the second sense. We give some applications for numerical inequalities involving special means. The integrals exist because of the following result, which is analogous to the local Lipschitz property of convex functions.

THEOREM 1.4. *Let f be a function on $[a,b]$ which is s-convex in the second sense. Then for $a < y < z < b$ we have*

$$|f(y) - f(z)| \leq (z-y)^s \max\left(f(b)/(b-y)^s, f(a)/(z-a)^s\right) \tag{1.2}$$

so the f is locally Hölder continuous of order s on (a,b). Thus f is Riemann integrable on $[a,b]$.

Proof. Put $t := (z-y)/(b-y)$ so that $z = (1-t)y + tb$ and by (1.1)

$$f(z) \leq (1-t)^s f(y) + t^s f(b) \leq f(y) + t^s f(b).$$

Thus $f(z) - f(y) \leq t^s f(b) = \left((z-y)^s/(b-y)^s\right) f(b)$.

Similarly we have $f(y) - f(z) \leq (z-y)^s f(a)/(z-a)^s$ by choosing t so that $y = ta + (1-t)z$. That establishes (1.2) and the other assertions follow easily.

In the case where f actually takes on its least possible value, 0, we have monotonicity on either side of the zero.

THEOREM 1.5. *Let f be a function on $[a,b]$ which is s-convex in the second sense. If $f(c) = 0$ for some $c \in [a,b]$ then $f(x) \leq f(y)$ if $c \leq x \leq y \leq b$ and $f(x) \geq f(y)$ if $a \leq x \leq y \leq c$.*

Proof. If $c \leq x \leq y \leq b$ then $f(x) \leq t^s f(c) + (1-t)^s f(y)$ by (1.1) if $x = tc + (1-t)y$. So $f(x) \leq (1-t)^s f(y) \leq f(y)$. The inequality on the other side of c is similar.

2. Hadamard's Inequality

Our first result is a generalization of Hadamard's Inequalities which reduces to it in the case $s = 1$.

THEOREM 2.1. *Let f be a s-convex function in the second sense on an interval $I \subset [0, \infty)$ and let $a, b \in I$ with $a < b$. Then*

$$2^{s-1} f((a+b)/2) \leq \int_a^b f(x)\,dx/(b-a) \leq (f(a)+f(b))/(s+1). \tag{2.1}$$

Proof. As f is s-convex on I we have

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

for all $t \in [0,1]$. Integrating this inequality we get

$$\int_0^1 f(ta+(1-t)b)\,dt \leq f(a)\int_0^1 t^s\,dt + f(b)\int_0^1 (1-t)^s\,dt = (f(a)+f(b))/(s+1)$$

and the second inequality in (2.1) follows.

To prove the first inequality, observe that for all $x, y \in I$ we have

$$f((x+y)/2) \leq (f(x)+f(y))/2^s. \tag{2.2}$$

Then put $x := ta + (1-t)b$ and $y := tb + (1-t)a$ to get

$$f((a+b)/2) \leq (f(ta+(1-t)b) + f(tb+(1-t)a))/2^s.$$

Integrating this inequality we get the first part of (2.1).

REMARK 2.2. For any $s \in (0,1]$ the second inequality in (2.1) is sharp.

Indeed by Theorem 1.3 the function $f(x) := x^s$ is s-convex on $[0,1]$ and we have $\int_0^1 x^s\,dx = 1/(s+1) = (f(0)+f(1))/(s+1)$.

3. The mapping H and its properties

Let $f \in L^1[a,b]$ and define

$$H(t) := \frac{1}{b-a}\int_a^b f(tx + (1-t)(a+b)/2)\,dx \tag{3.1}$$

fot $t \in [0,1]$.

THEOREM 3.1. *Let f be a s-convex function in the second sense on an interval $[a,b]$. Then H is s-convex on $[0,1]$ and if $0 \leq t \leq 1$ then*

$$H(t) \geq 2^{s-1} f((a+b)/2). \tag{3.2}$$

Proof. Let $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Then

$$H(\alpha t_1 + \beta t_2)$$

$$= \int_a^b f[(\alpha t_1 + \beta t_2)x + (1 - (\alpha t_1 + \beta t_2))(a+b)/2)]\, dx/(b-a)$$

$$= \int_a^b f(\alpha[t_1 x + (1-t_1)(a+b)/2] + \beta[t_2 x + (1-t_2)(a+b)/2])\, dx/(b-a)$$

$$\leq \int_a^b [\alpha^s f(t_1 x + (1-t_1)(a+b)/2) + \beta^s f(t_2 x + (1-t_2)(a+b)/2)]\, dx/(b-a)$$

$$= \alpha^s H(t_1) + \beta^s H(t_2)$$

which shows that H is s-convex in the second sense.

Now let $t \in (0,1]$. Put $u := tx + (1-t)(a+b)/2$ to get

$$H(t) = \int_q^p f(u)\, du/(p-q)$$

where $p := tb + (1-t)(a+b)/2$ and $q := ta + (1-t)(a+b)/2$.

Applying (2.1) we get

$$\int_q^p f(u)\, du/(p-q) \geq 2^{s-1} f((p+q)/2) = 2^{s-1} f((a+b)/2)$$

and the inequality (3.2) follows.

For $t = 0$ we use Theorem 1.2 which says that $f((a+b)/2) \geq 0$. Define

$$H_1(t) := t^s \int_a^b f(x)\, dx/(b-a) + (1-t)^s f((a+b)/2)$$

and

$$H_2(t) := [f(ta + (1-t)(a+b)/2) + f(tb + (1-t)(a+b)/2)]/(s+1)$$

THEOREM 3.2. *Let f be a s-convex function in the second sense on an interval $[a,b]$. Then for $0 \leq t \leq 1$*

$$H(t) \leq \min(H_1(t), H_2(t)). \tag{3.3}$$

Proof. Applying the second half of (2.1) we have

$$\int_p^q f(u)\, du \leq (f(p) + f(q))/(s+1)$$

$$= (f(tb + (1-t)(a+b)/2) + f(ta + (1-t)(a+b)/2))/(s+1)$$

while for $t = 0$ it again reduces to Theorem 1.2 which says that $f((a+b)/2) \geq 0$.

For H_2, we have

$$f(tx + (1-t)(a+b)/2) \leq t^s f(x) + (1-t)^s f((a+b)/2)$$

and integrating this inequality we get the remaining inequality needed for (3.3).

On the other hand if we let $\tilde{H} := \max(H_1, H_2)$ then we can also get bounds.

THEOREM 3.4. *Let f be a s-convex function in the second sense on an interval $[a,b]$. If $\tilde{H} := \max(H_1, H_2)$ for $0 \leq t \leq 1$ we have then*

$$\tilde{H}(t) \leq t^s(f(a) + f(b))/(s+1) + (1-t)^s 2f((a+b)/2)/(s+1).$$

Proof. We have

$$\begin{aligned} H_2(t) &\leq (t^s f(a) + (1-t)^s f((a+b)/2) + t^s f(b) \\ &\quad + (1-t)^s f((a+b)/2))/(s+1) \\ &= t^s(f(a) + f(b))/(s+1) + (1-t)^s 2f((a+b)/2)/(s+1). \end{aligned}$$

On the other hand, by (2.1) we know that

$$\int_a^b f(x)\,dx/(b-a) \leq (f(a) + f(b))/(s+1)$$

and

$$(1-t)^s f((a+b)/2) \leq (1-t)^s 2f((a+b)/2)/(s+1)$$

for $0 < t < 1$ so that

$$H_1(t) \leq t^s(f(a) + f(b))/(s+1) + (1-t)^s 2f((a+b)/2)/(s+1)$$

as required.

REMARK 3.2. If f is a convex function on $[a,b]$ and H is as above, then we get

$$\inf\{H(t) : t \in [0,1]\} = H(0) = f((a+b)/2)$$

and

$$\sup\{H(t) : t \in [0,1]\} = H(1) = \int_a^b f(x)\,dx/(b-a)$$

which recovers some results from [4] (see also [5] and [9]).

Also we get the inequalities

$$H(t) \le \min\Big(t\int_a^b f(x)\,dx/(b-a) + (1-t)f((a+b)/2),$$
$$[f(ta+(1-t)(a+b)/2) + f(tb+(1-t)(a+b)/2)]/2\Big)$$

and

$$\tilde{H}(t) \le t(f(a)+f(b))/2 + (1-t)f((a+b)/2)$$

for all $t \in [0,1]$, which complement the results from [9].

4. The function F and its properties

Assume that f is Lebesque integrable on $[a,b]$. Consider the function defined by

$$F(t) := \int_a^b\int_a^b f(tx+(1-t)y)\,dx\,dy/(b-a)^2$$

for $t \in [0,1]$. The following result can be proved similarly to Theorem 3.1.

THEOREM 4.1. *Let f be s-convex in the second sense on $[a,b]$. Then F is also s-convex in the second sense and $F(1/2+t) = F(1/2-t)$ for $t \in [0,1]$.*

Now we prove some inequalities regarding this double integral.

THEOREM 4.2. *Let f be s-convex in the second sense on $[a,b]$. Then for $t \in [0,1]$ we have*:

$$2^{1-s}F(t) \ge \int_a^b\int_a^b f((x+y)/2)\,dx\,dy/(b-a)^2, \tag{4.1}$$

$$F(t) \ge 2^{s-1}\max(H(t), H(1-t)), \tag{4.2}$$

$$F(t) \le (t^s + (1-t)^s)\int_a^b f(x)\,dx/(b-a) \tag{4.3}$$

and

$$F(t) \le (f(a)+f(ta+(1-t)b)+f(b)+f(tb+(1-t)a))/(s+1)^2. \tag{4.4}$$

Proof. Since f is s-convex in the second sense we have

$$(f(tx+(1-t)y)+f(ty+(1-t)x))/2^s \ge f((x+y)/2)$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. Integrating over $[a,b]^2$ we get

$$\int_a^b\int_a^b (f(tx+(1-t)y)+f(ty+(1-t)x))/2^s\,dx\,dy \ge \int_a^b\int_a^b f((x+y)/2)\,dx\,dy$$

and as

$$\int_a^b \int_a^b f(tx + (1-t)y)\,dx\,dy = \int_a^b \int_a^b f(ty + (1-t)x)\,dx\,dy$$

this yields (4.1).

Now for $y \in [a, b]$ define

$$H_y(t) := \int_a^b f(tx + (1-t)y)\,dx/(b-a)$$

so that, as in Theorem 3.1 we have

$$H_y(t) = \int_p^q f(u)\,du/(p-q)$$

where $p := tb + (1-t)y$ and $q := ta + (1-t)y$. Applying Hadamard's inequality we get

$$\int_p^q f(u)\,du/(p-q) \geq 2^{s-1} f((p+q)/2) = 2^{s-1} f(t(a+b)/2 + (1-t)y)$$

and integrating over $y \in [a, b]$ we find $F(t) \geq 2^{s-1} H(1-t)$. Since $F(t) = F(1-t)$ we get (4.2).

To get (4.3) we integrate the inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

over $[a, b]^2$.

Now observe that, in the notation above, we have

$$H_y(t) = \int_p^q f(u)\,du/(p-q) \leq (f(tb + (1-t)y) + f(ta + (1-t)y))/(s+1)$$

so that integrating over $[a, b]$ we get

$$F(t) \leq \int_a^b (f(tb + (1-t)y) + f(ta + (1-t)y))\,dy/((s+1)(b-a)).$$

As above we have

$$\int_a^b f(tb + (1-t)y)\,dy/(b-a) \leq (f(b) + f(tb + (1-t)a))/(s+1)$$

and

$$\int_a^b f(ta+(1-t)y)\,dy/(b-a) \le (f(a)+f(tb+(1-t)a))/(s+1)$$

and we add those inequalities to get (4.4).

5. Applications

Suppose that f is a concave function on an interval $[a,b]$ which is also s-convex in the second sense. Then we have

$$\int_a^b f(x)\,dx/(b-a) \le f((a+b)/2) \le 2^{1-s}\int_a^b f(x)\,dx/(b-a) \tag{5.1}$$

which suggests that we need conditions which guarantee that f has those properties.

THEOREM 5.1. *Let Φ be a ϕ-function on $[0,\infty)$ which is twice differentiable on $(0,\infty)$. If $0<s<1$ and*

$$0 \le \Phi(t)\Phi''(t) \le (1-s)[\Phi'(t)]^2$$

for all t then Φ^s is a concave function which is also s-convex in the second sense.

Proof. Note that if $\Phi(t) \ne 0$ then $\Phi''(t) \ge 0$, so Φ is convex. By Theorem 1.3 the function $g := \Phi^s$ is s-convex in the second sense. Then

$$g''(t) = s[\Phi(t)]^{s-2}[\Phi(t)\Phi''(t) - (1-s)(\Phi'(t))^2] \le 0$$

which shows that g is also concave.

COROLLARY 5.2. *Let $s \in (0,1)$. Then for $1 \le p \le 1/s$ the function $g(x) := x^{ps}$ is concave and s-convex in the second sense on $[0,\infty)$.*

Now if we choose $f(t) := t^{ps}$ for $1 \le p \le 1/s$ then we have for $0<a<b$

$$\begin{aligned} 0 &\le (b^{ps+1}-a^{ps+1})/((b-a)(ps+1)) \le ((a+b)/2)^{ps} \\ &\le 2^{1-s}(b^{ps+1}-a^{ps+1})/((b-a)(ps+1)) \end{aligned}$$

so that using (2.1) to get

$$(b^{ps+1}-a^{ps+1})/((ps+1)(b-a)) \le (a^{ps}+b^{ps})/(s+1)$$

we see that in the notation above

$$\begin{aligned} H(t) &= \Big(\int_a^b (tx+(1-t)(a+b)/2)^{ps}dx\Big)/(b-a) \\ &= [(tb+(1-t)(a+b)/2)^{ps+1} \\ &\quad - (ta+(1-t)(a+b)/2)^{ps+1}]/((ps+1)t(b-a)) \end{aligned}$$

for all $t \in (0,1]$. Thus using the results of Section 3 we have the inequalities:

$$\begin{aligned}
&2^{s-1}((a+b)/2)^{ps} \\
&\quad \le [(tb+(1-t)(a+b)/2)^{ps+1} \\
&\qquad - (ta+(1-t)(a+b)/2)^{ps+1}]/((ps+1)t(b-a)) \\
&\quad \le \min(t^s(b^{ps+1}-a^{ps+1})/((ps+1)(b-a)) + (1-t)^s((a+b)/2)^{ps}, \\
&\qquad [(ta+(1-t)(a+b)/2))^{ps} + (tb+(1-t)(a+b)/2))^{ps}]/(s+1)
\end{aligned}$$

and

$$\begin{aligned}
&\max\{t^s(b^{ps+1}-a^{ps+1})/((ps+1)(b-a)) + (1-t)^s((a+b)/2)^{ps}, \\
&\qquad [(ta+(1-t)(a+b)/2))^{ps} + (tb+(1-t)(a+b)/2))^{ps}]/(s+1)\} \\
&\qquad \le t^s(a^{ps}+b^{ps})/(s+1) + (1-t)^s 2((a+b)/2)^{ps}/(s+1)
\end{aligned}$$

for $0 < t < 1$, $0 < s < 1$ and $1 \le p \le 1/s$.

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