DEMONSTRATIO MATHEMATICA
Vol. XXXII No 3 1999

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SOME PROPERTIES OF ALMOST WEAKLY CONTINUOUS MULTIFUNCTIONS

Abstract. In [13], the authors defined a multifunction $F: X \to Y$ to be upper (lower) almost weakly continuous if for each $x \in X$ and each open set V containing $F(x)$ (resp. $F(x) \cap V \neq \emptyset$) $x \in \mathrm{Int}(\mathrm{Cl}(F^{+}(\mathrm{Cl}(V))))$ (resp. $x \in \mathrm{Int}(\mathrm{Cl}(F^{-}(\mathrm{Cl}(V))))$). In this paper, further characterizations and several properties concerning upper (lower) almost weakly continuous multifunctions are obtained.

1. Introduction

In 1978, Smithson [25] and Popa [18] introduced independently the concept of almost continuous (in the sense of Husain) multifunctions in topological spaces. In 1988, Popa [21] introduced the notion of precontinuous multifunctions and showed that H-almost continuity and precontinuity are equivalent for multifunctions. Janković [5] defined almost weakly continuous functions as a generalization of both weakly continuous functions due to Levine [7] and almost continuous functions in the sense of Husain [4]. Noiri and Popa studied several properties of almost weakly continuous functions in [11] and [22]. Paul and Bhattacharyya [16] introduced the notion of quasi precontinuous functions. Popa and Noiri [22] had shown that the quasi-precontinuity is equivalent to the almost weak continuity. Recently, Noiri and Popa [13] have defined and investigated the notion of almost weakly continuous multifunctions. The purpose of the present paper is to obtain further characterizations of upper (resp. lower) almost weakly continuous multifunctions and investigate other properties of such multifunctions.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A

1991 *Mathematics Subject Classification*: 54C60.
Key words and phrases: preopen, multifunctions, precontinuous, almost weakly continuous.

is said to be *preopen* [9] (resp. *semi-open* [8],*α-open* [10]) if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$). The family of all preopen (resp. semi-open) sets of X containing a point $x \in X$ is denoted by $\mathrm{PO}(X,x)$ (resp. $\mathrm{SO}(X,x)$). The family of all preopen (resp. semi-open) sets in X is denoted by $\mathrm{PO}(X)$ (resp. $\mathrm{SO}(X)$). The complement of a preopen (resp. semi-open) set is said to be *preclosed* [3] (resp. *semi-closed*). The intersection of all preclosed sets of X containing A is called the *preclosure* [4] of A and is denoted by $\mathrm{pCl}(A)$. Similarly, the semi-closure $\mathrm{sCl}(A)$ is defined. The union of all preopen sets of X contained in A is called the *preinterior* of A and is denoted by $\mathrm{pInt}(A)$. It is obvious that X - $\mathrm{pCl}(A) = \mathrm{pInt}(X - A)$ for every subset A of X. The *θ-closure* [26] of A, denoted by $\mathrm{Cl}_\theta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Cl}(U) \neq \emptyset$ for every open neighborhood U of x. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be *θ-closed*. The complement of a θ-closed set is said to be *θ-open*. It is shown in [26] that $\mathrm{Cl}_\theta(A)$ is closed in X for any subset A of X and that $\mathrm{Cl}(U) = \mathrm{Cl}_\theta(U)$ for each open set U of X. A subset A of X is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(A)) = A$ (resp. $\mathrm{Cl}(\mathrm{Int}(A)) = A$).

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces and $F : X \to Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function. For a multifunction $F : X \to Y$, following [2], we shall denote the upper and lower inverse of a subset B of a space Y by $F^+(B)$ and $F^-(B)$, respectively, that is,

$$F^+(B) = \{x \in X : F(x) \subset B\} \text{ and } F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

DEFINITION 1. A multifunction $F : X \to Y$ is said to be
(1) *upper H-almost continuous* [18, 25] if for each point $x \in X$ and each open set V containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(V)))$,
(2) *lower H-almost continuous* [18, 25] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(V)))$.

DEFINITION 2. A multifunction $F : X \to Y$ is said to be
(1) *upper precontinuous* [21] if $F^+(V) \in \mathrm{PO}(X)$ for each open set V of Y,
(2) *lower precontinuous* [21] if $F^-(V) \in \mathrm{PO}(X)$ for each open set V of Y.

DEFINITION 3. A multifunction $F : X \to Y$ is said to be
(1) *upper almost weakly continuous* (briefly *u.a.w.c.*) [13] if for each point $x \in X$ and each open set V containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{Cl}(V))))$,
(2) *lower almost weakly continuous* (briefly *l.a.w.c.*)[13] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{Cl}(V))))$.

LEMMA 1 (Noiri and Popa [13]). *A multifunction $F : X \to Y$ is u.a.w.c. (resp. l.a.w.c.) if and only if for each $x \in X$ and each open set V such that*

$F(x) \subset V$ (resp. $F(x) \cap V \neq \emptyset$), there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$ (resp. $F(U) \cap \mathrm{Cl}(V) \neq \emptyset$).

3. Characterizations

THEOREM 1. *For a multifunction $F : X \to Y$ the following statements are equivalent:*
(1) *F is u.a.w.c.;*
(2) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset B of Y;*
(3) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$ *for every open set V of Y;*
(4) $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$ *for every preopen set V of Y;*
(5) $\mathrm{pCl}(F^-(\mathrm{Int}(K))) \subset F^-(K)$ *for every regular closed set K of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Suppose that $x \in X - F^-(\mathrm{Cl}_\theta(B))$. Then $x \in F^+(Y - \mathrm{Cl}_\theta(B))$ and $\mathrm{Cl}_\theta(B)$ is closed set of Y. Therefore, by Lemma 1 for some $U \in \mathrm{PO}(X, x)$ we have

$$U \subset F^+(\mathrm{Cl}(Y - \mathrm{Cl}_\theta(B))) = F^+(Y - \mathrm{Int}(\mathrm{Cl}_\theta(B))) = X - F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))).$$

Thus, we obtain $U \cap F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))) = \emptyset$ and $x \in X - \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B))))$. This shows that $\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^-(\mathrm{Cl}_\theta(B))$ *for every subset B of Y.*

(2) $\Rightarrow$ (3): The proof is obvious since $\mathrm{Cl}_\theta(V) = \mathrm{Cl}(V)$ for every open set V of Y.

(3) $\Rightarrow$ (4): Let V be any preopen set of Y. Then we have $V \subset \mathrm{Int}(\mathrm{Cl}(V))$ and hence

$$\begin{aligned}\mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) &= \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))))) \\ &\subset F^-(\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(V)))) = F^-(\mathrm{Cl}(V)).\end{aligned}$$

(4) $\Rightarrow$ (5): Let K be any regular closed set of Y. Then, we have $\mathrm{Int}(K) \in \mathrm{PO}(Y)$ and hence $\mathrm{pCl}(F^-(\mathrm{Int}(K))) = \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(K))))) \subset F^-(\mathrm{Cl}(\mathrm{Int}(K))) = F^-(K)$.

(5) $\Rightarrow$ (1): Let V be any open set of Y. Then $\mathrm{Cl}(V)$ is regular closed in Y. Therefore, we obtain $\mathrm{pCl}(F^-(V)) \subset \mathrm{pCl}(F^-(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^-(\mathrm{Cl}(V))$. It follows from [13, Theorem 3.1] that F is *u.a.w.c.*.

THEOREM 2. *For a multifunction $F : X \to Y$ the following statements are equivalent:*
(1) *F is l.a.w.c.;*
(2) $\mathrm{pCl}(F^+(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset F^+(\mathrm{Cl}_\theta(B))$ *for every subset B of Y;*
(3) $\mathrm{pCl}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^+(\mathrm{Cl}(V))$ *for every open set V of Y;*
(4) $\mathrm{pCl}(F^+(\mathrm{Int}(\mathrm{Cl}(V)))) \subset F^+(\mathrm{Cl}(V))$ *for every preopen set V of Y;*
(5) $\mathrm{pCl}(F^+(\mathrm{Int}(K))) \subset F^+(K)$ *for every regular closed set K of Y.*

Proof. The proof is similar to that of Theorem 1.

A function $f : X \to Y$ is said to be *almost weakly continuous* [5] if for every open set V of Y, $f^{-1}(V) \subset \mathrm{Int}(\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))))$.

COROLLARY 1 (Popa and Noiri [22])). *For a function $f : X \to Y$ the following statements are equivalent:*
(1) *f is almost weakly continuous;*
(2) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}_\theta(B)))) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ *for every subset B of Y;*
(3) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set V of Y;*
(4) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(\mathrm{Cl}(V)))) \subset f^{-1}(\mathrm{Cl}(V))$ *for every preopen set V of Y;*
(5) $\mathrm{pCl}(f^{-1}(\mathrm{Int}(K))) \subset f^{-1}(K)$ *for every regular closed set K of Y.*

THEOREM 3. *Let Y be a regular space. For a multifunction $F : X \to Y$ the following properties are equivalent:*
(1) *F is upper precontinuous;*
(2) *$F^-(\mathrm{Cl}_\theta(\mathrm{B}))$ is preclosed in X for every subset B of Y;*
(3) *$F^-(K)$ is preclosed in X for every θ-closed set K of Y;*
(4) *$F^+(V) \in \mathrm{PO}(X)$ for every θ-open set V of Y.*

Proof. (1) $\Rightarrow$ (2): Let B be any subset of Y. Then $\mathrm{Cl}_\theta(B)$ is closed in Y and it follows from [21, Theorem 2.5] that $F^-(\mathrm{Cl}_\theta(B))$ is preclosed in X.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (4): Let V be any θ-open set of Y. By (3), $F^-(Y - V)$ is preclosed in X and $F^-(Y - V) = X - F^+(V)$. Therefore, we obtain $F^+(V) \in \mathrm{PO}(X)$.

(4) $\Rightarrow$ (1): Let V be any open set of Y. Since Y is regular, V is θ-open in Y and by (4) we have $F^+(V) \in \mathrm{PO}(X)$. Therefore, F is upper precontinuous.

THEOREM 4. *Let Y be a regular space. For a multifunction $F : X \to Y$ the following properties are equivalent:*
(1) *F is lower precontinuous;*
(2) *$F^+(\mathrm{Cl}_\theta(\mathrm{B}))$ is preclosed in X for every subset B of Y;*
(3) *$F^+(K)$ is preclosed in X for every θ-closed set K of Y;*
(4) *$F^-(V) \in \mathrm{PO}(X)$ for every θ-open set V of Y;*
(5) *F is l.a.w.c.*

Proof. We prove only the implication (5) $\Rightarrow$ (1), the proof of the other being similar to that of Theorem 3 we omit. The proof of the implication (4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1): Let V be any open set of Y and $x \in F^-(V)$. Then, we have $F(x) \cap V \neq \emptyset$. Since Y is regular, there exists an open set W of Y such that $F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. Since F is *l.a.w.c.*, by Lemma 1 there exists $U \in \mathrm{PO}(X, x)$ such that $U \subset F^-(\mathrm{Cl}(W)) \subset F^-(V)$. Therefore, we have $x \in U \subset \mathrm{Int}(\mathrm{Cl}(U)) \subset \mathrm{Int}(\mathrm{Cl}(F^-(V)))$ and hence $F^-(V) \subset \mathrm{Int}(\mathrm{Cl}(F^-(V)))$. This shows that $F^-(V) \in \mathrm{PO}(X)$ and that F is lower precontinuous.

DEFINITION 4. A multifunction $f : X \to Y$ is said to be
(1) *upper almost precontinuous* [24] if for each point $x \in X$ and each open set V containing $F(x)$, $x \in \mathrm{Int}(\mathrm{Cl}(F^+(\mathrm{sCl}(V))))$,
(2) *lower almost precontinuous* [24] if for each point $x \in X$ and each open set V such that $F(x) \cap V \neq \emptyset$, $x \in \mathrm{Int}(\mathrm{Cl}(F^-(\mathrm{sCl}(V))))$.

COROLLARY 2 (Popa, Noiri and Ganster [24]). *Let Y be a regular space. For a multifunction $F : X \to Y$ the following statements are equivalent:*
(1) *F is lower precontinuous;*
(2) *F is lower almost precontinuous;*
(3) *F is l.a.w.c.*

COROLLARY 3 (Popa and Noiri [22]). *Let Y be a regular space. For a function $f : X \to Y$ the following properties are equivalent :*
(1) *f is almost continuous (in the sense of Husain);*
(2) *$f^{-1}(\mathrm{Cl}_\theta(B))$ is preclosed in X for every subset B of Y;*
(3) *$f^{-1}(K)$ is preclosed in X for every θ-closed set K of Y;*
(4) *$f^{-1}(V) \in \mathrm{PO}(X)$ for every θ-open set V of Y;*
(5) *f is almost weakly continuous.*

In order to obtain further characterizations of *u.a.w.c.* (resp. *l.a.w.c.*) multifunctions, we recall some definitions. For a multifunction $F : X \to Y$, by $\mathrm{Cl}F : X \to Y$ [1] (resp. $\mathrm{pCl}F : X \to Y$ [21]) we denote a multifunction defined as follows: $(\mathrm{Cl}F)(x) = \mathrm{Cl}(F(x))$ (resp. $(\mathrm{pCl}F)(x) = \mathrm{pCl}(F(x))$) for each point $x \in X$.

DEFINITION 5. A subset A of a space X is said to be
(1) *α-paracompact* [27] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X,
(2) *α-regular* [6] if for each $a \in A$ and each open set U of X containing a, there exists an open set G of X such that $a \in G \subset \mathrm{Cl}(G) \subset U$.

LEMMA 2 (Noiri and Popa [14]). *If $F : X \to Y$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y $(\mathrm{Cl}F)^+(V) = (\mathrm{pCl}F)^+(V) = F^+(V)$.*

THEOREM 5. *Let $F : X \to Y$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following statements are equivalent:*
(1) *F is u.a.w.c.;*
(2) $\mathrm{pCl}F$ *is u.a.w.c.;*
(3) $\mathrm{Cl}F$ *is u.a.w.c.*

Proof. We set $G = \mathrm{pCl}F$ or $\mathrm{Cl}F$. Suppose that F is *u.a.w.c.* Let $x \in X$ and V be any open set of Y containing $G(x)$. By Lemma 2, we have $x \in$

$G^+(V) = F^+(V)$ and hence by Lemma 1 there exists $U \in$PO(X, x) such that $F(u) \subset$ Cl(V) for each $u \in U$; hence $G(u) \subset$ Cl(V) for each $u \in U$. Therefore, we obtain $G(U) \subset$ Cl(V). This shows that G is *u.a.w.c.*

Conversely, suppose that G is *u.a.w.c.* Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 2, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. By Lemma 1, there exists $U \in$PO(X, x) such that $G(U) \subset$ Cl(V). Therefore, we obtain $F(U) \subset$ Cl(V). This shows that F is *u.a.w.c.*

LEMMA 3 (Noiri and Popa [14]). *For a multifunction $F : X \to Y$, it follows that* $(\mathrm{Cl}F)^-(V) = (\mathrm{pCl}F)^-(V) = F^-(V)$ *for each open set V of Y.*

THEOREM 6. *For a multifunction $F : X \to Y$, the following statements are equivalent*:
(1) *F is l.a.w.c.*;
(2) pClF *is l.a.w.c.*;
(3) ClF *is l.a.w.c.*

Proof. By using Lemma 3 this can be shown similarly to that of Theorem 5.

4. Some properties

In this section, we investigate several properties of *u.a.w.c.* multifunctions into Urysohn (or Hausdorff) spaces.

DEFINITION 6. A multifunction $F : X \to Y$ is said to be
(1) *upper almost α-continuous* [23] if for each point $x \in X$, each $U \in$ SO(X, x) and each open set V containing $F(x)$, there exists a nonempty open set $G \subset U$ such that $F(G) \subset$ sCl(V),
(2) *upper weakly continuous* [19, 25] if each point $x \in X$ and each open set V containing $F(x)$, there exists an open set U containing x such that $F(U) \subset$ Cl(V).

THEOREM 7. *Let $F, G : X \to Y$ be multifunctions into a Hausdorff space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X$. If F is u.a.w.c., G is upper almost α-continuous and $F(x) \cap G(x) \neq \emptyset$ on a dense semi-open set D of X, then $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.*

Proof. Let $A = \{ x \in X : F(x) \cap G(x) \neq \emptyset \}$. Suppose that $x \in X - A$. Then we have $F(x) \cap G(x) = \emptyset$. Since $F(x)$ and $G(x)$ are compact sets of a Hausdorff space Y, there exist open sets V and W of Y such that $F(x) \subset V, G(x) \subset W$ and $V \cap W = \emptyset$; hence Cl$(V)\cap$Int(Cl$(W)) = \emptyset$. Since G is upper almost α-continuous, there exists an α-open set U_1 containing x such that $F(U_1) \subset$ Int(Cl(W)) [23, Theorem 3]. Since F is *u.a.w.c.*, by Lemma 1 there exists $U_2 \in$PO(X, x) such that $G(U_2) \subset$Cl(W). Now, put $U = U_1 \cap U_2$, then we have $U \in$PO(X, x) [3, Lemma 4.2] and $U \cap A = \emptyset$. Therefore, we have $x \in X -$ pCl(A) and hence A is preclosed in X. On the other hand, $F(x) \cap G(x) \neq \emptyset$ on D and hence $D \subset A$. Since D is semi-open and dense in X, we have $\mathrm{pCl}(D) = D \cup \mathrm{Cl}(\mathrm{Int}(D)) = D \cup \mathrm{Cl}(D) = \mathrm{Cl}(D)$ and $X = \mathrm{Cl}(D) = \mathrm{pCl}(D) \subset \mathrm{pCl}(A) = A$. Therefore, we obtain $F(x) \cap G(x) \neq \emptyset$ for each $x \in X$.

A function $f : X \to Y$ is said to be *almost α-continuous* [12] if for each regular open set V of Y, $f^{-1}(V)$ is α-open in X.

COROLLARY 4 (Paul and Bhattacharyya [17]). *Let $f, g : X \to Y$ be functions into a Hausdorff space Y. If f is almost weakly continuoous, g is almost α-continuous and $f = g$ on a dense open set D on X, then $f = g$ on X.*

LEMMA 4 (Smithson [25]). *If A and B are disjoint compact subsets of a Urysohn space X, then there exist open sets U and V of X such that $A \subset U$, $B \subset V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.*

THEOREM 8. *Let $F, G : X \to Y$ be multifunctions into an Urysohn space Y and $F(x)$, $G(x)$ compact in Y for each $x \in X$. If F is upper weakly continuous and G is u.a.w.c., then $A = \{ x \in X : F(x) \cap G(x) \neq \emptyset \}$ is preclosed in X.*

Proof. Let $x \in X - A$. Then we have $F(x) \cap G(x) = \emptyset$. By Lemma 4, there exist open sets V and W such that $F(x) \subset V, G(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since F is upper weakly continuous, there exists an open neighborhood U_1 of x such that $F(U_1) \subset \mathrm{Cl}(V))$. Since G is *u.a.w.c.*, by Lemma 1 there exists $U_2 \in \mathrm{PO}(X, x)$ such that $G(U_2) \subset \mathrm{Cl}(W)$. Seting $U = U_1 \cap U_2$, then we have $U \in \mathrm{PO}(X, x)$ and $U \cap A = \emptyset$. Therefore, A is preclosed in X.

A function $f : X \to Y$ is said to be *weakly continuous* [7] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Cl}(V)$.

COROLLARY 5 (Paul and Bhattacharyya [17]). *Let $f, g : X \to Y$ be functions into an Urysohn space Y. If f is weakly continuous and g is almost weakly continuous, then $\{x \in X : f(x) = g(x)\}$ is preclosed in X.*

THEOREM 9. *Let $F_1 : X_1 \to Y$ and $F_2 : X_2 \to Y$ be multifunctions into an Urysohn space Y and $F_i(x)$ compact in Y for each $x \in X_i$ and each $i = 1, 2$. If F_1 and F_2 are u.a.w.c., then $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is a preclosed set in the product space $X_1 \times X_2$.*

Proof. We shall show that $X_1 \times X_2 - A$ is preopen in $X_1 \times X_2$. Let $(x_1, x_2) \in X_1 \times X_2 - A$. Then we have $F_1(x_1) \cap F_2(x_2) = \emptyset$. By Lemma 4, there exist open sets V_i such that $F_i(x_i) \subset V_i$ for $i = 1,2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_i is *u.a.w.c.*, by Lemma 1 there exists $U_i \in \mathrm{PO}(X_i, x_i)$ such that $F(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Now, set $U = U_1 \times U_2$, then we have $U \in$

$\mathrm{PO}(X_1 \times X_2)$ [21, Lemma 2] and $(x_1, x_2) \in U \subset X_1 \times X_2 - A$. Therefore, $X_1 \times X_2 - A$ is preopen in $X_1 \times X_2$ and A is preclosed in $X_1 \times X_2$.

A multifunction $F : X \to Y$ is said to be *injective* if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$. A space X is said to be *pre-T_2* [15] if for each distinct points x_1, x_2 of X there exist preopen sets U_i such that $x_i \in U_i$ for $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$.

THEOREM 10. *If $F : X \to Y$ is an u.a.w.c. injective multifunction into a Urysohn space Y and $F(x)$ is compact in Y for each $x \in X$, then X is pre-T_2.*

Proof. For any distinct points x_1, x_2 of X, we have $F(x_1) \cap F(x_2) = \emptyset$ since F is injective. Since $F(x_i)$ is compact in the Urysohn space Y, by Lemma 4 there exist open sets V_i such that $F(x_i) \subset V_i$ for $i = 1,2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F is *u.a.w.c.*, by Lemma 1 there exist open sets $U_i \in \mathrm{PO}(X, x_i)$ such that $F(U_i) \subset \mathrm{Cl}(V_i)$ for $i = 1, 2$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$ and hence X is pre-T_2.

COROLLARY 6 (Paul and Bhattacharyya [17]). *If $f : X \to Y$ is an almost weakly continuous injection and Y is an Urysohn space, then X is pre-T_2.*

THEOREM 11. *Let $F_1, F_2 : X \to Y$ be multifunctions into an Urysohn space Y and $F_i(x)$ compact in Y for each $x \in X$ and each $i = 1, 2$. If $F_1(x) \cap F_2(x) \neq \emptyset$ for each $x \in X$, F_1 is u.a.w.c. and F_2 is upper weakly continuous, then a multifunction $F : X \to Y$, defined as follows $F(x) = F_1(x) \cap F_2(x)$ for each $x \in X$, is u.a.w.c.*

Proof. Let $x \in X$ and V be an open set of Y such that $F(x) \subset V$. Then, $A = F_1(x) - V$ and $B = F_2(x) - V$ are disjoint compact sets. By Lemma 4, there exist open sets V_1 and V_2 such that $A \subset V_1, B \subset V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since F_1 is *u.a.w.c.*, by Lemma 1 there exists $U_1 \in \mathrm{PO}(X, x)$ such that $F_1(U_1) \subset \mathrm{Cl}(V_1 \cup V)$. Since F_2 is upper weakly continuous, there exists an open set U_2 containing x such that $F_2(U_2) \subset \mathrm{Cl}(V_2 \cup V)$. Set $U = U_1 \cap U_2$. Then we have $U \in \mathrm{PO}(X, x)$. If $y \in F(x_0)$ for any $x_0 \in U$, then $y \in \mathrm{Cl}(V_1 \cup V) \cap \mathrm{Cl}(V_2 \cup V) = (\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)) \cup \mathrm{Cl}(V)$. Since $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$, we have $y \in \mathrm{Cl}(V)$ and hence $F(U) \subset \mathrm{Cl}(V)$. Therefore, by Lemma 1 F is *u.a.w.c.*

For a multifunction $F : X \to Y$, the graph $G(F) = \{(x, F(x)) : x \in X\}$ is said to be *strongly preclosed* if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in \mathrm{PO}(X, x)$ and $V \in \mathrm{PO}(Y, y)$ such that $[U \times \mathrm{pCl}(V)] \cap G(F) = \emptyset$.

LEMMA 5. *A multifunction $F : X \to Y$ has a strongly preclosed graph if and only if for each $(x, y) \in (X \times Y) - G(F)$, there exist $U \in \mathrm{PO}(X, x)$ and $V \in \mathrm{PO}(Y, y)$ such that $F(U) \cap \mathrm{pCl}(V) = \emptyset$.*

Proof. This proof is obvious.

THEOREM 12. *If $F : X \to Y$ is an u.a.w.c. multifunction such that $F(x)$ is compact for each $x \in X$ and Y is an Urysohn space, then $G(F)$ is strongly preclosed.*

Proof. Let $(x, y) \in (X \times Y) - G(F)$, then $y \in Y - F(x)$. By Lemma 4, there exist open sets V and W of Y such that $y \in V, F(x) \subset W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since F is *u.a.w.c.*, by Lemma 1 there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(W)$. Therefore, we have $F(U) \cap \mathrm{pCl}(V) \subset \mathrm{Cl}(W) \cap \mathrm{Cl}(V) = \emptyset$ and by Lemma 5 $G(F)$ is strongly preclosed.

COROLLARY 7 (Paul and Bhattacharyya [17]). *If $f : X \to Y$ is an almost weakly continuous function and Y is an Urysohn space, then $G(f)$ is strongly preclosed.*

The *prefrontier* [24] of a subset A of a space X, denoted by $\mathrm{pFr}(A)$, is defined by $\mathrm{pFr}(A) = \mathrm{pCl}(A) \cap \mathrm{pCl}(X - A) = \mathrm{pCl}(A) - \mathrm{pInt}(A)$.

THEOREM 13. *The set of all points x of X at which a multifunction $F : X \to Y$ is not u.a.w.c. (resp. l.a.w.c.) is identical with the union of the prefrontier of the upper (resp. lower) inverse images of the closures of open sets containing (resp. meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not *u.a.w.c.* Then, by Lemma 1 there exists an open set V containing $F(x)$ such that $U \cap (X - F^+(\mathrm{Cl}(V))) \neq \emptyset$ for every $U \in \mathrm{PO}(X, x)$. Therefore, we have $x \in \mathrm{pCl}(X - F^+(\mathrm{Cl}(V)))$. Since $x \in F^+(V)$, we have $x \in \mathrm{pCl}(F^+(\mathrm{Cl}(V)))$ and hence $x \in \mathrm{pFr}(F^+(\mathrm{Cl}(V)))$.

Conversely, if F is *u.a.w.c.* at x, then for any open set V of Y containing $F(x)$ there exists $U \in \mathrm{PO}(X, x)$ such that $F(U) \subset \mathrm{Cl}(V)$; hence $U \subset F^+(\mathrm{Cl}(V))$. Therefore, we obtain $x \in U \subset \mathrm{pInt}(F^+(\mathrm{Cl}(V)))$. This contradicts with the fact that $x \in \mathrm{pFr}(F^+(\mathrm{Cl}(V)))$. Thus F is not *u.a.w.c.* at x. The case of *l.a.w.c.* can be similarly shown.

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Received June 30, 1998.