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SOME SEQUENCE SPACES DEFINED BY $|\overline{N}, p_n|$ SUMMABILITY

Abstract. The object of this paper is to introduce some new sequence spaces which arise from the notion of $|\overline{N}, p_n|$ summability. Some topological results, certain inclusion relations and a result on matrix transformations have been discussed.

1. Introduction

Given an infinite series $\sum_{n=0}^{\infty} a_n$, let

$$x_n = a_0 + a_1 + \cdots + a_n. \tag{1}$$

Denote the sequence (a_n) by a and the sequence (x_n) by x. We will suppose throughout that a, x are related by (1) (where no limits are stated, sums throughout are to be taken from 1 to ∞). Denote by $(p_n)_{n\geq 0}$ a sequence of positive real numbers, and write $P_n = \sum_{k=0}^{n} p_k$. It is well-known that the series $\sum_{n=0}^{\infty} a_n$ (or the sequence x) is said to be summable $(\overline{N}, p_n)$ to the sum s (finite), if $t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu x_\nu \to s$ as $n \to \infty$, and is said to be absolutely summable $(\overline{N}, p_n)$, or summable $|\overline{N}, p_n|$, if also the sequence $(t_n) \in BV$, that is $\sum_n |t_n - t_{n-1}| < \infty$. Let $|\overline{N_p}|$ and $\overline{N_p}$ denote, respectively, the set of all sequences which are summable $|\overline{N}, p_n|$ and $(\overline{N}, p_n)$. If $p_n = 1$ for all n, then $|\overline{N_p}|$ and $\overline{N_p}$, respectively, become $|C_1|$ (the set of all sequences which are absolutely Cesàro summable of order 1) and C_1 (the set of all sequences which are Cesàro summable of order 1).

The main object of this paper is to study a new sequence space $|\overline{N_p}|(r)$ which emerges naturally as an extension of $|\overline{N_p}|$ in the same way as ℓ, the space of absolutely convergent sequences, is extended to $\ell(p)$ (see Bourgin [1], Landsberg [4], Maddox [8, p. 30] and Simons [9]). The definition of

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$|\overline{N_p}|(r)$ is given in the following section. In §3, we propose to study linear topological structure of $|\overline{N_p}|(r)$ endowed with natural paranorm. In this section we also discuss uniform convexity, k-convexity and local boundedness of $|\overline{N_p}|(r)$ spaces. In §4, certain inclusion relations have been discussed. In §5, we give a criterion for compactness of a subset $K \subset |\overline{N_p}|(r)$, which is completely analogous to the classical theorem (see [6]). Finally in §6, we state without proof a result on matrix transformations.

2. Notation and definitions

The following inequalities (see, e.g., [8]) are needed throughout the paper.

Let p $= (p_k)$ be a bounded sequence of strictly positive real numbers. If $H = \sup p_k$, then

(2) $$|a_k + b_k|^{p_k} \leq C\left(|a_k|^{p_k} + |b_k|^{p_k}\right),$$

where C $= \max\left(1, 2^{H-1}\right)$. Also for any complex λ,

(3) $$|\lambda|^{p_k} \leq \max\left(1, |\lambda|^H\right).$$

Given a sequence $a = (a_k)$, we write, for $n \geq 1$, $\phi_n(a) = t_n - t_{n-1}$, where $t_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu x_\nu$. By an application of Abel's transformation we have

(4) $$\phi_n(a) = \frac{p_n}{P_n P_{n-1}} \sum_{k=1}^{n} P_{k-1} a_k (n \geq 1).$$

Note that for any sequences a, b and scalar λ, we have

(5) $$\phi_n(a+b) = \phi_n(a) + \phi_n(b) \quad \text{and} \quad \phi_n(\lambda a) = \lambda \phi_n(a).$$

We now extend the definition of $|\overline{N_p}|$ to a more general space $|\overline{N_p}|(r)$ in the same way as ℓ is extended to $\ell(p)$.

DEFINITION 2.1. Let $r = (r_n)$ be a bounded sequence of positive real numbers. We define $|\overline{N_p}|(r) = \{a = (a_k) : \sum_n |\phi_n(a)|^{r_n} < \infty\}$.

If r_n is a constant (which we will denote by r), we write $|\overline{N_p}|_r$ in place of $|\overline{N_p}|(r)$. We omit the suffix r in the case r = 1; note that this agrees with the definition of $|\overline{N_p}|$ already given. If $p_n = r_n = 1$ for all n, then $|\overline{N_p}|(r)$ reduces to $|C_1|$.

Note that $|\overline{N_p}|(r)$ is a linear space since for $a, b \in |\overline{N_p}|(r)$ and λ , $\mu \in C$ by (2), (3) and (5) we have

$$\sum_n |\phi_n(\lambda a + \mu b)|^{r_n}$$

$$\leq \max\left(1, 2^{H-1}\right)\left(\max\left(1, |\lambda|^H\right)\sum_n |\phi_n(a)|^{r_n} + \max\left(1, |\mu|^H\right)\sum_n |\phi_n(b)|^{r_n}\right)$$

where $H = \sup r_n$.

3. Topological structure of $|\overline{N}_p|(r)$

In this section we propose to study linear topological structure of $|\overline{N}_p|(r)$.

THEOREM 3.1. (i) *$|\overline{N}_p|(r)$ is a complete topological linear space paranormed by $g_r(a) = (\sum_n |\phi_n(a)|^{r_n})^{1/M}$, where $M = \max(1, H)$, $H = \sup r_n$.*

(ii) *If r is a constant sequence, then $|\overline{N}_p|_r$ is a Banach space for $r \geq 1$ and a complete r-normed space for $r < 1$ (see [8, p.94]). In these cases , we write $\|a\|_r$ in place of $g_r(a)$.*

(iii) *$|\overline{N}_p|_2$ is a Hilbert space with the inner product given by $(a,b) = \sum_n \phi_n(a)\phi_n(\overline{b})$.*

The proof is a routine verification by using 'standard' techniques and hence is omitted.

In the following theorem we shall discuss uniform convexity (see, e.g., [3], [10]) of $|\overline{N}_p|_r$. In case r = 2 , we have just shown that $|\overline{N}_p|_2$ is a Hilbert space and hence is uniformly convex. We now discuss the general case when $1 < r < \infty$.

THEOREM 3.2. *$|\overline{N}_p|_r$ is uniformly convex for $1 < r < \infty$.*

In order to prove this theorem we require the following lemmas.

LEMMA 3.3. *For $r \geq 2$ and $a, b \in |\overline{N}_p|_r$ we have*

$$\left\|\frac{a+b}{2}\right\|_r^r + \left\|\frac{a-b}{2}\right\|_r^r \leq \frac{1}{2}(\|a\|_r^r + \|b\|_r^r).$$

The proof follows from the well-known inequality [3, p. 224].

LEMMA 3.4. *If $1 < r < 2$, $\frac{1}{r} + \frac{1}{s} = 1$ and $a, b \in |\overline{N}_p|_r$, then*

$$\left\|\frac{a+b}{2}\right\|_r^s + \left\|\frac{a-b}{2}\right\|_r^s \leq \left(\frac{1}{2}\|a\|_r^r + \frac{1}{2}\|b\|_r^r\right)^{s/r}.$$

Proof. Using the well-known inequality

$$|z+w|^s + |z-w|^s \leq 2(|z|^r + |w|^r)^{\frac{1}{r-1}}$$

[3, p. 227], we get

$$\left(\left|\frac{\phi_n(a)+\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}} + \left(\left|\frac{\phi_n(a)-\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}} \leq \left(\frac{1}{2}|\phi_n(a)|^r + \frac{1}{2}|\phi_n(b)|^r\right)^{\frac{1}{r-1}}$$

whence, since $0 < r - 1 < 1$, we have

$$(6) \quad \left(\sum_n \left(\left(\left|\frac{\phi_n(a)+\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}} + \left(\left|\frac{\phi_n(a)-\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}}\right)^{r-1}\right)^{\frac{1}{r-1}}$$

$$\leq \left(\frac{1}{2}\sum_n |\phi_n(a)|^r + \frac{1}{2}\sum_n |\phi_n(b)|^r\right)^{\frac{1}{r-1}} = \left(\frac{1}{2}\|a\|_r^r + \frac{1}{2}\|b\|_r^r\right)^{s/r}.$$

By Minkowski's inequality for $0 < r-1 < 1$, we have

$$\left(\sum_n \left(\left(\left|\frac{\phi_n(a)+\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}} + \left(\left|\frac{\phi_n(a)-\phi_n(b)}{2}\right|^r\right)^{\frac{1}{r-1}}\right)^{r-1}\right)^{\frac{1}{r-1}} \tag{7}$$

$$\geq \left(\sum_n \left|\frac{\phi_n(a)+\phi_n(b)}{2}\right|^r\right)^{s/r} + \left(\sum_n \left|\frac{\phi_n(a)-\phi_n(b)}{2}\right|^r\right)^{s/r}$$

$$= \left\|\frac{a+b}{2}\right\|_r^s + \left\|\frac{a-b}{2}\right\|_r^s.$$

So Lemma 3.4 follows by (6) and (7).

Proof of Theorem 3.2. Let $a, b \in |\overline{N}_p|_r$ be such that $\|a\|_r = \|b\|_r = 1$ and $\|a-b\|_r > \epsilon$ for some ϵ satisfying $0 < \epsilon \leq 2$. By Lemma 3.3, for $r \geq 2$ we have $\|\frac{a+b}{2}\|_r^r + \|\frac{a-b}{2}\|_r^r \leq 1$, which implies that

$$\left\|\frac{a+b}{2}\right\|_r^r \leq 1 - \left\|\frac{a-b}{2}\right\|_r^r \leq 1 - (\epsilon/2)^r. \tag{8}$$

Again for $1 < r < 2$, by Lemma 3.4, we get

$$\left\|\frac{a+b}{2}\right\|_r^s \leq 1 - \left\|\frac{a-b}{2}\right\|_r^s \leq 1 - (\epsilon/2)^s. \tag{9}$$

From (8) and (9) it follows that we can find for each $r > 1$, a number δ, $0 < \delta < 1$, which depends only on ϵ such that $\|\frac{a+b}{2}\|_r \leq 1 - \delta$, and so $|\overline{N}_p|_r$ is uniformly convex.

COROLLARY 3.5. *$|\overline{N_p}|_r$ is reflexive for $1 < r < \infty$.*

The result follows from Theorem 3.1 (ii), Theorem 3.2 and the fact that a uniformly convex Banach space is reflexive (see [10, p.109]).

In the following theorem we discuss the condition for k-convexity and local boundedness of $|\overline{N_p}|(r)$. We refer to [7, 9] for the definitions of k-convexity and local boundedness.

THEOREM 3.6. (i) *For $r > 1$, if $|\overline{N}_p|_r$ is k-convex then $|\overline{N}_p|_k \subset |\overline{N}_p|_r$.*

(ii) *$|\overline{N_p}|(r)$ is locally bounded (that is, there exists a bounded neighbourhood of 0) if $\liminf r_n > 0$.*

Proof. (i) If $|\overline{N}_p|_r$ is k - convex then there is an absolutely k-convex set U and $\delta > 0$ such that $\{a : \|a\|_r \leq \delta\} \subset U \subset \{a : \|a\|_r \leq 1\}$. Thus if $a^N (N = 1, 2, \ldots)$ be defined by $\phi_n(a^N) = \delta/N^{1/r}$ for $n \leq N$, 0 for $n > N$, then $a^N \in U$ for all N. Thus by absolute k-convexity of U, the condition

$\sum_{N=1}^{m} \lambda_N^k \leq 1(\lambda_N$ real and non-negative) implies that $\sum_{N=1}^{m} \lambda_N a^N \in U$. Hence $\| \sum_{N=1}^{m} \lambda_N a^N \|_r \leq 1$, that is

$$\Big(\sum_n \Big| \phi_n \Big(\sum_{N=1}^{m} \lambda_N a^N \Big) \Big|^r \Big)^{1/r} \leq 1,$$

and, since $r > 1$, we have

$$\sum_n \Big| \phi_n \Big(\sum_{N=1}^{m} \lambda_N a^N \Big) \Big|^r \leq 1,$$

which by (5) implies that

$$\sum_n \Big(\sum_{N=1}^{m} \lambda_N \phi_n(a^N) \Big)^r \leq 1.$$

Thus, using (2.12.1) of [2, p. 32], we get

$$1 > \sum_n \Big(\lambda_1 \phi_n(a^1)\Big)^r + \sum_n \Big(\lambda_2 \phi_n(a^2)\Big)^r + \ldots + \sum_n \Big(\lambda_m \phi_n(a^m)\Big)^r$$

$$= \lambda_r^r \phi_1^r(a^1) + \lambda_2^r \sum_{n=1}^{2} \phi_n^r(a^2) + \ldots + \lambda_m^r \sum_{n=1}^{m} \phi_n^r(a^m),$$

since $\phi_n(a^N) = 0$ for $n > N$. Consequently, $1 > \lambda_1^r \delta^r + \lambda_2^r 2 \left(\delta / 2^{1/r}\right)^r + \ldots + \lambda_m^r m \left(\delta / m^{1/r}\right)^r = \delta^r \sum_{N=1}^{m} \lambda_N^r$, that is $\sum_{N=1}^{m} \lambda_N^r \leq 1/\delta^r$. Thus, we have shown that

$$\text{(10)} \qquad \sum_{N=1}^{m} \lambda_N^k \leq 1 \quad \text{implies that} \quad \sum_{N=1}^{m} \lambda_N^r \leq 1/\delta^r.$$

If now $a \in |\overline{N}_p|_k$ and $S = \sum_n |\phi_n(a)|^k$ then, for any m, $\sum_{N=1}^{m} \frac{|\phi_N(a)|^k}{S} \leq 1$.

Hence by (10) we have $\sum_{N=1}^{m} \left| \frac{\phi_N(a)}{S^{1/k}} \right|^r \leq 1/\delta^r$. Since this holds for any m, it follows that $\sum_n |\phi_n(a)|^r \leq S^{r/k}/\delta^r$. We have thus proved that $|\overline{N}_p|_k \subset |\overline{N}_p|_r$.

(ii) If $\inf r_n = \rho > 0$ then for any $K > 0$ and $\epsilon > 0$, we choose an integer $N > 1$ so that $N^{\rho/M} \geq K/\epsilon$ where $M = \max(1, \sup r_n)$. Now $g_r(a) \leq K$ implies that $g_r(a/N) \leq K/N^{\rho/M} \leq \epsilon$. In other words $\{a : g_r(a) \leq K\} \subset N\{a : g_r(a) \leq \epsilon\}$; and since for every $\epsilon > 0$, there exists an N for which this relationship holds, $\{a : g_r(a) \leq K\}$ is bounded. It is immediate from this that any metrically bounded set is bounded. In particular $\{a : g_r(a) \leq 1\}$ is bounded, and it follows that $|\overline{N_p}|(r)$ is locally bounded.

4. Inclusion between $|\overline{N_p}|(r)$ spaces

The folowing result gives inclusion relation between $|\overline{N_p}|(r)$ spaces.

THEOREM 4.1. (i) *If r_n and s_n are bounded sequences of positive real numbers such that $r_n \le s_n$ for each n, then $|\overline{N_p}|(r) \subset |\overline{N_p}|(s)$.*

(ii) *If (t_n) and (u_n) are bounded sequences of positive real numbers and if $r_n = \min(t_n, u_n)$, $q_n = \max(t_n, u_n)$, then $|\overline{N_p}|(r) = |\overline{N_p}|(t) \bigcap |\overline{N_p}|(u)$ and $|\overline{N_p}|(q) = w$ where w is the subspace generated by $|\overline{N_p}|(t) \bigcup |\overline{N_p}|(u)$.*

Proof. We only prove (i). Proof of (ii) is similar to that of the corresponding result for $\ell(p)$ given by Simons [9, Lemma 3].

(i) Let $a \in |\overline{N_p}|(r)$. Then $|\phi(a)| \le 1$ for large n. Since $r_n \le s_n$, $|\phi_n(a)|^{s_n} \le |\phi_n(a)|^{r_n}$ for large n. This shows that $a \in |\overline{N_p}|(s)$ and the proof is complete.

COROLLARY 4.2. *The three conditions $|\overline{N_p}|(t) \subset |\overline{N_p}|(u)$, $|\overline{N_p}|(r) = |\overline{N_p}|(t)$ and $|\overline{N_p}|(u) = |\overline{N_p}|(q)$ are equivalent.*

COROLLARY 4.3. *$|\overline{N_p}|(t) = |\overline{N_p}|(u)$ if and only if $|\overline{N_p}|(r) = |\overline{N_p}|(q)$.*

5. Compactness

The next result gives characterization of compact sets in $|\overline{N_p}|(r)$.

THEOREM 5.1. *A set $K \subset |\overline{N_p}|(r)$ is compact if and only if*

(i) *K is closed and bounded,*

(ii) *given any $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon)$ (depending only on ϵ) such that $\left(\sum_{n=n'+1}^{\infty} |\phi_n(a)|^{r_n}\right)^{1/M} < \epsilon$, for all $a \in K$ whenever $n' \ge n_0$, and*

(iii) *if $f_k : |\overline{N_p}|(r) \to C$ is given by $f_k(a) = \phi_k(a)$ for all $a \in |\overline{N_p}|(r)$, then $f_k(K)$ is compact for $k \ge 1$.*

Proof. One may readily adapt the arguments of Leonard [5, Theorem 5.1] to prove the necessity. We consider only the sufficiency. Suppose (i), (ii) and (iii) hold. Since K is closed and $|\overline{N_p}|(r)$ is complete, it suffices to show that K is totally bounded. Given $\epsilon > 0$, there exists an $n_0 = n_0(\epsilon)$, depending only on ϵ, such that $\left(\sum_{n=n'+1}^{\infty} |\phi_n(a)|^{r_n}\right)^{1/M} < \frac{\epsilon}{2^{1/M}}$, for all $a \in K$, whenever $n' \ge n_0$. Now $f_k(K)$ is compact for all $k \ge 1$, hence, totally bounded; so for each $n = 1, 2, \ldots, n_0$, there exist $\phi_n(a_1), \phi_n(a_2), \ldots \phi_n(a_{n'_n}) \in f_n(K)$ such that if $\phi_n(a) \in f_n(K)$ then there exists i, $1 \le i \le n'_n$ such that $|\phi_n(a) - \phi_n(a_i)| < \frac{\epsilon}{2^{\frac{1}{r_n}} n_0^{\frac{1}{r_n}}}$. Now let

$$K_{n_0} = \{b : b = \left(\phi_1(a_{i_1}), \phi_2(a_{i_2}), \ldots, \phi_{n_0}(a_{i_{n_0}}), 0, 0, \ldots,\right),$$
$$1 \le i_1 \le n'_1,\ 1 \le i_2 \le n'_2,\ \ldots,\ 1 \le i_{n_0} \le n'_{n_0}\}.$$

Then K_{n_0} is a finite set containing $n'_1 n'_2 \ldots n'_{n_0}$ elements. If $a \in K$, then $\phi_n(a) \in f_n(K)$ for all $n \ge 1$. Let $b \in K_{n_0}$ be given by $b = (\phi_1(a_{i_1}), \phi_2(a_{i_2}), \ldots$

$\ldots, \phi_{n_0}(a_{i_{n_0}}), 0, 0, \ldots, 0, \ldots)$, where $|\phi_n(a) - \phi_n(a_{i_n})| < \left(\frac{\epsilon^M}{2n_0}\right)^{\frac{1}{r_n}}$, $n = 1, 2, \ldots, n_0$.

Then

$$g_r^M(a-b) = \sum_{n=1}^{n_0} |\phi_n(a) - \phi_n(a_{i_n})|^{r_n} + \sum_{n=n_0+1}^{\infty} |\phi_n(a)|^{r_n}$$
$$< \sum_{n=1}^{n_0} \frac{\epsilon^M}{2n_0} + \sum_{n=n_0+1}^{\infty} |\phi_n(a)|^{r_n} = \frac{\epsilon^M}{2} + \sum_{n=n_0+1}^{\infty} |\phi_n(a)|^{r_n}.$$

But $\sum_{n=n_0+1}^{\infty} |\phi_n(a)|^{r_n} < \frac{\epsilon^M}{2}$ for all $a \in K$. Therefore, $g_r(a-b) < \epsilon$; and since $a \in K$ is arbitrary and K_{n_0} is a finite set,it follows that K is totally bounded.

6. A result on matrix transformations

Finally, we state without proof a result on matrix transformations. If X, Y are any two sets of sequences, we denote by (X, Y) the set of those matrices $A = (a_{nk})$ which have the property that Ax exists and belongs to Y for any $x \in X$.

Write $\phi_n(Ax) = \frac{p_n}{P_n P_{n-1}} \sum_{i=1}^{n} P_{i-1} A_i(x) = \sum_k b_{nk} x_k$, where $b_{nk} = \frac{p_n}{P_n P_{n-1}} \sum_{i=1}^{n} P_{i-1} a_{ik}$.

With this notation we have the following result.

THEOREM 6.1. *Let* $r \geq 1$. *Then* $A \in (\ell, |\overline{N_p}|(r))$ *if and only if* $\sup_k \sum_n |b_{nk}|^r < \infty$.

The proof uses ideas similar to those used (e.g.) in [8, p.167] .

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