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ON A CLASS OF GENERALIZED FREDHOLM OPERATORS, IV

Abstract. In the present paper we investigate generalized Fredholm operators (see [15], [16] and [17]) on a complex Hilbert space H. The main results of this paper read as follows:

1. $T \in \Phi_g(H)$ if and only if $T = T_1 \oplus T_2$, where T_1 is a Fredholm operator such that $\alpha(T_1 - \lambda)$ and $\beta(T_1 - \lambda)$ are constant for $|\lambda|$ small and T_2 is a finite-dimensional nilpotent operator.
2. If $T \in \Phi_g(H)$ is not finite-dimensional, then

$$\operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}) = \lim_{n\to\infty} \gamma_e(T^n)^{1/n},$$

where $\gamma_e(T)$ denotes the essential minimum modulus of T.

1. Preliminaries and notations

The present paper is a continuation of our previous papers [15], [16] and [17]. Notations and definitions not explicitly given are taken from [15], [16] and [17]. In this section, X always denotes a complex, infinite-dimensional Banach space. $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on X. We will use the following notations:

$$\mathcal{F}(X) = \{T \in \mathcal{L}(X) : \dim T(X) < \infty\},$$
$$\mathcal{K}(X) = \{T \in \mathcal{L}(X) : T \text{ is compact}\},$$
$$\Phi(X) = \{T \in \mathcal{L}(X) : T \text{ is Fredholm}\},$$
$$\Phi_g(X) = \{T \in \mathcal{L}(X) : T \text{ is generalized Fredholm}\}.$$

Let $T \in \Phi(X)$. It is well-known (see [5]) that there are $\delta > 0$ and integers $c_1, c_2 \geq 0$ such that

$$T - \lambda I \in \Phi(X) \text{ for } |\lambda| < \delta,$$

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$$\alpha(T-\lambda I) = c_1 \leq \alpha(T) \text{ for } 0<|\lambda|<\delta,$$
$$\beta(T-\lambda I) = c_2 \leq \beta(T) \text{ for } 0<|\lambda|<\delta$$

and

$$\alpha(T)-\alpha(T-\lambda I) = \beta(T)-\beta(T-\lambda I) \text{ for } |\lambda|<\delta.$$

We define the *jump* $j(T)$ of $T \in \Phi(X)$ by

$$j(T) = \alpha(T) - c_1 \qquad (= \beta(T) - c_2).$$

An operator $T \in \mathcal{L}(X)$ is called an *operator of Saphar type* if T is relatively regular and $N(T) \subseteq \bigcap_{n=1}^{\infty} T^n(X)$. By $\mathcal{S}(X)$ we denote the set of all operators of Saphar type.

In [12] we have introduced the following concepts:

$$\rho_{rr}(T) = \{\lambda \in \mathbb{C} : T-\lambda I \in \mathcal{S}(X)\},$$
$$\sigma_{rr}(T) = \mathbb{C} \setminus \rho_{rr}(T).$$

For properties of $\rho_{rr}(T)$ and $\sigma_{rr}(T)$ see [8], [10], [12], [13] and [14].

The following proposition is due to T. Kato [5].

PROPOSITION 1.1. *For $T \in \Phi(X)$ we have*

$$j(T) = 0 \iff T \in \mathcal{S}(X).$$

PROPOSITION 1.2. *Let $T \in \Phi_g(X)$. Then:*

(1) *$T^n(X)$ is closed for each $n \in \mathbb{N}$.*
(2) *If $T \in \mathcal{S}(X)$ then $T \in \Phi(X)$ and $j(T) = 0$.*
(3) *There is an integer $m \geq 0$ such that*
   (i) *$N(T) \cap T^m(X) = N(T) \cap T^{m+k}(X)$ for $k \geq 0$,*
   (ii) *$N(T^m) + T(X) = N(T^{m+k}) + T(X)$ for $k \geq 0$,*
   (iii) *$N(T^m) + T(X)$ is closed.*

Proof. (1) follows from [15], Proposition 4.8 (b).

(2): Since $N(T) \subseteq T(X)$ and $\dim N(T) \cap T(X) < \infty$ ([15], Theorem 4.8 (a)), we have $\alpha(T) < \infty$. Theorem 3.22 in [16] shows that $T \in \Phi(X)$. By Proposition 1.1 we get $j(T) = 0$.

(3): (i) and (ii) follow from Proposition 1.6 in [17]. To show (iii) take a sequence (y_n) in $N(T^m) + T(X)$ with $y_n \to y_0$ $(n \to \infty)$. Then there are sequences (z_n) and (x_n) such that

$$y_n = Tz_n + x_n, \quad z_n \in X, \quad x_n \in N(T^m) \quad (n \in \mathbb{N}).$$

This gives

$$T^m y_n = T^{m+1} z_n \to T^m y_0 \quad (n \to \infty).$$

By (1), $T^{m+1}(X)$ is closed, thus $T^m y_0 \in T^{m+1}(X)$, hence $T^m y_0 = T^{m+1} z_0$ for some $z_0 \in X$. It follows that $y_0 - Tz_0 \in N(T^m)$. Therefore $y_0 \in T(X) + N(T^m)$. ∎

By definition, the *minimum modulus* $\gamma(T)$ of $T \in \mathcal{L}(X)$ is the supremum of all real numbers $\gamma \geq 0$ such that

$$\|Tx\| \geq \gamma \ \mathrm{dist}(x, N(T)) \ \text{ for all } \ x \in X.$$

It is well known that

$$T(X) \ \text{is closed} \iff \gamma(T) > 0.$$

PROPOSITION 1.3. *Let $T \in \mathcal{L}(X)$.*

(1) *If $T \in \mathcal{L}(X)^{-1}$, then $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \mathrm{dist}(0, \sigma(T))$.*
(2) *If $T \in \Phi(X)$ then $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and is equal to the supremum of all $\delta > 0$ such that $T - \lambda I \in \Phi(X)$ and $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are constant on $0 < |\lambda| < \delta$.*
(3) *If 0 is a pole of the resolvent $(T - \lambda I)^{-1}$, then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \mathrm{dist}(0, \sigma(T) \setminus \{0\}).$$

Proof. (1) Since T is invertible in $\mathcal{L}(X)$, $\gamma(T) = \|T^{-1}\|^{-1}$. It follows that $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \lim_{n\to\infty} (\|(T^{-1})^n\|^{1/n})^{-1} = r(T^{-1})^{-1}$. Then it is easy to see that $\lim_{n\to\infty} \gamma(T^n)^{1/n} = \mathrm{dist}(0, \sigma(T))$.

(2) [3], Theorem 5.

(3) [1], Corollary 5.2. ∎

PROPOSITION 1.4. *Let $T \in \mathcal{L}(X)$ and $U \in \mathcal{L}(X)^{-1}$. Then*

$$\gamma(UT) \leq \|U\| \gamma(T).$$

Proof. Since $N(T) = N(UT)$, we get for $x \notin N(UT)$ that

$$\frac{\|UTx\|}{\mathrm{dist}(x, N(UT))} \leq \|U\| \frac{\|Tx\|}{\mathrm{dist}(x, N(T))},$$

thus $\gamma(UT) \leq \|U\| \dfrac{\|Tx\|}{\mathrm{dist}(x, N(T))}$ for all $x \notin N(T)$. This shows that $\gamma(UT) \leq \|U\|\gamma(T)$. ∎

REMARK. In [15] we have defined genaralized Fredholm operators only on infinite-dimensional Banach spaces. For a finite-dimensional Banach space X we define the class $\Phi_g(X)$ as in Definition 1.2 in [15]. In this case we have $\Phi_g(X) = \mathcal{L}(X)$.

2. Quasi-Fredholm operators

For the rest of this paper, H always denotes a complex, infinite-dimensional Hilbert space.

For $T \in \mathcal{L}(H)$ it is well-known that

$$T(H) \text{ is closed } \iff T \text{ is relatively regular.}$$

Furthermore, we have

$$T \in \Phi_g(H) \iff T^* \in \Phi_g(H).$$

Quasi-Fredholm operators have been defined by J.P. Labrousse [7] as a generalization of semi-Fredholm operators.

DEFINITION 2.1. $T \in \mathcal{L}(H)$ is called a *quasi-Fredholm operator* if there is an integer $m \geq 0$ such that

$$T^m(H) \cap N(T) = T^{m+k} \cap N(T) \text{ for all } k \geq 0$$

and

$$T^m(H) \cap N(T) \text{ and } T(H) + N(T^m) \text{ are closed.}$$

We denote the set of all quasi-Fredholm operators on H by $q\Phi(H)$.

PROPOSITION 2.2. $\Phi_g(H) \subseteq q\Phi(H)$.

Proof. Proposition 1.2 (1) and (3). ∎

DEFINITION 2.3 (Kato's decomposition). We say that $T \in \mathcal{L}(H)$ has a *Kato decomposition* if there exist two closed, T-invariant subspaces H_1 and H_2 such that

$$(2.1) \qquad H = H_1 \oplus H_2, \quad T|_{H_1} \in \mathcal{S}(H_1) \quad \text{and} \quad T|_{H_2} \quad \text{is nilpotent.}$$

Notation. Let $T \in \mathcal{L}(H)$. If there are closed, T-invariant subspaces H_1 and H_2 of H with $H = H_1 \oplus H_2$, then we always denote the operators $T|_{H_1}$ and $T|_{H_2}$ by T_1 and T_2, respectively. In this case the operator T can be written in the form $T = T_1 \oplus T_2$. We say that T has the *Kato decomposition* (H_1, H_2) if the subspaces H_1 and H_2 also satisfy (2.1).

REMARK. T. Kato has shown in [5] that each semi-Fredholm operator has a Kato decomposition.

THEOREM 2.4. *For $T \in \mathcal{L}(H)$, Definitions* 2.1 *and* 2.3 *are equivalent.*

Proof. [7], Théorème 3.2.2. ∎

In [10] M. Mbekhta has characterized operators in $q\Phi(H)$ as follows:

THEOREM 2.5. *For $T \in \mathcal{L}(H)$ the following assertions are equivalent:*

(1) $T \in q\Phi(H)$.

(2) *0 is an isolated point of* $\sigma_{rr}(T)$ *and* $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ *exists and is strictly positive.*

PROPOSITION 2.6. *Let* $T \in \Phi_g(H)$ $(\subseteq q\Phi(H))$. *For each Kato decomposition* (H_1, H_2) *of* T *we have*

$$T_1 \in \Phi(H_1), \quad j(T_1) = 0 \quad \text{and} \quad T_2 \in \mathcal{F}(H_2). \tag{2.2}$$

Proof. By [17], Proposition 1.5, we have $T_i \in \Phi_g(H_i)$ $(i = 1, 2)$. Since $T_1 \in \mathcal{S}(H_1)$, we get from Proposition 1.2 (2) that $T_1 \in \Phi(H_1)$ and $j(T_1) = 0$. If $\dim H_2 < \infty$, then it is clear that $T_2 \in \mathcal{F}(H_2)$. If $\dim H_2 = \infty$, then we get $T_2 \in \mathcal{F}(H_2)$ from Theorem 3.7 in [16] as follows: T_2 is nilpotent, thus $\sigma(T_2) = \sigma_\Phi(T_2) = \{0\}$, hence T_2 is a Riesz operator and so $T_2 \in \mathcal{F}(H_2)$. ∎

Now we are in a position to characterize operators in $\Phi_g(H)$.

THEOREM 2.7. *For* $T \in \mathcal{L}(H)$ *the following conditions are equivalent:*

(1) $T \in \Phi_g(H)$.
(2) $T = T_1 \oplus T_2$ *with* T_1 *Fredholm and* T_2 *finite-dimensional.*
(3) $T = T_1 \oplus T_2$ *with* T_1 *Fredholm,* $j(T_1) = 0$ *and* T_2 *finite-dimensional.*
(4) $T \in q\Phi(H)$, *and* (2.2) *holds for each Kato decomposition* (H_1, H_2) *of* T.
(5) $T \in q\Phi(H)$, *and there is a Kato decomposition* (H_1, H_2) *of* T *such that* (2.2) *holds.*

Proof. It follows from Proposition 2.2, Theorem 2.4 and Proposition 2.6 that (1) implies (2), (3), (4) and (5).

(3) "$\Longrightarrow$" (2): Clear.

(4) "$\Longrightarrow$" (5): Clear.

Now suppose that (2) or (5) holds. Then T_1 is Fredholm and T_2 is finite-dimensional. Therefore, T_1 and T_2 are generalized Fredholm. Then it follows from [17], Proposition 1.5, that $T \in \Phi_g(H)$. ∎

Recall that an operator $T \in \mathcal{L}(H)$ is said to have the *single valued extension property* (SVEP) in $\xi \in \mathbb{C}$ if for any analytic function $f : D \to H$, D is a neighbourhood of ξ, with $(T - \lambda I)f(\lambda) \equiv 0$ on D, we have $f \equiv 0$.

PROPOSITION 2.8. *Let* $T \in \mathcal{L}(H)$. *If* $T = T_1 \oplus T_2$ *and* $\xi \in \mathbb{C}$, *then*

$$T \text{ has the SVEP in } \xi \iff T_1 \text{ and } T_2 \text{ have the SVEP in } \xi.$$

Proof. [2], Proposition 1.1.3. ∎

THEOREM 2.9. *Let* $T \in \Phi_g(H)$.

(1) *If* $T = T_1 \oplus T_2$ *with* T_1 *Fredholm and* T_2 *finite-dimensional, then*
   (i) T *has the SVEP in* 0 $\iff$ T_1 *has the SVEP in* 0 $\iff$ $p(T_1) < \infty$. *In this case we have* $\mathrm{ind}(T_1) \leq 0$;

(ii) T^* *has the SVEP in* $0 \iff T_1^*$ *has the SVEP in* $0 \iff q(T_1) < \infty$. *In this case we have* $\mathrm{ind}(T_1) \geq 0$.

(2) *If* $T = T_1 \oplus T_2$ *with* T_1 *Fredholm,* $j(T_1) = 0$ *and* T_2 *finite-dimensional, then*

(i) T *has the SVEP in* $0 \iff \alpha(T_1) = 0 \iff T_1$ *is left invertible;*

(ii) T^* *has the SVEP in* $0 \iff \beta(T_1) = 0 \iff T_1$ *is right invertible;*

(iii) T *and* T^* *have the SVEP in* $0 \iff T_1$ *is invertible.*

Proof. (1) (i): Since T_2 is finite-dimensional, T_2 has the SVEP (in each $\xi \in \mathbb{C}$). From Proposition 2.8 we therefore derive that T has the SVEP in 0 if and only if T_1 has the SVEP in 0. Theorem 2.5 in [17] gives

$$T_1 \text{ has the SVEP in } 0 \iff p(T_1) < \infty.$$

If $p(T_1) < \infty$ then we see from [4], Satz 104.6, that $\mathrm{ind}(T_1) \leq 0$.

(1) (ii): From Proposition 2.8 and [17], Theorem 2.5, we get

$$T^* \text{ has the SVEP in } 0 \iff T_1^* \text{ has the SVEP in } 0 \iff q(T_1) < \infty.$$

Use again [4], Satz 104.6, to derive $\mathrm{ind}(T_1) \geq 0$ if $q(T_1) < \infty$.

(2) (i): Since $j(T_1) = 0$, we see from (1) and [17], Theorem 2.3 (1), that

$$T \text{ has the SVEP in } 0 \iff p(T_1) < \infty$$
$$\iff \alpha(T_1) = 0 \iff T_1 \text{ is left invertible.}$$

(2) (ii): Similar.

(2) (iii) follows (i) and (ii) ∎

3. The minimum modulus in C^*-algebras

In this section, $\mathcal{B}$ always denotes a complex C^*-algebra with identiy $e \neq 0$. Without loss of generality, we assume $\|e\| = 1$. Fix $t \in \mathcal{B}$ and define the linear operator $\mathcal{T} \in \mathcal{L}(\mathcal{B})$ by

$$\mathcal{T}b = tb \qquad (b \in \mathcal{B}).$$

We define the *minimum modulus* $\gamma(t)$ of t by

$$\gamma(t) = \gamma(\mathcal{T}).$$

Proposition 3.1. *Let* $t \in \mathcal{B}$ *and* $\mathcal{T} \in \mathcal{L}(\mathcal{B})$ *as above.*

(1) $\sigma(t) = \sigma(\mathcal{T})$.
(2) $\gamma(t^n) = \gamma(\mathcal{T}^n)$ *for each* $n \in \mathbb{N}$.
(3) $\gamma(t) = \inf\{\sigma(|t|) \setminus \{0\}\}$.
(4) $\gamma(t)^2 = \gamma(|t|)^2 = \gamma(t^*t) = \gamma(tt^*) = \gamma(|t^*|)^2 = \gamma(t^*)^2$.
(5) *If* $u \in \mathcal{B}^{-1}$ *then* $\gamma(ut) \leq \|u\|\gamma(t)$.

Proof. (1) We only have to show that $0 \in \rho(t) \iff 0 \in \rho(\mathcal{T})$. Take $0 \in \rho(t)$, put $s = t^{-1}$ and define the operator $\mathcal{S} \in \mathcal{L}(\mathcal{B})$ by $\mathcal{S}b = sb$ $(b \in \mathcal{B})$. Then, for each $b \in \mathcal{B}$,

$$\mathcal{T}\mathcal{S}b = tsb = b = stb = \mathcal{S}\mathcal{T}b,$$

thus $\mathcal{S} = \mathcal{T}^{-1}$ and $0 \in \rho(\mathcal{T})$.

If $0 \in \rho(\mathcal{T})$, put $s = \mathcal{T}^{-1}(e)$, then $ts = \mathcal{T}\mathcal{T}^{-1}(e) = e$. Define the operator $\mathcal{S} \in \mathcal{L}(\mathcal{B})$ by $\mathcal{S}b = sb$ $(b \in \mathcal{B})$. It follows that $\mathcal{T}\mathcal{S}b = tsb = b$, thus $\mathcal{S}$ is a right inverse of $\mathcal{T}$. Since $\mathcal{T}$ as a unique right inverse in $\mathcal{L}(\mathcal{B})$, it follows that $\mathcal{S} = \mathcal{T}^{-1}$. Therefore, $e = \mathcal{S}\mathcal{T}(e) = \mathcal{S}(t) = st$. Hence $ts = e = st$, thus $0 \in \rho(t)$.

(2) Clear, since $\mathcal{T}^n b = t^n b$ $(n \in \mathbb{N})$.

(3) and (4) follow from (0.6) and (0.7) in [11].

(5) follows from Proposition 1.4. ■

Recall from [15] that the set of *generalized invertible* elments $\mathcal{B}^g$ of $\mathcal{B}$ is given by

$$\mathcal{B}^g = \{t \in \mathcal{B} : \text{ there is } s \in \mathcal{B} \text{ with } tst = t \text{ and } e - st - ts \in \mathcal{B}^{-1}\}.$$

By Proposition 3.9 in [15] we have:

$$\text{(3.1)} \qquad \begin{array}{l} \text{if } t \in \mathcal{B}^g, \text{ then there is a unique } s \in \mathcal{B} \\ \text{with } tst = t,\ sts = s \text{ and } ts = ts. \end{array}$$

PROPOSITION 3.2. *Let $t \in \mathcal{B}^g$ and $s \in \mathcal{B}$ such that* (3.1) *holds. Then:*

(1) *t and s are not quasinilpotent.*
(2) *$0 \in \rho(t)$ or 0 is a pole of order* 1 *of $(t - \lambda e)^{-1}$.*
(3) *$\lim_{n\to\infty} \gamma(t^n)^{1/n}$ exists and*

$$\lim_{n\to\infty} \gamma(t^n)^{1/n} = \operatorname{dist}(0, \sigma(t) \setminus \{0\} = \left(\lim_{n\to\infty} \|s^n\|^{1/n}\right)^{-1}.$$

Proof. (1) and (2): [16], Proposition 2.7.

(3) In view of [16], Proposition 2.7, we only have to show that $\lim_{n\to\infty} \gamma(t^n)^{1/n} = \operatorname{dist}(0, \sigma(t)\setminus\{0\})$. By Proposition 3.1 we have $\gamma(t^n) = \gamma(\mathcal{T}^n)$ $(n \in \mathbb{N})$, $0 \in \rho(\mathcal{T})$ or 0 is a pole of order 1 of $(\mathcal{T} - \lambda)^{-1}$. From Proposition 1.3 (1) and (3) we derive

$$\lim_{n\to\infty} \gamma(t^n)^{1/n} = \lim_{n\to\infty} \gamma(\mathcal{T}^n)^{1/n} = \operatorname{dist}(0, \sigma(\mathcal{T}) \setminus \{0\}) = \operatorname{dist}(0, \sigma(t) \setminus \{0\}). \ ■$$

REMARK. If $t \in \mathcal{B}^g \setminus \{0\}$, then $\sigma(t) \setminus \{0\}$ is compact and $\neq \emptyset$ since t is not quasinilpotent.

4. Stability radii for operators in $\Phi_g(H)$

A proof of the following result can be found in [6], [8] or [14].

PROPOSITION 4.1. *If* $T \in \mathcal{S}(H)$ *then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \operatorname{dist}(0, \sigma_{rr}(T)).$$

Notation. We write $\widetilde{\mathcal{L}}$ for the C^*-algebra $\mathcal{L}(H)/\mathcal{K}(H)$ and $\widetilde{T}$ for the coset $T + \mathcal{K}(H)$ of T in $\widetilde{\mathcal{L}}$. Then we have $\sigma(\widetilde{T}) = \sigma_\Phi(T)$.

The *essential minimum modulus* $\gamma_e(T)$ of $T \in \mathcal{L}(H)$ is defined by

$$\gamma_e(T) = \gamma(\widetilde{T}).$$

From Proposition 3.1 (4) we get

$$\gamma_e(T) = \gamma_e(|T|) = \inf\{\sigma_\Phi(|T|) \setminus \{0\}\}.$$

PROPOSITION 4.2. *Let* $T \in \mathcal{L}(H)$.

(1) $\gamma_e(T) = \max_{K\in\mathcal{K}(H)} \gamma(T+K) \geq \gamma(T)$.

(2) *If* $T(H)$ *is closed, then* $\gamma_e(T) = \sup_{F\in\mathcal{F}(H)} \gamma(T+F) \geq \gamma(T) > 0$.

Proof. (1) [11], Théorème 1 and Théorème 2.

(2) [11], Théorème 7. ∎

For the next result recall from [15], Proposition 1.3, that if $T \in \Phi_g(H)$ there is some $\delta > 0$ such that $T - \lambda I \in \Phi(H)$ for $0 < |\lambda| < \delta$. Furthermore, by [16], (3.6), there is some $S \in \mathcal{L}(H)$ with

$$TST = T, \quad STS = S, \quad ST - TS \in \mathcal{F}(H) \tag{4.1}$$

and, if $T \notin \mathcal{F}(H)$,

$$\operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}) = r(\widetilde{S})^{-1}. \tag{4.2}$$

THEOREM 4.3. *If* $T \in \Phi_g(H) \setminus \mathcal{F}(H)$, *then* $\lim_{n\to\infty} \gamma_e(T^n)^{1/n}$ *exists, and*

$$\operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}) = \lim_{n\to\infty} \gamma_e(T^n)^{1/n}.$$

Proof. Since $T \notin \mathcal{F}(H)$ and $T(H)$ is closed, we have $T \notin \mathcal{K}(H)$, thus $\widetilde{T} \in \widetilde{\mathcal{L}}^g$ and $\widetilde{T} \neq \widetilde{0}$. From Proposition 3.2 we get that $\lim_{n\to\infty} \gamma_e(T^n)^{1/n} = \lim_{n\to\infty} \gamma(\widetilde{T}^n)^{1/n}$ exists, and

$$\lim_{n\to\infty} \gamma_e(T^n)^{1/n} = \operatorname{dist}(0, \sigma(\widetilde{T}) \setminus \{0\}) = \operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}).$$ ∎

COROLLARY 4.4. *Let $T \in \Phi_g(H) \setminus \mathcal{F}(H)$ and $S \in \mathcal{L}(H)$ such that* (4.1) *holds. Then*

$$\lim_{n\to\infty} \Big(\max_{K\in\mathcal{K}(H)} \gamma(T^n - K)\Big)^{1/n} = \Big[\lim_{n\to\infty} \Big(\inf_{K\in\mathcal{K}(H)} \|S^n - K\|\Big)^{1/n}\Big]^{-1}$$
$$= \Big[\lim_{n\to\infty} \Big(\inf_{F\in\mathcal{F}(H)} \|S^n - F\|\Big)^{1/n}\Big]^{-1} = \lim_{n\to\infty} \Big(\sup_{F\in\mathcal{F}(H)} \gamma(T^n - F)\Big)^{1/n}.$$

Proof. Proposition 4.2, (4.2), Theorem 4.3 and [16], Corollary 3.13. ∎

COROLLARY 4.5. *If $T \in \mathcal{L}(H)\setminus\mathcal{F}(H)$ is normal and if $T(H)$ is closed, then $T \in \Phi_g(H)$ and there is some $K \in \mathcal{K}(H)$ such that $TK = KT$ and*

$$\operatorname{dist}(0, \sigma_\Phi(T) \setminus \{0\}) = \gamma_e(T) = \gamma(T+K).$$

Proof. From [15], Remark 4.7 (c), we derive $T \in \Phi_g(H)$. It is shown in the proof of Corollaire 3 in [11] that, if T is normal, then $\gamma_e(T^n) = \gamma_e(T)^n$ for $n \in \mathbb{N}$. By Corollaire 3 (3.2) in [11], there is some $K \in \mathcal{K}(H)$ with $TK = KT$ and $\gamma_e(T) = \gamma(T+K)$. ∎

REMARKS. (1) If $T \in \mathcal{F}(H)$, then $T \in \Phi_g(H)$ and $T - \lambda I \in \Phi(H)$ for each $\lambda \neq 0$. Thus $\sigma_\Phi(T) \setminus \{0\} = \emptyset$.

(2) If $T \in \mathcal{L}(H)$ is *normal* and $T(H)$ is closed, then $H = T(H) \oplus N(T)$. Therefore, 0 is a pole of order 1 of the resolvent $(T - \lambda I)^{-1}$ or $0 \in \rho(T)$. It is easy to see that for each $n \in \mathbb{N}$, $\gamma(T^n) = \gamma(T)^n$. Hence we get from Proposition 1.3 (3) that

$$\gamma(T) = \operatorname{dist}(0, \sigma(T) \setminus \{0\}). \tag{4.3}$$

(3) In [16] we have defined the *generalized Fredholm spectrum* $\sigma_{\Phi_g}(T)$ for $T \in \mathcal{L}(H)$ as follows:

$$\sigma_{\Phi_g}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin \Phi_g(H)\}.$$

In view of Theorem 4.3 one might conjecture that if $T \in \Phi_g(H) \setminus \mathcal{F}(H)$, then

$$\lim_{n\to\infty} \gamma_e(T^n)^{1/n} = \operatorname{dist}(0, \sigma_{\Phi_g}(T)).$$

But this is not true in general. Take a projection $P \in \mathcal{L}(H)$ with $P \notin \Phi(H) \cup \mathcal{F}(H)$. Then

$$\sigma(P) = \sigma_\Phi(P) = \{0, 1\}. \tag{4.4}$$

By Remark 1.7 (c) in [15], $P, I - P \in \Phi_g(H)$, hence $\sigma_{\Phi_g}(P) = \emptyset$, but

$$1 = \operatorname{dist}(0, \sigma_\Phi(P) \setminus \{0\}) = \lim_{n\to\infty} \gamma_e(P^n)^{1/n}. \tag{4.5}$$

(4) If $P \in \mathcal{L}(H)$ is an *orthogonal* projection and if $P \notin \Phi(H) \cup \mathcal{F}(H)$, then it follows from Corollary 4.5, (4.3), (4.4) and (4.5) that there is some

$K \in \mathcal{K}(H)$ with $PK = KP$ and

$$\gamma_e(P) = \gamma(P) = \gamma(P+K) = 1.$$

Since $\Phi_g(H) \subseteq q\Phi(H)$ (Proposition 2.2), we derive from Theorem 2.5 that if $T \in \Phi_g(H)$, then there is some $\varepsilon > 0$ with

$$T - \lambda I \in \mathcal{S}(H) \text{ for } 0 < |\lambda| < \varepsilon.$$

From [8], Corollaire 3.9, we get the following theorem:

THEOREM 4.6. *If $T \in \Phi_g(H)$ then*

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \operatorname{dist}(0, \sigma_{rr}(T) \setminus \{0\}).$$

If $T \in \Phi_g(H)$, then there is $\delta > 0$ such that $T - \lambda I \in \Phi(H)$ for $0 < |\lambda| < \delta$ ([15], Proposition 1.3) and $T - \lambda I \in \mathcal{S}(H)$ for $0 < |\lambda| < \delta$. Let $d(T)$ be the supremum of all $\varepsilon > 0$ such that $T - \lambda I \in \Phi(H)$ and $\alpha(T - \lambda I)$ and $\beta(T - \lambda I)$ are constant for $0 < |\lambda| < \varepsilon$.

Our next result is a generalization of Proposition 1.3 (2) in the Hilbert space case.

THEOREM 4.7. *If $T \in \Phi_g(H)$ then*

$$d(T) = \lim_{n\to\infty} \gamma(T^n)^{1/n}.$$

Proof. Put $d_1 = \lim_{n\to\infty} \gamma(T^n)^{1/n}$, $D_1 = \{\lambda \in \mathbb{C} : 0 < |\lambda| < d_1\}$, $d_2 = d(T)$ and $D_2 = \{\lambda \in \mathbb{C} : 0 < |\lambda| < d_2\}$.

Let $\lambda \in D_2$. Then $T - \lambda I \in \Phi(H)$ and $j(T - \lambda I) = 0$, hence $T - \lambda I \in \mathcal{S}(H)$, by Proposition 1.1. This gives $D_2 \subseteq \rho_{rr}(T)$ and $d_2 \leq \operatorname{dist}(0, \sigma_{rr}(T) \setminus \{0\})$. Theorem 4.6 shows now that $d_2 \leq d_1$.

It remains to show that $d_1 \leq d_2$. By Theorem 4.6, $D_1 \subseteq \rho_{rr}(T)$. In [18], M.A. Shubin has shown that there is a holomorphic function $F : D_1 \to \mathcal{L}(H)$ such that

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \text{ for all } \lambda \in D_1.$$

Since $T - \lambda I \in \Phi(H)$ for $0 < |\lambda| < d_2$ $(\leq d_1)$, $\tilde{T} - \lambda \tilde{I}$ is invertible for $\lambda \in D_2$ $(\subseteq D_1)$. Therefore we get $\widetilde{F(\lambda)} = (\tilde{T} - \lambda\tilde{I})^{-1}$ for each $\lambda \in D_2$, thus

$$(\tilde{T} - \lambda\tilde{I})\widetilde{F(\lambda)} = \tilde{I} = \widetilde{F(\lambda)}(\tilde{T} - \lambda\tilde{I}) \text{ for } \lambda \in D_2.$$

Define the holomorphic functions $G_1, G_2 : D_1 \to \mathcal{L}(H)$ by

$$G_1(\lambda) = (T - \lambda I)F(\lambda) - I \quad \text{and} \quad G_2(\lambda) = F(\lambda)(T - \lambda I) - I.$$

Then we have

$$\widetilde{G_1(\lambda)} = \tilde{0} = \widetilde{G_2(\lambda)} \text{ for each } \lambda \in D_2.$$

Since the mappings $\lambda \mapsto \widetilde{G_i(\lambda)}$ $(i = 1,2)$ are holomorphic on D_1, it results that

$$\widetilde{G_1(\lambda)} = \widetilde{0} = \widetilde{G_2(\lambda)} \text{ for all } \lambda \in D_1.$$

Therefore, $\widetilde{T} - \lambda\widetilde{I}$ is invertible for each $\lambda \in D_1$, hence $T - \lambda I \in \Phi(H)$ for each $\lambda \in D_1$. Since $D_1 \subseteq \rho_{rr}(T)$, it follows that $j(T - \lambda I) = 0$ for $\lambda \in D_1$. This shows that $D_1 \subseteq D_2$. Thus $d_1 \le d_2$. ∎

For our next result we need some additional notations. We denote the algebra $\mathcal{L}(H)/\mathcal{F}(H)$ by $\widehat{\mathcal{L}}$, and we write $\widehat{T}$ for the coset $T + \mathcal{F}(H)$ of T in $\widehat{\mathcal{L}}$.

If $R \in \mathcal{L}(H)$ is relatively regular, then the set of all pseudo-inverses of R is denoted by $\mathcal{P}(R)$.

Let $T \in \Phi_g(H)$, $R \in \Phi(H)$ and $TR - RT \in \mathcal{F}(H)$. Then, by [15], Theorem 4.10, there is some $\delta > 0$ such that $T - \lambda R \in \Phi(H)$ for $0 < |\lambda| < \delta$. Put

$$d(T,R) = \sup\{\varepsilon > 0 : T - \lambda R \in \Phi(H) \text{ for } 0 < |\lambda| < \varepsilon\}.$$

If $T \in \mathcal{F}(H)$ $(\subseteq \Phi_g(H))$, then $\widehat{T} - \lambda\widehat{R} = \lambda\widehat{R} \in \widehat{\mathcal{L}}^{-1}$ for each $\lambda \neq 0$, hence $T - \lambda R \in \Phi(H)$ for each $\lambda \neq 0$, thus $d(T,R) = \infty$.

The following theorem deals with the case where $T \in \Phi_g(H) \setminus \mathcal{F}(H)$. This result improves Theorem 4.10 in [15] (in the Hilbert space case).

THEOREM 4.8. *Suppose that $T \in \Phi_g(H) \setminus \mathcal{F}(H)$, $R \in \Phi(H)$ and $TR - RT \in \mathcal{F}(H)$. Put $\Phi(T,R) = \{\lambda \in \mathbb{C} : T - \lambda R \in \Phi(H)\}$. Then we have for each $U \in \mathcal{P}(R)$:*

(1) $UT, TU \in \Phi_g(H) \setminus \mathcal{F}(H)$.
(2) $\sigma_\Phi(UT) = \sigma_\Phi(TU) = \mathbb{C} \setminus \Phi(T,R)$.
(3) $d(T,R) = \lim\limits_{n\to\infty} \gamma_e((UT)^n)^{1/n} = \lim\limits_{n\to\infty} \gamma_e((TU)^n)^{1/n}$
$= \operatorname{dist}(0, \sigma_\Phi(TU) \setminus \{0\})$.

Proof. (1) Let $U \in \mathcal{P}(R)$. Then $H = (RU)(H) \oplus (I - RU)(H)$ and $(I - UR)(H) = N(R)$. It follows that $\widehat{R}\widehat{U} = \hat{I} = \widehat{U}\widehat{R}$, hence $U \in \Phi(H)$. From $\widehat{T}\widehat{R} = \widehat{R}\widehat{T}$ we get $\widehat{T} = \widehat{T}\widehat{R}\widehat{U} = \widehat{R}\widehat{T}\widehat{U}$, thus $\widehat{U}\widehat{T} = \widehat{U}\widehat{R}\widehat{T}\widehat{U} = \widehat{T}\widehat{U}$, hence $UT - TU \in \mathcal{F}(H)$. Since $U \in \Phi(H) \subseteq \Phi_g(H)$, we see from [15], Theorem 4.5, that $UT, TU \in \Phi_g(H)$. Since $\widehat{T} \neq \widehat{0}$, we see from $\widehat{T} = \widehat{R}\widehat{T}\widehat{U} = \widehat{R}\widehat{U}\widehat{T}$ that $\widehat{T}\widehat{U} \neq \widehat{0} \neq \widehat{U}\widehat{T}$. This gives $UT, TU \notin \mathcal{F}(H)$.

(2) From $UT - TU \in \mathcal{F}(H)$, we derive $\widetilde{U}\widetilde{T} = \widetilde{T}\widetilde{U}$ and so

$$\sigma_\Phi(UT) = \sigma(\widetilde{U}\widetilde{T}) = \sigma(\widetilde{T}\widetilde{U}) = \sigma_\Phi(TU).$$

Let $\lambda \in \mathbb{C}$. Then (observe that $\widehat{R}\widehat{U} = \widehat{U}\widehat{R} = \hat{I}$)

$$\begin{aligned}\lambda \in \Phi(T,R) &\iff T - \lambda R \in \Phi(H) \iff \widehat{T} - \lambda\widehat{R} \in \widehat{\mathcal{L}}^{-1}\\ &\iff \widehat{R}\widehat{U}\widehat{T} - \lambda\widehat{R} = \widehat{R}(\widehat{U}\widehat{T} - \lambda\hat{I}) \in \widehat{\mathcal{L}}^{-1}\end{aligned}$$

$$\iff \widehat{U}\widehat{T} - \lambda\widehat{I} \in \widehat{\mathcal{L}}^{-1} \iff UT - \lambda I \in \Phi(H)$$
$$\iff \lambda \in \mathbb{C} \setminus \sigma_\Phi(UT).$$

(3) It follows from (2) that $d(T, R) = \operatorname{dist}(0, \sigma(UT) \setminus \{0\}) = \operatorname{dist}(0, \sigma(TU) \setminus \{0\})$. Now use Theorem 4.3 and (1) to obtain

$$d(T, R) = \lim_{n\to\infty} \gamma_e((UT)^n)^{1/n} = \lim_{n\to\infty} \gamma_e((TU)^n)^{1/n}. \qquad \blacksquare$$

COROLLARY 4.9. *Let T and R as in Theorem 4.8. Then*

$$\lim_{n\to\infty} \gamma_e(T^n)^{1/n} r(\widetilde{R})^{-1} \leq d(T, R) \leq \lim_{n\to\infty} \gamma_e(T^n)^{1/n} \left(\lim_{n\to\infty} \gamma_e(R^n)^{1/n}\right)^{-1}.$$

Proof. Take $U \in \mathcal{P}(R)$. Since $\widehat{T}\widehat{R} = \widehat{R}\widehat{T}$, $\widehat{U}\widehat{R} = \widehat{I} = \widehat{R}\widehat{U}$ and $\widehat{T}\widehat{U} = \widehat{U}\widehat{T}$, we get $\widetilde{T}\widetilde{U} = \widetilde{U}\widetilde{T}$, $\widetilde{R} \in \widetilde{\mathcal{L}}^{-1}$ and $\widetilde{U} = \widetilde{R}^{-1}$. From Proposition 3.1 we see that

$$\begin{aligned}\gamma_e(T^n) = \gamma(\widetilde{T}^n) = \gamma(\widetilde{T}^n\widetilde{U}^n\widetilde{R}^n) \leq \|\widetilde{R}^n\|\gamma((\widetilde{TU})^n)\\ = \|\widetilde{R}^n\|\gamma_e((TU)^n)\end{aligned}$$

for each $n \in \mathbb{N}$, thus

$$\frac{\gamma_e(T^n)^{1/n}}{\|\widetilde{R}^n\|^{1/n}} \leq \gamma_e((TU)^n)^{1/n} \qquad (n \in \mathbb{N}).$$

From Theorem 4.8 (3) we derive

$$\lim_{n\to\infty} \gamma_e(T^n)^{1/n} r(\widetilde{R})^{-1} \leq d(T, R).$$

Use again Proposition 3.1 to see that

$$\gamma_e((TU)^n) = \gamma(\widetilde{T}^n\widetilde{U}^n) \leq \|\widetilde{U}^n\|\gamma(\widetilde{T}^n) = \|\widetilde{U}^n\|\gamma_e(T^n)^{1/n}$$

for each $n \in \mathbb{N}$. This gives

$$d(T, R) \leq r(\widetilde{U}) \lim_{n\to\infty} \gamma_e(T^n)^{1/n}.$$

Theorem 3.10 in [16] shows that

$$\operatorname{dist}(0, \sigma_\Phi(R) \setminus \{0\}) = r(\widetilde{U})^{-1}.$$

Since $R \in \Phi(H) \subseteq \Phi_g(H) \setminus \mathcal{F}(H)$, it follows from Theorem 4.3 that

$$\lim_{n\to\infty} \gamma_e(R^n)^{1/n} = \operatorname{dist}(0, \sigma_\Phi(R) \setminus \{0\}),$$

hence $r(\widetilde{U}) = \left(\lim_{n\to\infty} \gamma_e(R^n)^{1/n}\right)^{-1}$. This completes the proof. $\blacksquare$

THEOREM 4.10. *Let $T \in \Phi_g(H) \setminus \mathcal{F}(H)$, $D = \{\lambda \in \mathbb{C} : |\lambda| < \lim_{n\to\infty} \gamma(T^n)^{1/n}\}$ and $\dot{D} = D \setminus \{0\}$. Then there is a meromorphic function $F : D \to \mathcal{L}(H)$*

such that

(4.6) $$(T-\lambda I)F(\lambda)(T-\lambda I)=T-\lambda I \quad \textit{for} \quad \lambda\in\dot{D}.$$

Proof. By Theorem 2.7 there is a Kato decomposition (H_1, H_2) of T with $T_1 \in \Phi(H_1)$, $j(T_1)=0$, $T_2 \in \mathcal{F}(H_2)$, T_2 nilpotent and $T = T_1 \oplus T_2$. Take $m \in \mathbb{N}\cup\{0\}$ with $T_2^m = 0$. We denote the identity on H_i by I_i $(i=1,2)$. Since $T \notin \mathcal{F}(H)$, $H_1 \neq \{0\}$.

Case 1: $H_2=\{0\}$. Thus $H_1 = H$ and $T = T_1 \in \mathcal{S}(H)$ (Proposition 1.1). Theorem 4.6 shows that $D \subseteq \rho_{rr}(T)$. It follows from [18] that there is a *holomorphic* function $F : D \to \mathcal{L}(H)$ with

$$(T-\lambda I)F(\lambda)(T-\lambda I)=T-\lambda I \quad \text{for each} \quad \lambda\in D.$$

Case 2: $H_2 \neq \{0\}$. M. Mbekhta has shown in the proof of [8], Corollaire 3.9, that

$$\lim_{n\to\infty}\gamma(T^n)^{1/n}=\lim_{n\to\infty}\gamma(T_1^n)^{1/n}.$$

Therefore, we get from Theorem 4.6 that $D \subseteq \rho_{rr}(T_1)$. As in Case 1, we see that there is a holomorphic function $F_1 : D \to \mathcal{L}(H_1)$ with

$$(T_1-\lambda I_1)F(\lambda)(T_1-\lambda I_1)=T_1-\lambda I_1 \quad \text{for} \quad \lambda\in D.$$

Since T_2 is nilpotent, we have $T_2 - \lambda I_2 \in \mathcal{L}(H_2)^{-1}$ for each $\lambda \neq 0$ and $(T_2-\lambda I_2)^{-1} = -\sum_{k=0}^{\infty}\frac{T_2^k}{\lambda^{k+1}} = -\sum_{k=0}^{m-1}\frac{T_2^k}{\lambda^{k+1}}$ for $\lambda \neq 0$. Put $F_2(\lambda)=(T_2-\lambda I_2)^{-1}$ for $\lambda \neq 0$. Then it is clear that

$$(T_2-\lambda I_2)F_2(\lambda)(T_2-\lambda I_2)=T_2-\lambda I_2 \quad \text{on} \quad \mathbb{C}\setminus\{0\}.$$

If we define the function $F : \dot{D} \to \mathcal{L}(H)$ by $F(\lambda) = F_1(\lambda)\oplus F_2(\lambda)$, then it is clear that (4.6) holds. ∎

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