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ON SOME NEW PROPERTIES OF THE EXPONENTIAL DISTRIBUTION

1. Introduction

In 1960–1970, there appeared many papers in which the density of finite products of continuous random variables with distributions of type gamma, beta, normal, Bessel and others were defined.

The first aim of this paper is to show how to present the density function of a random variable with an exponential distribution as a density of a finite product of independent random variables X_k where $k \in \{1, \ldots, n\}$.

We have proved the convergence of the series $\sum_1^\infty \ln X^{(k)}$ (formula (11)) to $\ln X$, with the probability one. This result will be used to represent the density of a r.v. X as a density of an infinite product r.v.'s $X^{(k)}$ (formula (18)).

The presentation of a r.v. with a gamma distribution, in the form of a infinite product of r.v.'s with the same distributions, was used by Lu and Richards [10], to define square of the Vandermonde determinant with random elements.

Further we have applied the modified Rogozin [16] and Mieshalkin-Rogozin [15] theorems to evaluate the difference of some distribution functions from the difference of their characteristic functions.

We consider the exponential distribution of a r.v. X with the density function

$$(1) \qquad f_X(x|\alpha) = \alpha^{-1} \exp(-x/\alpha), \quad x \geq 0, \quad \alpha > 0,$$

where α is a scaling parameter.

In a way similar to Zolotarev [18], we use the Mellin transform $M_X(s) = EX^S$, where s is complex, which gives

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$$M_x(s) = \alpha^S \Gamma(s+1), \quad \operatorname{Re} s > -1. \tag{2}$$

The r.v.X can be replaced by a finite (Theorem 1) or an infinite product (Theorem 3), or as the series of Theorem 4.

2. The case of a finite product

Applying the formula 8.335 from [3]

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-0,5} \prod_{k=1}^{n} \Gamma\left(x + \frac{k-1}{n}\right), n \in N, \tag{3}$$

to the gamma function in (2), we can after some simplifications rewrite (2) as

$$M_X(s) = \prod_{k=1}^{n} \left[\alpha^{s/n} (2\pi)^{\frac{1-n}{2n}} n^{\frac{s+0,5}{n}} \Gamma\left(\frac{s+k}{n}\right)\right] = \prod_{k=1}^{n} g_k(s). \tag{4}$$

To each factor $g_k(s)$ of the finite product $\prod_{k=1}^{n} g_k(s)$ we shall use the inverse Mellin transform. The result can be represented as

$$f_{X_k}(x|\alpha, n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s-1} g_k(s) ds, \quad \operatorname{Re} s > -1.$$

Because $c = \operatorname{Re} s > -1$, then assuming $c = 0$ we obtain

$$f_{X_k}(x|\alpha, n) = \frac{n}{(\alpha n)^{k/n} \Gamma(\frac{k}{n})} x^{k-1} \exp\left(-\frac{x^k}{n\alpha}\right), \quad k = 1, \ldots, n. \tag{5}$$

It is easy to check that, for each $k = 1, 2, \ldots, n$ the condition $\int_0^\infty f_k(X|\alpha) dx = 1$ holds. Therefore, by the non-negativity of the integral function, we conclude that the formula (5) has defined the probability density function of r. v. X_k on the interval $[0, \infty)$, and consequently that each factor $g_k(s)$, $k = 1, 2, \ldots, n$ of the product $\prod_{k=1}^{n} g_k(s)$ is the Mellin transform of the r. v. X_k with the density (5).

Since the finite product of the Mellin transform of independent random variables is equal to the Mellin transform of the product of these r.v.'s

$$\prod_{k=1}^{n} M_{X_k}(s) = M_{\prod_{k=1}^{n} X_k}(s) \tag{6}$$

and, by (4), we have

$$M_X(s) = \prod_{k=1}^{n} M_{X_k}(s), \tag{7}$$

then it follows that from (6) and (7) that

$$M_X(s) = M_{\prod_{k=1}^{n} X_k}(s). \tag{8}$$

If we now apply, the inverse Mellin transform to both sides of the last relation, we obtain the following stochastic equality

$$X \stackrel{\text{st}}{=} \prod_{k=1}^{n} X_k, \tag{9}$$

which means that we have proved the following theorem.

THEOREM 1. *The density of the exponential distribution of a r. v. X formula* (1) *is equal to the density of a product $\prod_{x=1}^{n} X_k$ of independent and nonnegative r.v.'s X_k, $k = 1, 2, \dots, n$ with generalized gamma distributions defined by* (5), *with a scaling parameter $(n\alpha)^{1/k}$ and with the shape parameter equal to k, respectively.*

Let us apply the formula (3) to the gamma function in (4) putting there $n_1 x = (s+k)/n$. We obtain then $M_X(s)$ as a double finite product $\prod_{k=1}^{n} \prod_{k_1=1}^{n_1} g_{k,k_1}(s)$, and using (5) we obtain

$$X \stackrel{\text{st}}{=} \prod_{k=1}^{n} \prod_{k_1=1}^{n_1} X_{k,k_1}. \tag{10}$$

These decompositions can be repeated consecutively.

3. Convergence with probability one

Consider the following series $\sum_1^{\infty} \ln X^{(k)}$ where r.v. $X^{(k)}$ has the density

$$f_{X^{(k)}}(x|\alpha, k) = 2^{k-1}\alpha^{-\frac{1}{2}}\Gamma^{-1}\left(\frac{1}{2}\right)x^{2^{k-1}-1}\exp(-x^{2^k}4^{-1}\alpha^{-1}), \quad x \in (0, \infty). \tag{11}$$

We shall prove the following

THEOREM 2. *The series $\sum_1^{\infty} \ln X^{(k)}$ is convergent to $\ln X$ with probability one, where X is r. v. with the density function* (1).

Proof. Let us recall the Marcinkiewicz-Zygmund [13] and Loéve [9] theorems: if r. v.'s are independent and the series $\sum_1^{\infty} EZ_k$, $\sum_1^{\infty} \operatorname{Var} Z_k$ are convergent then $\sum_1^{\infty} Z_k$ is convergent to r. v. Z with probability one and

$$\sum_1^{\infty} EZ_k = EZ, \quad \sum_1^{\infty} \operatorname{Var} Z_k = \operatorname{Var} Z_k. \tag{12}$$

Let us compute $E\ln X^{(k)}$. We have $E\ln X^{(k)} = \int_0^{\infty} \ln x f_{X^{(k)}}(x|\alpha, k)dx$.

Making the following change of variable $x^{2^k} \to 4\alpha u$ we obtain $\ln X = 2^{-k}[\ln(4\alpha) + \ln u]$. Next making use of the formula 3,4,352,1 from [3] we obtain

$$2^{-k}\int_0^\infty \ln u u^{\frac{1}{2}-1}\exp(-u)du = 2^{-k}\Gamma(2^{-1})\psi(2^{-1}),$$

therefore

$$\sum_1^\infty E\ln X^{(k)} = \sum_1^\infty 2^{-k}[\ln(4\alpha)+\psi(2^{-1}]$$
$$= 2\ln 2 + \ln\alpha + (-C - 2\ln 2) = \ln\alpha - C,$$

where C is Euler-Mascheron constant, and $\psi(x) = d/dx \ln\Gamma(x)$ is psi function of Euler. Finally we obtain

$$\sum_1^\infty E\ln X^{(k)} = \ln(4\alpha) + \psi(2^{-1}) = \ln\alpha - C. \tag{13}$$

For the computation of $\sum_1^\infty E\ln^2 X^{(k)}$ see Appendix. Because

$$\sum_1^\infty (E\ln X^{(k)})^2 = \sum_1^\infty 4^{-k}(\ln(4\alpha) - C - 2\ln 2)^2 = 3^{-1}(\ln\alpha - C)^2$$

therefore $\operatorname{Var}\ln X = \pi^2/6 = \sum_1^\infty \operatorname{Var}\ln X^{(k)}$.

Finally, the following equivalence theorem holds: for series of independent r.v.'s convergence in pr., convergence of laws and a.s. convergence are equivalent.

4. The case of an infinite product

Applying the Knar formula 8,324 from [3]

$$\Gamma(x+1) = 4^x\prod_{k=1}^\infty\left[\Gamma\left(\frac{1}{2}+\frac{x}{2^k}\right)\Gamma^{-1}\left(\frac{1}{2}\right)\right], \quad \operatorname{Re} x > -1. \tag{14}$$

to the gamma function in (2), we can rewrite (2) as

$$M_X(s) = \prod_{k=1}^\infty\left[(4\alpha)^{s/2^k}\Gamma\left(\frac{1}{2}+\frac{x}{2^k}\right)\Gamma^{-1}\left(\frac{1}{2}\right)\right] = \prod_{k=1}^\infty h_k(s). \tag{15}$$

We now apply the inverse Mellin transform to every factor $h_k(s)$.

$$f_{X^{(k)}}(x|\alpha,k) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s-1}h_k(s)ds, \quad \operatorname{Re} s > -1.$$

Since $\operatorname{Re} s > -1$ we can assume $c = 0$ and we obtain

$$f_{X^{(k)}}(x|\alpha,k) = \frac{2^k}{2\sqrt{\alpha}\Gamma(\frac{1}{2})}x^{2^{k-1}-1}\exp\left(-\frac{x^{2^k}}{4\alpha}\right). \tag{16}$$

It is easy to check that, for each $k \in N$ the condition $\int_0^\infty f_{X^{(k)}}(x|\alpha,k)dx = 1$ holds. Therefore since $f_X^{(k)}(x|\alpha,k) \geq 0$ we conclude that the formula (16) has defined the probability density function of the r. v. $X^{(k)}$ on the interval $[0,\infty)$ as the two-parametric gamma distribution. Finally we shall prove that $\prod_1^\infty h_k(s)$ is the Mellin transform of $\prod \infty_1 X^{(k)}$.

Since $M_x(it) = \varphi_{\ln X}(t)$, where $\varphi_X(t)$ is characteristic function of X, we have also (infinite convolutions [12] 3.7)

$$\varphi_{\ln X}(t) = \prod_1^\infty h_k(it) = \prod_1^\infty \varphi_{\ln X^{(k)}}(t) = \varphi_{\sum_1^\infty \ln X^{(k)}}(t) \tag{17}$$
$$= \varphi_{\ln \prod_{k=1}^n X^{(k)}}(t).$$

So from one to one correspondence of the characteristic functions and r.v.'s, it follows that

$$X \stackrel{\text{st}}{=} \prod_1^\infty X^{(k)}.$$

Thus, we have proved the following theorem:

THEOREM 3. *The density of the random variable X with the exponential density (formula (1)) is equal to the density of an infinite product $\prod_{k=1}^n X^{(k)}$ of independent random variables $X^{(k)}$ with two-parameter gamma densities defined by (16) with a scaling parameter $(4\alpha)^{1/2^k}$ respectively and a parameter of shape 2^{k-1} respectively, so*

$$X \stackrel{\text{st}}{=} \prod_1^\infty X^{(k)}. \tag{18}$$

In this case we shall apply the Knar (14) formula to both the gamma functions in (15) and we replace $(4\alpha)^{1/2^k}$ by $(4\alpha)^{\frac{2}{2^k 2^{k_1}}}$. Applying, to each factor of the obtained product the inverse Mellin transform we obtain the density functions of r.v.'s X_{k,k_1} and the result can be described as

$$X \stackrel{\text{st}}{=} \prod_1^\infty X^{(k)} = \prod_{k=1}^\infty \prod_{k_1=1}^\infty X^{(k),(k_1)} \tag{19}$$

We can repeat this method any number of times.

5. Infinite convolution of distribution functions (d.f.'s)

Let us consider the sequence of factors $u_n = \prod_1^n (1 - i\alpha t)^{-\frac{1}{2^k}}$. It is easy to check that the factors of the product u_n are the characteristic functions of the two parameter r.v'.s $\lambda_{(k)}$ with density function given by the following formula (20)

$$X_{(k)} \sim \Gamma_2(x|\alpha,k) = \alpha^{-\frac{1}{2^k}}\Gamma^{-1}(2^{-k})x^{2^{-k}-1}\exp(-x\alpha^{-1}), \quad x \geq 0, \quad k \in N \tag{20}$$

the d.f'.s of which are $F_k(x)$. The sequence u_n is uniformly convergent in each finite interval, to the characteristic function

$$\varphi_x(t) = (1 - i\alpha t)^{-1} \tag{21}$$

of the density function (1). The latter property is a sufficient and necessary for the infinite convolution $F_1 * F_2 * \ldots * F_n * \ldots = \prod_1^\infty *F_n$ to be convergent [12, Theorem (3.6.2)] to d.f. $F(x) = \int_{-\infty}^{X} \alpha^{-1}\exp(-y\alpha^{-1})dy$. As it was shown by Titchmarsh [17] and Markushewitsch [14] the latter convergence is uniform in sense of theory of infinite products [12, formula after non numerable formula after (3.7.3)].

Therefore

$$\varphi_X(t) = \prod_1^\infty \varphi_{X_{(k)}}(t) = \varphi_{\sum_{k=1}^\infty X_{(k)}}(t). \tag{22}$$

From (22) it follows that

$$X \stackrel{\text{st}}{=} \sum_{k=1}^{\infty} X_{(k)}. \tag{23}$$

THEOREM 4. *The density of the exponential distribution of a r.v. X (formula* (1)) *is equal to the density of an infinite sum* $\sum_1^\infty X_{(k)}$ *of independent and nonnegative r.v.'s* $X_{(k)}$. *The two parameter r.v.'s* $X_{(k)}$ *have densities determined by formula* (20).

6. The modified Rogozin theorem for exponential distribution

Next our aim is to determine an estimation of supremum of the difference between the two d.f.'s $F(x)$ and $G(x)$ if the sup $|f(t) - g(t)|$ is known, where $f(t)$ and $g(t)$ are the corresponding characteristic functions. We shall assume that $F(x)$ is the d.f. of an exponential distribution (formula 1) and $G(x)$ is unknown, d.f. (see below). This problem was treated for the first time by Gnedenko and Kolmogorov [2] where they gave an estimation of such a difference, using the integral $\int_{-T}^{T} |f(t) - g(t)|/tdt$. Next, Dyson showed [1] that it is not possible to determine, for any $\delta > 0$ such an $\epsilon > 0$ being dependent on δ only, that $\sup_x |F(x) - G(x)| < \delta$ results from $\sup_t |f(t) - g(t)| < \epsilon$. A full solution of this problem was given by Rogozin [16]. In my paper, $F(x) = 1 - \exp(-x/\alpha)$ and for $G(x)$ we adopt the d.f. of a finite sum of n r.v.'s with densities given by formula (20).

Let us take any number γ from the interval $(0, 0.1]$ (γ could be chosen from the interval $(0,1)$ but then the difference of two d.f.'s may be greater than unity, what makes the estimation trivial). Let us divide the interval $[0,\infty)$ for $M+1$ intervals [14]

$$[0,a_1),[a_1,a_2),\ldots,[a_{M-1},a_M),[a_M,\infty)$$

of the same lengths $a_i - a_{i-1} = \gamma/c$ for $i = 1,2,\ldots,M$, $a_0 = 0$ where c is the supremum of the density (1), so $c = 1/a$, a_1 and a_M are chosen in such a way that

$$F(a_1) \le 0,5\gamma, F(a_M) \ge 1 - 0,5\gamma.$$

Applying the method of Rogozin [16] to the intervals on halfline we obtain

$$\text{(24)} \qquad \sup|F(x) - F_{\sum_1^n X_{(k)}}(x)| \le N(\gamma)[\epsilon + 16c(\pi\gamma T)^{-1}] + 3\gamma.$$

and $N(\gamma) = 2M$ is function γ dependent on $F(x)$, decreasing and $\to \infty$ when $\gamma \to 0$. As an estimation of $2M$ Rogozin accept

$$\text{(25)} \qquad M = [(F^{-1}(1-\gamma/2) - F^{-1}(\gamma/2))(\alpha\gamma)^{-1}] + 1,$$

where $[A]$ is the integer part of A.

Let us calculate the value of $2M$ for the exponential distribution function

$$F(x) = 1 - \exp\left(\frac{-x}{\alpha}\right) = y, \text{ so } x = -\alpha\ln(1-y) = F^{-1}(y)$$

and finally

$$[F^{-1}(1-\gamma/2) - F^{-1}(\gamma/2)](\alpha\gamma)^{-1} + 1 = \gamma^{-1}\ln(2/\gamma - 1) + 1, \gamma \in (0,01).$$

Then the supremum of LHS of (24) can be expressed as

$$\text{(26)} \qquad \sup_X |1 - \exp(-x/\alpha) - F_{\sum_1^n X_{(k)}}(x) \\ \le 2[1 + 2/\gamma\ln(2/\gamma - 1)](16/(\pi\alpha\gamma T) + \epsilon_1) + 3\gamma$$

what proves the following:

THEOREM 5. *If the absolute value of the difference of two characteristic functions:* $\varphi_X(t)$ *determined by formula (21) and* $\varphi_{\sum_1^n X_0^{(k)}}(t)$ *where* $X_{(k)} \sim \Gamma_2(x|\alpha,k)$ *(formula (20)) satisfies the condition*

$$\text{(27)} \qquad |\varphi_X(t) - \varphi_{\sum_1^n X_{(k)}}(t)| < \epsilon_1 \text{ for } n > n_1 \text{ and } |t| < T$$

then the supremum of the absolute value of the difference of the corresponding distribution functions fulfils the inequality (26), where γ *is any number from the interval* $(0, 0.1)$ *and* T *where existence see text after formula (33).*

For ilustration we present values of the first factor of the RHS of (26), $N(\gamma) = 1 + 2/\gamma \ln(2/\gamma - 1)$ in the table below:

γ	0,001	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09	0,1
$N(\gamma)$	15202	1060	461	281	196	148	117	96	80	69	60

As it is seen from (26) the second factor of the RHS, for fixed α and γ, depends on ϵ and on T (see text after formula (33)).

7. The modified Mieszalkin-Rogozin theorem

Five years later Mieszalkin and Rogozin published a paper [15] extending the results from [16]. We shall use here the theorem 1 from the paper [15] in a somewhat modified form, concerning functions determined in the interval $[0, \infty)$.

Let us assume that d.f. $F(x)$ and a function of bounded variation (f.b.v.) $G(x) = F_{\sum_1^n X_{(k)}}(x)$ fulfil the following conditions:

C1. $F(0) = G(0) = 0$

C2. $G'(x)$ exists for any $x > 0$ and $|G'(x)| \leq A < \infty$

C3. $|\varphi_X(t) - \varphi_{\sum_1^n X_{(k)}}(t)| < \epsilon$ for $|t| < T$, where $\varphi(t)$, $\varphi_{\sum_1^n X_{(k)}}(t)$, $g(t)$ are the characteristic functions of $F(x)$ and $G(x)$ correspondingly.

Then for $A, T, \epsilon > 0$ and $L > \frac{2}{T}$, the following inequality holds

$$(28) \quad |F(x) - G(X)| < 16(\ln(LT) + 2^{-3}\pi^{-1} + 1)^{-1}\left[\epsilon \ln(LT) + \frac{A}{T} + \gamma(L)\right],$$

where

$$(29) \qquad \gamma(L) = \operatorname{var} G(x)_{0 \leq x < \infty} - \sup_x \operatorname{var} G(y)_{x \leq y \leq x+L}.$$

The $G(x) = 1 - \exp(-\frac{x}{\alpha})$ is a f.b.v, so $\operatorname{var} G(x) = 1$, and

$$(30) \qquad \begin{aligned} \sup_x \operatorname{var} G(y)_{x \leq y \leq x+L} &= \sup[G(x+L)_x - G(x)] \\ &= \sup \exp_x(-x/\alpha)[-\exp(-L/\alpha)] \\ &= 1 - \exp(-L/\alpha). \end{aligned}$$

Thus $\gamma(L) = \exp(-L/\alpha)$. From the condition $(C2)$ we have $|G'(x)| \leq \frac{1}{\alpha} = A$, so the R.H.S side of (28) becomes the estimation of the unknown d.f. $F_{\sum_1^n X_{(k)}}(x)$

$$(31) \qquad \begin{aligned} &\sup |1 - \exp(-x/\alpha) - F_{\sum_1^n X_{(k)}}(x)| \\ &\leq 16[\ln(LT) + 2^{-3}\pi^{-1} + 1]^{-1} \cdot [\epsilon \ln(LT) + (\alpha T)^{-1} + \exp(-L/\alpha)], \end{aligned}$$

where $LT > 2$ and for the choise of T see text after formula (33).

We known that

$$\varphi_X(t) - \varphi_{\sum X_k}(0) = 0. \tag{32}$$

Because of the fact that two functions of L.H.S of (31) are absolutely continuous it follows from the Riemann-Lebesgue lemma [12] that

$$\lim_{t\to\infty}(\varphi_X(t) - \varphi_{\sum_1^n X_k}(t)) = 0. \tag{33}$$

Then the two cases are possible:
1. for some $\sup_t$ of the LHS of $(C3)$ is larger than ϵ_n and then there exists such $T > 0$ that for $|t| < T$ the inequality (31) is fulfiled
2. for all n the supremum over t of the LHS of $(C3)$ is $\le \epsilon_n$ and then for any $T > 0$ the inequality (31) is fulfiled.

This ends our proof.

8. Appendix

Let us compute $\sum_1^\infty E\ln^2 X_{(k)}$. Making the same change of previous variable we obtain
$\ln^2 X = 4^{-k}[\ln^2(4\alpha) + 2\ln(4\alpha)\ln u + u^2]$ and
$E\ln^2 X_k = 2^{k-1}\alpha^{-1/2}\Gamma^{-1}(2^{-1})[J_k^{(1)} + J_k^{(2)} + J_k^{(3)}]$, where
$J_k^{(1)} = \ln^2(4\alpha)8^{-k}\int_0^\infty u^{-1/2}\exp(-u)du = 8^{-k}2\sqrt{\alpha}\ln^2(4\alpha)\Gamma(2^{-1})$
$J_k^{(2)} = 2\ln(4\alpha)\int_0^\infty \ln u u^{-1/2}\exp(-u)du = 8^{-k}2\sqrt{\alpha}2\ln(4\alpha)\psi(2^{-1})$ (formula [3.4.352,4])
$J_k^{(3)} = 8^{-k}2\sqrt{\alpha}\Gamma(2^{-1})[\psi^2(2^{-1}) + \xi(2,-2^{-1})]$ (formula [3.4.353,2])
where $\xi(2,-2^{-1}) = 4\sum_1^\infty(2n-1)^{-2} = \pi^2/2 + \psi(2^{-1}) = \pi^2/2 - C - 2\ln 2$.
We obtain

$$\begin{aligned}&\sum_1^\infty E\ln^2 X_k\\ &= \sum_1^\infty 4^{-k}\left[\ln^2(4\alpha) + 2\ln(4\alpha)\psi(2^{-1}) + \psi^2(2^{-1}) + 4\sum_1^\infty(2n-1)^2\right]\\ &= 3^{-1}[\ln(4\alpha) + \psi(2^{-1})]^2 + \pi^2/6 = 3^{-1}(\ln\alpha - C)^2 + \pi^2/6.\end{aligned}$$

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