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FARTHEST POINTS AND BOUNDED 2-FUNCTIONALS

Abstract. In this paper, we give some theorems on further characterizations and existences of ϵ-farthest points in linear 2-normed spaces in terms of bounded linear 2-functionals.

I. Introduction

Let X be a linear space of dimension greater than 1 and $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ which satisfies the following conditions:

(N_1) $\|a,b\| = 0$ if and only if a and b are linearly dependent,
(N_2) $\|a,b\| = \|b,a\|$,
(N_3) $\|\alpha a,b\| = |\alpha|\|a,b\|$, where α is real,
(N_4) $\|a+b,c\| \leq \|a,c\| + \|b,c\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ is called a *linear 2-normed space* ([3]). Note that the 2-norm is non-negative and $\|a,b\| = \|a+b,b\|$.

The following definitions and Theorem 1.1 are given in [5] and [9]:

DEFINITION 1.1. A *2-functional* f is a real-valued mapping with domain $A \times C$, where A and C are linear manifolds of a linear 2-normed space $(X, \|\cdot,\cdot\|)$.

DEFINITION 1.2. A 2-functional f is said to be *linear* if

(1) $f(a+c, b+d) = f(a,b) + f(a,d) + f(c,b) + f(c,d)$,
(2) $f(\alpha a, \beta b) = \alpha\beta f(a,b)$, where α and β are real.

DEFINITION 1.3. A 2-functional f with domain $D(f)$ is said to be *bounded* if there is a real constant $K > 0$ such that $|f(a,b)| \leq K\|a,b\|$ for $(a,b) \in D(f)$.

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If f is bounded, define the norm of f, $\|f\|$, by

$$\|f\| = \inf\{K : |f(a,b)| \leq K\|a,b\| \quad \text{for all} \quad (a,b) \in D(f)\}.$$

If f is not bounded, define $\|f\| = +\infty$.

For $x \in X$, let $V(x)$ denote the subspace of X generated by x.

THEOREM 1.1 ([5]). *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $x_o \in X$ be a non-zero element. Let $z \in X$ be such that x_o and z are linearly independent. Then there exists a bounded linear 2-functional f with the domain $X \times V(z)$ such that*

(1) $f(x_o, z) = \|x_o, z\|$,
(2) $\|f\| = 1$.

Additional properties of bounded linear 2-functionals may be found in [5] and [8].

DEFINITION 1.4 ([6]). Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. For $x, y \in X, x \neq y$ and $y \neq 0$, the set $L(x,y) = \{x + ty : t \in R\}$ is called the *algebraic line* determined by x and y.

It is possible to define the unit cylinder with central axis $L(0,z)$ for every non-zero point $z \in X$.

DEFINITION 1.5 ([6]). Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. For a non-zero point $z \in X$, the set $B(z,1) = \{x \in X : \|x,z\| \leq 1\}$ is called the *unit cylinder* with central axis $L(0,z)$.

In 1979, C. Franchetti and I. Singer ([2]) studied the concept of farthest points in normed linear spaces and obtained some results on characterizations and existences of farthest points in normed linear spaces. In 1984, M. Janc ([4]) gave also some characterizations of farthest points in normed linear spaces and I. Singer ([8], p162) gave a characterization of the elements of ϵ-approximation. Recently, some characterizations of farthest points in linear 2-normed spaces have been obtained by S. Elumalai and S. Ravi ([1]) and S. Ravi ([7]).

In this paper, we give some theorems on further characterizations and existences of ϵ-farthest points in linear 2-normed spaces.

2. Characterizations of ϵ–farthest points

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space with dim $X \geq 2$, G be a subset of X and let ϵ be a real number ≥ 0. For $x \in X$ and $G \subset X$, we shall denote by $[G, x]$, the subspace of X generated by some elements of the set G and x. A nonempty subset G of a linear 2-normed space X is *bounded* if

$\sup_{g\in G}\|g,z\| < \infty$ for $z \in X\setminus[G]$, where $[G]$ denotes the subspace of X generated by the elements of G.

The *deviation* of G from x is defined by

$$\delta_G(x,z) = \sup_{g\in G}\|g-x,z\| \quad \text{for} \quad z\in X\setminus[G,x].$$

Let $x\in X$ such that $[G,x]\subsetneq X$. An element $g_o\in G$ is called a *farthest point* to x in G(with respect to z) if

$$\|g_o - x, z\| = \sup_{g\in G}\|g-x,z\|$$

for every $z\in X\setminus[G,x]$.

The set of all farthest points to x in G(with respect to z) is denoted by $F_G(x,z)$.

DEFINITION 2.1. Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X, $x\in X$ such that $[G,x]\subsetneq X$, $g_o\in G$ and $\epsilon\geq 0$. Then an element g_o is called an *ϵ-farthest point* to x in G(with respect to z) if

$$\|g_o-x,z\| \geq \delta_G(x,z)-\epsilon = \sup_{g\in G}\|g-x,z\|-\epsilon$$

for every $z\in X\setminus[G,x]$.

The set of all ϵ-farthest points to x in G(with respect to z) is denoted by $F_{G,\epsilon}(x,z)$. Of course, $F_{G,0}(x,z)=F_G(x,z)$.

The following theorems give the necessary and sufficient condition for $g_o\in G$ to be an element of $F_{G,\epsilon}(x,z)$.

THEOREM 2.1. *Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X, $x\in X$ such that $[G,x]\subsetneq X$ and $g_o\in G$. If $g_o\in F_{G,\epsilon}(x,z)$, then $g_o\in F_{G,\epsilon}(\alpha x+(1-\alpha)g_o,z)$ for any $\alpha\geq 1$.*

Proof. Suppose that $g_o\in F_{G,\epsilon}(x,z)$. Then we have

$$\|g_o-x,z\|+\epsilon\geq\|g-x,z\|$$

for any $g\in G$. Let $x_\alpha=\alpha x+(1-\alpha)g_o$ with $\alpha\geq 1$. For $z\in X\setminus[G,x]$,

$$\begin{aligned}\|g-x_\alpha,z\| &\leq \|x-x_\alpha,z\|+\|g-x,z\|\\ &\leq (\alpha-1)\|x-g_o,z\|+\|g_o-x,z\|+\epsilon\\ &= \|g_o-(\alpha x+(1-\alpha)g_o),z\|+\epsilon.\end{aligned}$$

Thus, we have $g_o\in F_{G,\epsilon}(\alpha x+(1-\alpha)g_o,z)$. This completes the proof.

COROLLARY 2.2. *Let $(X,\|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X, $x\in X$ such that $[G,x]\subsetneq X$ and $g_o\in G$. Then $g_o\in F_{G,\epsilon}(x,z)$ if and only if $\alpha g_o+\beta g\in F_{G,\alpha\epsilon}(\alpha x+\beta g,z)$ for any $\alpha\geq 1$, $\beta\leq 0$ with $\alpha+\beta=1$ and for all $g\in G$ with $\alpha g_o+\beta g\in G$.*

THEOREM 2.3. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a subset of X, $x \in X$ such that $[G,x] \subsetneq X$ and $\epsilon \geq 0$. Then $g_o \in F_{G,\epsilon}(x,z)$ if and only if $g \in B(z, \|x-g_o,z\|+\epsilon)+x$ for each $g \in G$.*

Proof. Let $g_o \in F_{G,\epsilon}(x,z)$. Suppose that there exists $g_1 \in G$ such that if $g_1 \notin B(z,\|x-g_o,z\|+\epsilon)+x$, then

$$\|g_o - x, z\| + \epsilon < \|g_1 - x, z\| \leq \delta_G(x,z),$$

which is a contradition.

Conversely, if $g_o \notin F_{G,\epsilon}(x,z)$, then $\|g_o - x, z\| < \delta_G(x,z) - \epsilon$. Then, by definition of $\delta_G(x,z)$, there exists $g_1 \in G$ such that $\|g_o - x,z\| + \epsilon < \|g_1 - x, z\|$. Therefore, we have

$$g_1 \notin B(z, \|x - g_o, z\| + \epsilon) + x.$$

This completes the proof.

3. ϵ–farthest points in term of bounded linear 2-functionals

In this section, we give some characterizations of ϵ-farthest points in linear 2-normed spaces in terms of linear bounded 2-functionals. For any non-zero element $z \in X$, we denote by $(X \times V(z))^*$ the space of all bounded linear 2-functionals f with domain $X \times V(z)$ and with the norm $\|\cdot\|$ defined by

$$\|f\| = \sup\{|f(x,z)| : \|x,z\| = 1, (x,z) \in X \times V(z)\}.$$

For $f \in (X \times V(z))^*$ and $G \subset X$, we shall write $\sup_{\|f\|=1}$ and $\sup f(G,z)$ for $\sup_{f \in (X\times V(z))^*, \|f\|=1}$ and $\sup_{g\in G} f(g,z)$ respectively.

THEOREM 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X, $x \in X$ such that $[G,x] \subsetneq X, g_o \in G$ and $\epsilon \geq 0$. Suppose that for every $g \in G$, there is a 2-functional $f_z \in (X \times V(z))^*$ such that*

(1) $\|f_z\| = 1$,
(2) $f_z(g - x, z) = \|g - x, z\|$,
(3) $f_z(g - g_o, z) \leq \epsilon$.

Then $g_o \in F_{G,\epsilon}(x,z)$.

Proof. Suppose that the conditions (1), (2) and (3) hold for $g \in G$. Then, for each $z \in X \setminus [G,x]$

$$\begin{aligned}\|g-x,z\| &= f_z(g-x,z)\\ &= f_z(g-g_o,z)+f_z(g_o-x,z)\\ &\leq \epsilon + \|g_o - x, z\|.\end{aligned}$$

Since $g \in G$ is arbitrary, passing to supremum over $g \in G$, we have

$$\delta_G(x,z) \leq \epsilon + \|g_o - x, z\|.$$

Therefore, $g_o \in F_{G,\epsilon}(x,z)$. This completes the proof.

THEOREM 3.2. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a subset of X, $x \in X$ such that $[G,x] \subsetneq X, g_o \in G$ and $\epsilon \geq 0$. Then $g_o \in F_{G,\epsilon}(x,z)$ if and only if for every $g \in G$, there is a 2-functional $f_z \in (X \times V(z))^*$ such that*

(1) $\|f_z\| = 1$
(2) $f_z(g_o - x, z) \geq \|g - x, z\| - \epsilon$.

Proof. Suppose that the conditions (1) and (2) hold. Then, for each $z \in X \setminus [G,x]$

$$\|g_o - x, z\| \geq |f_z(g_o - x, z)| \geq f_z(g_o - x, z) \geq \|g - x, z\| - \epsilon.$$

Since $g \in G$ is arbitrary, passing to supremum over $g \in G$, we have

$$\|g_o - x, z\| \geq \delta_G(x,z) - \epsilon.$$

Thus, $g_o \in F_{G,\epsilon}(x,z)$.

Conversely, suppose that the conditions do not hold. Then there exists $g_1 \in G$ such that for each $f \in (X \times V(z))^*$ with $\|f\| = 1$, $z \in X \setminus [G,x]$, we have

$$f(g_o - x, z) < \|g_1 - x, z\| - \epsilon.$$

By the Hahn-Banach theorem type ([9]), there exists $f_z \in (X \times V(z))^*$ with $\|f_z\| = 1$ such that $f_z(g_o - x, z) = \|g_o - x, z\|$. Thus we have

$$\|g_o - x, z\| = f_z(g_o - x, z) < \|g_1 - x, z\| - \epsilon \leq \delta_G(x,z) - \epsilon.$$

Therefore, $g_o \notin F_{G,\epsilon}(x,z)$. This completes the proof.

By using Theorem 3.2, we easily obtain the following:

COROLLARY 3.3. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a subset of X, $x \in X$ such that $[G,x] \subsetneq X, g_o \in G$ and $\epsilon \geq 0$. Then the following statements are equivalent:*

(1) $g_o \in F_{G,\epsilon}(x,z)$.
(2) *For every $g \in G$, there is a 2-functional $f_z \in (X \times V(z))^*$ such that*
  (i) $\|f_z\| = 1$,
  (ii) $f_z(g_o - x, z) \geq \sup_{g \in G} \|g - x, z\| - \epsilon$.

The following lemmas were proved by R. Ravi ([7]).

LEMMA 3.4. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X and $x \in X$ such that $[G,x] \subsetneq X$. Then, for each $z \in X \setminus [G,x]$*

$$\sup_{\|f_z\|=1} |\sup_{g \in G} f_z(g - x, z)| = \sup_{\|f_z\|=1} \sup_{g \in G} |f_z(g - x, z)|. \tag{3.1}$$

Proof. Let $x = 0$. Then, for each $z \in X \setminus [G]$

$$\sup_{\|f_z\|=1} |\sup f_z(G,z)| = \sup_{\|f_z\|=1} \sup |f_z(G,z)|. \tag{3.2}$$

Now, for each $z \in X \setminus [G]$, let $f_z \in (X \times V(z))^*$, $\|f_z\| = 1$.

Case (i). $\sup f_z(G,z) \geq 0$. Then

$$|\sup f_z(G,z)| = \sup f_z(G,z) \leq \sup |f_z(G,z)|. \tag{3.3}$$

Case (ii). $\sup f_z(G,z) < 0$. Then

$$\begin{aligned} |\sup f_z(G,z)| &= -\sup f_z(G,z) = \inf(-f_z)(G,z) \\ &\leq \sup(-f_z)(G,z) \leq \sup |f_z(G,z)|. \end{aligned} \tag{3.4}$$

Thus, by (3.3) and (3.4), we have for $z \in X \setminus [G]$

$$\sup_{\|f_z\|=1} |\sup f_z(G,z)| \leq \sup_{\|f_z\|=1} \sup |f_z(G,z)|. \tag{3.5}$$

Now, for each $z \in X \setminus [G]$ let $\epsilon_z > 0$ be given. Then, there exists an $\widetilde{f}_z \in (X \times V(z))^*$ with $\|\widetilde{f}_z\| = 1$ such that

$$\sup_{\|f_z\|=1} \sup |f_z(G,z)| \leq \sup |\widetilde{f}_z(G,z)| + \epsilon_z.$$

We may assume that $\sup |\widetilde{f}_z(G,z)| = |\sup \widetilde{f}_z(G,z)|$. Then, for $z \in X \setminus [G]$

$$\begin{aligned} \sup_{\|f_z\|=1} \sup |f_z(G,z)| &\leq \sup |\widetilde{f}_z(G,z)| + \epsilon_z \\ &= |\sup \widetilde{f}_z(G,z)| + \epsilon_z \\ &\leq \sup_{\|f_z\|=1} |\sup f_z(G,z)| + \epsilon_z. \end{aligned}$$

Hence, since $\epsilon_z > 0$ is arbitrary, we have

$$\sup_{\|f_z\|=1} \sup |f_z(G,z)| \leq \sup_{\|f_z\|=1} |\sup f_z(G,z)|, \quad z \in X \setminus [G]. \tag{3.6}$$

By (3.5) and (3.6), for each $z \in X \setminus [G]$, we have

$$\sup_{\|f_z\|=1} |\sup f_z(G,z)| = \sup_{\|f_z\|=1} |\sup f_z(G,z)|,$$

which is (3.2). This proves the lemma for $x = 0$.

Next, let $x \in X$ be arbitrary such that $[G,x] \subsetneq X$. Then, for each $z \in X \setminus [G,x]$ applying (3.2) to the set $G - x$, we have

$$\sup_{\|f_z\|=1} |\sup f_z(G-x,z)| \leq \sup_{\|f_z\|=1} \sup |f_z(G-x,z)|,$$

which is (3.1). This completes the proof.

LEMMA 3.5. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X and $x \in X$ such that $[G, x] \subsetneq X$. Then, for each $z \in X \setminus [G, x]$*

$$\sup_{g\in G} \|g - x, z\| = \sup_{\|f_z\|=1} |\sup f_z(G, z) - f_z(x, z)|.$$

Proof. For each $z \in X \setminus [G, x]$, let $f_z \in (X \times V(z))^*$. Then we have

$$\begin{aligned}\sup_{g\in G} \|g - x, z\| &= \sup_{g\in G} \sup_{\|f_z\|=1} |f_z(g - x, z)| \\ &= \sup_{\|f_z\|=1} \sup_{g\in G} |f_z(g - x, z)| \\ &= \sup_{\|f_z\|=1} |\sup_{g\in G} f_z(g - x, z)| \\ &= \sup_{\|f_z\|=1} |\sup f_z(G, z) - \sup f_z(x, z)|.\end{aligned}$$

This completes the proof.

From Theorem 3.2, Lemmas 3.4 and 3.5, we have the following:

THEOREM 3.6. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G be a bounded subset of X, $x \in X$ such that $[G, x] \subsetneq X$, $g_o \in G$ and $\epsilon \geq 0$. Consider the following statements:*

(1) $g_o \in F_{G,\epsilon}(x, z)$.
(2) *There is a 2-functional $f_o \in (X \times V(z))^*$ with $\|f_o\| = 1$, such that*
  (i) $f_o(g_o, z) \geq \sup f_o(G, z) - \epsilon$,
  (ii) $|\sup f_o(G, z) - f_o(x, z)| \geq \sup_{\|f_z\|=1} |\sup f_z(G, z) - f_z(x, z)| - \epsilon$.
(3) $g_o \in F_{G,2\epsilon}(x, z)$.

Then (1)$\Rightarrow$(2)$\Rightarrow$(3).

Proof. Suppose that (1) holds. Then by Theorem 3.2, for every $g \in G$ there is an $f_o \in (X \times V(z))^*$ such that $\|f_o\| = 1$, $f_o(g_o - x, z) \geq \sup \|g - x, z\| - \epsilon$ and so

$$f_o(g_o - x, z) \geq \delta_G(x, z) - \epsilon.$$

From Lemmas 3.4 and 3.5, it follows that

$$\begin{aligned}f_o(g_o, z) - f_o(x, z) &\geq \delta_G(x, z) - \epsilon \\ &= \sup_{g\in G} \|g - x, z\| - \epsilon \\ &= \sup_{\|f_z\|=1} |\sup f_z(G, z) - f_z(x, z)| - \epsilon.\end{aligned}$$

Since $\sup f_o(G, z) \geq f_o(g_o, z)$, (ii) follows.

On the other hand, since $g_o \in F_{G,\epsilon}(x, z)$,

$$f_o(g - x, z) \leq \|g - x, z\| \leq \|g_o - x, z\| + \epsilon$$

for every $z \in X \setminus [G, x]$ and thus

$$\begin{aligned} f_o(g_o, z) &= f_o(x, z) + \|g_o - x, z\| \\ &\geq f_o(x, z) + f_o(g - x, z) - \epsilon = f_o(g, z) - \epsilon \end{aligned}$$

for each $g \in G$ and so $f_o(g_o, z) \geq \sup f_o(G, z) - \epsilon$. Thus, (i) holds.

Next, suppose that (2) holds. From Lemma 3.5, the conditions (i) and (ii), we have

$$\begin{aligned} f_o(g_o, z) - f_o(x, z) &\geq \sup f_o(G, z) - f_o(x, z) - \epsilon \\ &\geq \sup_{\|f_z\|=1} |\sup f_z(G - x, z)| - 2\epsilon \\ &= \sup_{g \in G} \|g - x, z\| - 2\epsilon \end{aligned}$$

for every $z \in X \setminus [G, x]$. Thus, we have

$$\|g_o - x, z\| \geq f_o(g_o - x, z) \geq \delta_G(x, z) - 2\epsilon$$

and so $g_o \in F_{G,2\epsilon}(x)$. This completes the proof.

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