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ON MULTIVALUED f-NONEXPANSIVE MAPS

Abstract. In this paper we prove coincidence and common fixed points results for single-valued maps f and multivalued f-nonexpansive maps with star-shaped weakly compact domains in Banach spaces, which extend the theorems of [3], [6], [8] and others. Moreover weak convergence and strong convergence results for coincidence point sets have also been proved, extending a result in [1].

1. Introduction

For contraction maps there is the classical Banach-Caccioppoli fixed point result which asserts that every single-valued contraction selfmap of a complete metric space has a unique fixed point. A number of generalizations of this result have appeared in the literature. Among others Jungck [5] extended this result to single-valued f-contraction maps. Extensions of the Banach Contraction Principle to multivalued mappings were initiated independently by Markin [9] and Nadler [10]. Jungck's result was extended to the setting of multivalued mappings by Kaneko [7] who established the following theorem, which also generalizes a result of Nadler [10].

THEOREM 1.1. *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous map. Let T be a closed bounded valued f-contraction map on X which commutes with f and $T(X) \subseteq f(X)$. Then f and T have a coincidence point in X. Suppose moreover that one of the following holds: either (i) $f(x) \neq f^2(x)$ implies $f(x) \notin T(x)$ or (ii) $f(x) \in T(x)$ implies $\lim_{n\to\infty} f^n(x)$ exists. Then T and f have a common fixed point.*

(The referee of this paper has pointed out that condition (i) in the above result implies condition (ii), for if $f(x) \in T(x)$ then under (i) the sequence $\{f^n(x)\}$ is constant).

On the other hand, a natural generalization of the single-valued contraction map is a nonexpansive map, but it need not have a fixed point in the

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general metric space setting without additional hypotheses. Much work has been done on fixed points for such maps. The notion of nonexpansiveness has also been extended in many directions. Recently Jungck and Sessa [6] have used the idea of single-valued f-nonexpansiveness and the result of Jungck referred to above and proved the following common fixed point result.

THEOREM 1.2. *Let M be a weakly compact subset of a Banach space X which is star-shaped with respect to $q \in M$. Suppose $f, g : M \to M$, where f is continuous in the weak and strong topology on M, f is affine, $fg = gf$, $f(M) = M$, and $f(q) = q$. If for all x and y in M, $\|g(x) - g(y)\| \leq \|f(x) - f(y)\|$, then f and g have a common fixed point in M, provided that either (i) $f - g$ is demiclosed or (ii) X satisfies Opial's condition.*

In [8] Lami Dozo has used Nadler's result [10] and proved that it holds for multivalued nonexpansive maps under certain conditions, viz., each compact-valued nonexpansive map of a nonempty weakly compact convex subset of a Banach space satisfying Opial's condition has a fixed point. Recently Daffer and Kaneko [2] have studied f-nonexpansive multivalued maps and proved coincidence and fixed point theorems for compact-valued f-nonexpansive maps on connected metric spaces under some conditions; the concept of orbits was a major tool in their approach.

In section 2, one of our main purposes is to extend the result of Jungck and Sessa [6] to multivalued f-nonexpansive maps. This is done in Theorem 2.2, which includes results of Lami Dozo [8] and Dotson [3] as special cases. Section 3 deals with convergence of coincidence point sets of f-nonexpansive maps and is based on the papers of Pietramala [12] , Acedo and Xu [1]. Utilizing Lemma 2.2, we prove (Theorem 3.1) that under suitable conditions, it is possible to construct a sequence of coincidence point sets of f and multivalued f-contraction maps, converging weakly in the sense of Mosco to a coincidence point of f and a multivalued f-nonexpansive map; this contains the result in [1] as a special case.

We recall the following notions and definitions. Let M be a nonempty subset of a normed linear space X and let f be a single-valued mapping of M into X. We use $CB(X)$ to denote the collection of all nonempty closed bounded subsets of X, $K(X)$ for the collection of all nonempty compact subsets of X, and H for the Hausdorff metric on $CB(X)$ induced by the norm of X, i.e.

$$H(A,B) = \max\{\sup_{x\in A}\inf_{y\in B}\|x-y\|, \sup_{x\in B}\inf_{y\in A}\|x-y\|\},$$

for all A, B in $CB(X)$. A multivalued map $T : M \to CB(X)$ is said to be an f-contraction iff for a fixed constant $h \in (0,1)$ and for each $x, y \in M$,

$$H(T(x),T(y)) \leq h\|f(x) - f(y)\|.$$

Further, if T and f satisfy the inequality

$$H(T(x), T(y)) \leq \|f(x) - f(y)\|,$$

then T is said to be f-nonexpansive. In particular, if f is the identity map on M then a multivalued map is an f-contraction (resp. f-nonexpansive) iff it is a contraction (resp. nonexpansive). Note that each single-valued map is an f-contraction (resp. f-nonexpansive) iff it is a multivalued f-contraction (resp. f-nonexpansive). A point $x \in M$ is called a fixed point of the multivalued map T iff $x \in T(x)$ and it is called a coincidence point of f and T iff $f(x) \in T(x)$. We denote by $C(f \cap T)$ the set of coincidence points of f and T.

A Banach space X satisfies Opial's condition [11] if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \neq x$. Every Hilbert space [11] and the spaces $l_p (1 \leq p < \infty)$ [4] satisfy Opial's condition.

A subset M is said to be star-shaped with respect to $q \in M$ if $\{tx + (1 - t)q : 0 \leq t \leq 1\} \subset M$ for each $x \in M$. The point q is known as a star-centre of M. Clearly the star-shaped subsets include the convex subsets as a proper subclass.

2. Coincidence and fixed points

Before proving our main results of this section we begin with the following useful lemmas.

LEMMA 2.1. *Let X be a Banach space and M be a closed subset which is star-shaped with respect to $q \in M$. Let f be a continuous affine self-map of M such that $f(M) = M$ and $f(q) = q$. Let $T : M \to CB(M)$ be an f-nonexpansive map which commutes with f and has $T(M)$ bounded. Then for each positive integer n there are $x_n \in M$ and $w_n \in T(x_n)$ such that $f(x_n) - w_n \to 0$ as $n \to \infty$.*

Proof. Clearly, as a closed subset of a Banach space X, M is a complete metric space, and $T(M) \subset M = f(M)$. Take a sequence $\{h_n\}$ of real numbers for which $0 < h_n < 1$ and $h_n \to 1$ as $n \to \infty$. For each n, define a multivalued map J_n by setting

$$J_n(x) = h_n T(x) + (1 - h_n)q, \quad \text{for all } x \in M.$$

Then for each $n \geq 1$, J_n maps M into $CB(M)$ and $J_n(M) \subset f(M)$. Now we show that for all $n \geq 1$, J_n is an f-contraction and commutes with f. Let $x, y \in M$. Then

$$H(J_n(x), J_n(y)) = h_n H(T(x), T(y)),$$

so, by using the f-nonexpansiveness of T, we get

$$H(J_n(x), J_n(y)) \leq h_n \|f(x) - f(y)\|,$$

which proves that each J_n is an f-contraction . For each $x \in M$,

$$\begin{aligned} J_n f(x) &= h_n T f(x) + (1 - h_n) q \\ &= h_n f T(x) + (1 - h_n) f(q) \\ &= f\{h_n T(x) + (1 - h_n) q\} = f J_n(x), \end{aligned}$$

that is, each J_n commutes with f. All the conditions of Theorem 1.1 are satisfied and hence there is an $x_n \in M$ such that $f(x_n) \in J_n(x_n)$. So by the definition of $J_n(x_n)$, there is some $w_n \in T(x_n)$ such that

$$f(x_n) = h_n w_n + (1 - h_n) q.$$

Thus

$$\|f(x_n) - w_n\| = (1 - h_n)\|q - w_n\|.$$

Since $T(M)$ is bounded and $w_n \in T(x_n) \subset T(M)$, we have that $\|q - w_n\|$ is bounded and so by the fact that $h_n \to 1$ as $n \to \infty$, it follows that $f(x_n) - w_n \to 0$ as $n \to \infty$.

The following lemma is based on the important property of being demiclosed, generalizing Lami Dozo's result in [8]. We continue to use $\to$ to denote strong convergence, and we use $\xrightarrow{w}$ to denote weak convergence. A multivalued map $T : M \to 2^X$ (the collection of nonempty subsets of X) is said to be demiclosed if for every sequence $\{x_n\} \subset M$ and any $y_n \in T(x_n)$, $n = 1, 2, \ldots$, such that $x_n \xrightarrow{w} x$ and $y_n \to y$, we have $x \in M$ and $y \in T(x)$.

LEMMA 2.2. *Let M be a weakly compact subset of a Banach space X satisfying Opial's condition. Let $f : M \to X$ be a weakly continuous map and $T : M \to K(X)$ be an f-nonexpansive multivalued map. Then $f - T$ is demiclosed.*

Proof. Let $\{x_n\} \subset M$ and $y_n \in (f - T)x_n$ be such that $x_n \xrightarrow{w} x$ and $y_n \to y$. It is obvious that $x \in M$ and $f(x_n) \xrightarrow{w} f(x)$. Since $y_n \in f(x_n) - T(x_n)$, we get

$$(2.2.1) \qquad y_n = f(x_n) - u_n, \qquad \text{for some } u_n \in T(x_n).$$

Since $T(x)$ is a compact set, there is a $v_n \in T(x)$ such that

$$(2.2.2) \qquad \|u_n - v_n\| \leq H(T(x_n), T(x)) \leq \|f(x_n) - f(x)\|.$$

From (2.2.1) and (2.2.2), passing to the limit with respect to n, we obtain

$$(2.2.3) \quad \liminf_{n\to\infty} \|f(x_n) - f(x)\| \geq \liminf_{n\to\infty} \|u_n - v_n\| = \liminf_{n\to\infty} \|f(x_n) - y_n - v_n\|.$$

$T(x)$ being compact, for a convenient subsequence still denoted by $\{v_n\}$, we have $v_n \to v \in T(x)$. Then (2.2.3) yields

$$\liminf_{n\to\infty} \|f(x_n) - f(x)\| \geq \liminf_{n\to\infty} \|f(x_n) - y - v\|.$$

Since X satisfies Opial's condition and $f(x_n) \xrightarrow{w} f(x)$, this yields $f(x) = y+v$. Thus $y = f(x)-v \in f(x)-T(x)$, which proves that $f-T$ is demiclosed.

To see that not every continuous map f and compact valued f-nonexpansive map on a closed star-shaped subset M of a Banach space X have a coincidence point, we consider the following examples.

EXAMPLE 2.1. Let $X = \mathrm{R}$ with the usual norm and $M = [0,1]$. Define the maps $f : M \to M$ and $T : M \to K(M)$ as follows

$$f(x) = \frac{x+1}{2}, \qquad T(x) = \left[0, \frac{x+1}{3}\right] \qquad \forall x \in M.$$

Then clearly, T is an f-nonexpansive and f is a continuous affine map, but f and T have no coincidence point.

EXAMPLE 2.2. Let $X = \mathrm{R}$ with the usual norm and $M = [0,\infty)$. We define $f : M \to M$ and $T : M \to K(M)$ as follows

$$f(x) = 2x + 1, \qquad T(x) = \{x\} \qquad \forall x \in M.$$

Clearly, T is an f-nonexpansive map which commutes with f but f and T have no coincidence point.

Now utilizing the above lemmas we shall prove the following coincidence point results. In the rest of this section f denotes a *continuous affine* self-map of M.

THEOREM 2.1. *Let X be a Banach space and M be a closed subset which is star-shaped with respect to $q \in M$ such that $f(M) = M$ and $f(q) = q$. Let $T : M \to CB(M)$ be an f-nonexpansive map which commutes with f such that $(f - T)M$ is closed and $T(M)$ is bounded. Then $C(f \cap T) \neq \emptyset$.*

Proof. It follows from Lemma 2.1 that for each positive integer n there are $x_n \in M$ and $w_n \in T(x_n)$ such that

$$f(x_n) - w_n \to 0.$$

Since $f(x_n) - w_n \in (f - T)x_n \subset (f - T)M$ and $(f - T)M$ is closed, so $0 \in (f - T)M$. Hence there is a point $x \in M$ such that $f(x) \in T(x)$.

Now in the following we unify and generalize earlier stated results of Jungck and Sessa [6] and Lami Dozo [8] for multivalued f-nonexpansive maps.

THEOREM 2.2. *Let X be a Banach space and M be a weakly compact subset which is star-shaped with respect to $q \in M$. Let f be a weakly continuous map* *such that $f(M) = M$ and $f(q) = q$ and let $T : M \to K(M)$ be a multivalued f-nonexpansive map which commutes with f. Then $C(f \cap T) \neq \emptyset$ provided that one of the following holds: either*

(a) *$(f - T)$ is demiclosed, or*

(b) *X satisfies Opial's condition.*

Proof. Since the weak topology is Hausdorff and M is weakly compact and therefore weakly closed, it is also strongly closed. So by Lemma 2.1, for each positive integer n there are $x_n \in M$ and $w_n \in T(x_n)$ such that

$$f(x_n) - w_n \to 0.$$

M being weakly compact, for a convenient subsequence still denoted by $\{x_n\}$, we have $x_n \xrightarrow{w} x \in M$. Put

$$y_n = f(x_n) - w_n \in (f - T)x_n;$$

then $y_n \to 0$. Now if (a) holds then $0 \in (f - T)x$, that is $f(x) \in T(x)$. If (b) holds it follows from Lemma 2.2 that $(f - T)$ is demiclosed, and hence f and T have a coincidence point $x \in M$ as in the previous case.

REMARK 2.1 If $f = I$, the identity map on M, then Theorem 2.2(b) extends Theorem 3.2 of Lami Dozo [8] for star-shaped subsets and Theorem 2.2 generalizes Theorem 6 of Jungck and Sessa [6] for multivalued nonexpansive maps. Moreover Dotson [3] proved in his Theorem 2 that each single-valued nonexpansive selfmap T of a weakly compact star-shaped subset M of a Banach space has a fixed point in M provided that $I - T$ is demiclosed. Hence Theorem 2.2(a) contains Dotson's fixed point theorem as a special case.

Since for a compact set M in a Banach space, obviously,

$$0 = \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

if $\{x_n\} \subset M$, $x_n \xrightarrow{w} x$ and $y \neq x$, from the same technique adopted in Lemma 2.2 and Theorem 2.2 we derive the following theorem for general Banach spaces, which contains Theorem 1 of Dotson [3] as a special case.

THEOREM 2.3. *Let X be a Banach space and M be a compact subset which is star-shaped with respect to $q \in M$. Let f be a weakly continuous map such that $f(M) = M$ and $f(q) = q$ and let $T : M \to K(M)$ be a multivalued f-nonexpansive map which commutes with f. Then $C(f \cap T) \neq \emptyset$.*

Now we obtain the following common fixed point results.

THEOREM 2.4. *Suppose that M, f, T and q satisfy the assumptions of Theorem 2.2 and moreover the following condition holds*

$(*)$ $\qquad f(x) \in T(x)$ *implies* $\lim_n f^n(x)$ *exists.*

Then T and f have a common fixed point in M.

Proof. By Theorem 2.2, there exists a point $x_0 \in M$ such that $f(x_0) \in T(x_0)$. Then, clearly

$$f^n(x_0) = f^{n-1}f(x_0) \in f^{n-1}T(x_0) = T(f^{n-1}(x_0)).$$

Now, if $(*)$ holds, then by taking the limit as $n \to \infty$, we get

$$p \in T(p), \qquad \text{where } p = \lim_n f^n(x_0),$$

and clearly $p = f(p)$.

Similarly, using Theorem 2.1 we have another common fixed point result. And a proof of this is similar to that of Theorem 2.4, hence we omit it.

THEOREM 2.5. *Suppose that M, f, T, and q satisfy the assumptions of Theorem* 2.1 *and moreover the following condition holds*

$(*)$ $\qquad f(x) \in T(x)$ *implies* $\lim_n f^n(x)$ *exists.*

Then T and f have a common fixed point in M.

3. Convergence of coincidence point sets

In this section, following the technique in [1,12], we show that in a specific case it is also possible to construct a sequence of coincidence point sets of f and f-contraction maps T_λ, converging weakly in the sense of Mosco to a coincidence point of f and T.

First we recall the following [1,12]:

A sequence $\{A_n\}$ in $CB(X)$ is said to converge (converge weakly) to an element $A \in CB(X)$ in the sense of Mosco if

$$\liminf A_n = \limsup A_n = A \ \ (w - \liminf A_n = w - \limsup A_n = A),$$

where

$$\liminf A_n = \{x \in X : \exists \text{ a sequence } \{x_n\}, x_n \in A_n \text{ such that } x_n \to x\}$$

and

$$\limsup A_n = \{x \in X : \exists \text{ a subsequence } \{A_{n_k}\} \text{ of } \{A_n\} \text{ and a sequence } \{x_{n_k}\} \text{ with } x_{n_k} \in A_{n_k} \text{ such that } x_{n_k} \to x\};$$

in the case of $w - \liminf$ and $w - \limsup$, $\to$ is replaced by $\xrightarrow{w}$.

A net $\{A_\lambda\}_{\lambda \in (0,1)}$ of closed subsets of X converges (converges weakly) to A in the sense defined before if every sequence $\{A_{\lambda_n}\}$, $\lambda_n \to 1$ $(0 < \lambda_n < 1)$ as $n \to \infty$, converges (converges weakly) in the sense of Mosco to A.

Now, let M, f, T and q be the same as in Lemma 2.1. For $\lambda \in (0,1)$, we define the map $T_\lambda : M \to CB(M)$ by

$$T_\lambda(x) = \lambda T(x) + (1-\lambda)q, \quad \text{for all } \ x \in M.$$

Then T_λ is a multivalued f-contraction map, and hence [7], $C(f \cap T_\lambda) \neq \emptyset$.

Using Lemma 2.2 we prove the following convergence result.

THEOREM 3.1. *Let M be a weakly compact star-shaped subset of a Banach space X satisfying Opial's condition. Let f be a weakly continuous self-map of M and $T : M \to K(M)$ be an f-nonexpansive map such that $C(f \cap T) = \{z\}$. If for some star-centre q of M and for each $\lambda \in (0,1)$, $C(f \cap T_\lambda) \neq \emptyset$, then the net $\{C(f \cap T_\lambda)\}$ converges weakly to $C(f \cap T)$ in the sense of Mosco, as $\lambda \to 1$.*

Proof. We have to prove that

$$w - \limsup_{n\to\infty} C(f \cap T_{\lambda_n}) = w - \liminf_{n\to\infty} C(f \cap T_{\lambda_n}) = \{z\},$$

for every sequence $\lambda_n \to 1,\ 0 < \lambda_n < 1$. Since $w - \liminf \subseteq w - \limsup$, it is enough to show that

(i) $w - \limsup C(f \cap T_{\lambda_n}) \subseteq \{z\}$ and
(ii) $\{z\} \subseteq w - \liminf C(f \cap T_{\lambda_n})$.

For (i), let $x \in w - \limsup C(f \cap T_{\lambda_n})$. Then there exists a convenient subsequence still denoted by $\{\lambda_n\}$ and a convenient sequence $\{x_n\}$ such that $x_n \in C(f \cap T_{\lambda_n})$ and $x_n \xrightarrow{w} x$. Clearly $x \in M$; also since $f(x_n) \in T_{\lambda_n}(x_n)$, there exists $u_n \in T(x_n)$ such that

$$f(x_n) = \lambda_n u_n + (1-\lambda_n)q,$$

and hence

$$f(x_n) - u_n \to 0.$$

Since by Lemma 2.2, $f - T$ is demiclosed, it follows that $0 \in (f-T)x$, and therefore $x = z$.

To show (ii), for each $\lambda_n \in (0,1)$, choose any $x_{\lambda_n} = x_n \in C(f \cap T_{\lambda_n})$ and $u_n \in T(x_n)$ satisfying

$$f(x_n) = \lambda_n u_n + (1-\lambda_n)q.$$

Then by the same proof as above, we see that every weak cluster point of $\{x_n\}$ is a point of $C(f \cap T)$. Hence $x_{\lambda_n} \xrightarrow{w} z$ as $\lambda_n \to 1$.

If f and T have a unique coincidence point z such that $T(z) = \{z\}$, then we have the following strong convergence result in a real Hilbert space.

THEOREM 3.2. *Let M be a closed convex bounded subset of a real Hilbert space X and f, f^{-1} be continuous in the weak, respectively strong, topology on M. Let $T : M \to K(M)$ be an f-nonexpansive multivalued map such that*

$C(f\cap T)=\{z\}=T(z)$. Then for some fixed $q\in M$ and any $x_\lambda\in C(f\cap T_\lambda)$ for each $\lambda\in(0,1)$, $x_\lambda\to z$ as $\lambda\to 1$.

Proof. Since $f(x_\lambda)\in T_\lambda(x_\lambda)$, so there exists $y_\lambda\in T(x_\lambda)$ such that

$$f(x_\lambda)=\lambda y_\lambda+(1-\lambda)q. \tag{3.2.1}$$

Also, since

$$\|y_\lambda-z\|=\|y_\lambda-T(z)\|\le H(T(x_\lambda),T(z))\le\|f(x_\lambda)-f(z)\|, \tag{3.2.2}$$

so from (3.2.1) and (3.2.2) we have

$$\left\|\frac{f(x_\lambda)-(1-\lambda)q}{\lambda}-z\right\|\le\|f(x_\lambda)-z\|.$$

Thus

$$\left\|\frac{f(x_\lambda)-q}{\lambda}+(q-z)\right\|^2\le\|(f(x_\lambda)-q)+(q-z)\|^2,$$

which implies

$$\left\|\frac{f(x_\lambda)-q}{\lambda}\right\|^2+2\left\langle\frac{f(x_\lambda)-q}{\lambda},q-z\right\rangle\le\|f(x_\lambda)-q\|^2+2\langle f(x_\lambda)-q,q-z\rangle$$

and hence

$$\|f(x_\lambda)-q\|^2\le\frac{2\lambda}{1+\lambda}\langle f(x_\lambda)-q,z-q\rangle\quad\le\quad\langle f(x_\lambda)-q,z-q\rangle.$$

Finally, using the Cauchy-Schwarz inequality we obtain,

$$\|f(x_\lambda)-q\|\le\|z-q\|. \tag{3.2.3}$$

From the proof of the previous theorem, it is clear that $x_\lambda\xrightarrow{w}z$ as $\lambda\to 1$, so the weak continuity of f implies $f(x_\lambda)\xrightarrow{w}z$ and thus

$$\langle f(x_\lambda),q\rangle\to\langle z,q\rangle. \tag{3.2.4}$$

From the inequality (3.2.3) it follows that

$$\|f(x_\lambda)\|^2-2\langle f(x_\lambda),q\rangle\le\|z\|^2-2\langle z,q\rangle. \tag{3.2.5}$$

And hence (3.2.4) and (3.2.5) together imply that

$$\limsup_{\lambda\to 1}\|f(x_\lambda)\|\le\|z\|.$$

But, on the other hand $f(x_\lambda)\xrightarrow{w}z$ implies

$$\liminf_{\lambda\to 1}\|f(x_\lambda)\|\ge\|z\|.$$

Thus $\lim_{\lambda\to 1}\|f(x_\lambda)\|=\|z\|$ and since $f(x_\lambda)\xrightarrow{w}z$ we deduce that $f(x_\lambda)\to z$. Therefore, the continuity of f^{-1} implies $x_\lambda\to z$.

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