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Grzegorz Łysik

SINGULAR LINEAR DIFFERENTIAL EQUATIONS AND LAURENT TYPE SERIES

Abstract. Linear ordinary differential operators with meromorphic coefficients at zero are studied. It is well known that in the case when zero is a regular or regular singular point then fundamental system of solutions consists of convergent series of the Taylor type. On the other hand in the case of irregular singular point power series solution, in general, does not converge; however it can be asymptotically sum up in sectors to an exact solution. The aim of the paper is to show that for a class of operators with irregular singular point the fundamental system of solutions can be found in a form of convergent Laurent type series of a Gevrey order. Under suitable conditions the convergence of the approximation scheme for a functional equation $\sum_{j=-k}^{\infty} W_j(z-j)G(z-j) = H(z)$ is also derived and properties of its solution G are described.

Introduction

Let

$$P = P\Big(x, \frac{d}{dx}\Big) = \sum_{i=0}^{n} a^i(x)\frac{d^i}{dx^i} \tag{1}$$

be a linear operator with meromorphic coefficients at zero. Multiplying $a^i(x)$, $i = 0, \ldots, n$ by $x^n/a_n(x)$ we can assume that

$$a^i(x) = \begin{cases} \sum\limits_{j=j_0}^{\infty} a^i_j x^j \text{ for } 0 < |x| < r & \text{if } i = 0, \ldots, n-1, \\ x^n & \text{if } i = n. \end{cases} \tag{2}$$

with some $j_0 \in \mathbb{Z}$ and $r > 0$. The *Newton diagram* for P is defined by

$$\mathcal{N}_P = \{(i,j) \in \mathbb{N}_0 \times \mathbb{Z} : a^i_{i+j} \neq 0\}. \tag{3}$$

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We call the set $\{(i,0) : i \in \mathbb{N}_0\}$ as the *characteristic level of P* and define the *characteristic polynomial*

$$W(z) = W_P(z) = (z)_n + \sum_{i=0}^{n-1} a_i^i (z)_i. \tag{4}$$

where

$$(z)_i := \frac{\Gamma(z+1)}{\Gamma(z+1-i)} = \begin{cases} 1 & \text{if } i = 0, \\ z(z-1)\cdot \ldots \cdot (z-i+1) & \text{if } i \in \mathbb{N}. \end{cases} \tag{5}$$

The degree of W is equal to n. Note that $W(z)$ is equal to the coefficient $W_0(z)$ in the expression

$$x^{-z} P(x^z) = \sum_{j=-k}^{\infty} W_j(z) x^j$$

where $\quad k = \max_{(i,j)\in\mathcal{N}_P} (-j) \geq 0 \quad$ and

$$W_j(z) = \sum_{i=\max(0,j_0-j)}^{n-1} a_{i+j}^i (z)_i \text{ for } j \geq -k,\ j \neq 0. \tag{6}$$

Note also that P can be written in the form

$$P\left(x, \frac{d}{dx}\right) = \sum_{j=-k}^{\infty} x^j W_j\left(x \frac{d}{dx}\right).$$

We denote by κ_P the Katz invariant of P i.e. the smallest $\kappa \geq 0$ such that there are no points of $\mathcal{N}_P$ below the line $\{(i,j) : j = \kappa(i-n)\}$. If $k = \kappa_P = 0$ i.e. there are no points of $\mathcal{N}_P$ below the characteristic level then zero is *regular or regular singular point for P*. In the other case it is called *irregular singular point.* We have the well known

THEOREM (Fuchs, [H]). *Let zero be a regular or regular singular point for P and let f be analytic at zero. Put* $\rho_0 = 0$ *and let* $\rho_1, \ldots, \rho_m$ *be the roots of W. Then every solution of* $Pu = f$ *is given by a convergent series of the form*

$$\sum_{\mu=0}^{m} \sum_{p=0}^{q_\mu} \sum_{\gamma=0}^{\infty} C_{\mu,p,\gamma} \ln^p x \cdot x^{\rho_\mu + \gamma}$$

with some $q_\mu \in \mathbb{N}_0$ *for* $\mu = 0, 1, \ldots, m$.

In the case of an irregular singular point the Fuchs theorem is no longer true. However there exist formal power series solutions of $Pu = f$ (cf. [T]) they, in general, do not converge.

EXAMPLE 1 (Euler). The equation $x^2u' = u - x$ has a unique formal power series solution $\hat{u}(x) = \sum_{\gamma=1}^{\infty} a_\gamma x^\gamma$, where $a_\gamma = (\gamma - 1)!$

During the last few years a special procedure called *multisummability* was developpped to deal with divergent power series solutions of differential equations (Ecalle [E], Balser [B], Braaksma [Br], Malgrange [M],...). Applying this procedure to the Euler equation $x^2u' = u - x$ we need first to compute the Borel transform $B\hat{u}$ of its formal solution $\hat{u}$

$$B\hat{u}(\zeta) := \sum_{\gamma=1}^{\infty} \frac{a_\gamma}{\Gamma(\gamma)}\zeta^{\gamma-1} = \frac{1}{1-\zeta} \quad \text{for } |\zeta| < 1$$

and then analytically continue it to a function $g(\zeta) = 1/(\zeta - 1)$ for $\zeta \neq 1$. Now we note that g is of exponential growth along any ray $d_\theta := \{\zeta \in \mathbb{C} : \arg\zeta = \theta\}$ with $0 < \theta < 2\pi$ (in fact g is bounded on any d_θ). Next we compute the inverse to the Borel transform, which appears to be the *Laplace transform* of g along d_θ (in variable $1/x$)

$$\mathcal{L}_\theta g(x) = \int_{d_\theta} \frac{1}{1-\zeta}\exp\left(-\frac{\zeta}{x}\right) d\zeta.$$

$\mathcal{L}_\theta g$ is defined in a sector $S(\theta) := \{x : |\arg x - \theta| < \pi/2\}$. Finally, gluing together $\mathcal{L}_\theta g$ with $0 < \theta < 2\pi$ we get a function u defined in a sector $S = \{x : -\pi/2 < \arg x < 5\pi/2\}$. The u is a solution of the Euler equation and $\hat{u}$ is the asymptotic expansion of u in S.

On the other hand, since for the Euler equation $\mathcal{N}_P$ lies below the characteristic level it is natural to look for its solutions in the form

$$u(x) = \sum_{\gamma=-\infty}^{0} a_\gamma x^\gamma + \ln x \sum_{\gamma=-\infty}^{0} b_\gamma x^\gamma.$$

Comparing both sides of $x^2u' = u - x$ we get $b_0 = -1$, $b_{-1} = 1, \ldots,$ $b_\gamma = (-1)^{\gamma-1}/(-\gamma)!$ for $\gamma \in -\mathbb{N}_0$, $a_0 = C$, $a_{-1} = 1 - C, \ldots, a_\gamma = d_\gamma + (-1)^\gamma C/(-\gamma)!$ with $d_\gamma = (-1)^\gamma \sum_{i=1}^{-\gamma} \frac{(-1)^i}{i \cdot i!(-\gamma-i)!}$ for $\gamma \in -\mathbb{N}$, (so $|d_\gamma| \leq 1/(-\gamma-1)!$). Thus,

$$u(x) = Ce^{-1/x} - \ln x \cdot e^{-1/x} + \sum_{\gamma=-\infty}^{-1} d_\gamma x^\gamma$$

and the series converges for $x \in \widetilde{\mathbb{C} \setminus \{0\}}$.

It appears ([H], Ch. IV, Th. 10.1) that the fundamental system of solutions of the homogeneous equation $Pu = 0$ consists of Laurent type series.

However the coefficients arising in this kind of representation are very difficult to find as shows

EXAMPLE 2. Looking for a solution of the equation $x^2 \frac{du}{dx} = (1+x^2)u$ in the form of Laurent series $u = \sum_{\gamma=-\infty}^{\infty} a_\gamma x^\gamma$ we find

$$(\gamma - 1)a_{\gamma-1} = a_\gamma + a_{\gamma-2} \quad \text{for } \gamma \in \mathbb{Z}.$$

So we can fix arbitrarily $a_0 = C$. Putting $a_1 = a$ we find that, in general, both series $\sum_{\gamma=0}^{\infty} a_\gamma x^\gamma$ and $\sum_{\gamma=-\infty}^{-1} a_\gamma x^\gamma$ are divergent. However they will converge for a very special choice of $a/C = J_1(2)/J_0(2)$ (J_n – the Bessel function of order n), since the general solution is given by

$$u(x) = Ce^{x-1/x} = C \sum_{\gamma=-\infty}^{\infty} \Big(\sum_{i=\max(0,-\gamma)}^{\infty} \frac{(-1)^i}{i!(i+\gamma)!} \Big) x^\gamma.$$

In the paper we study the Cauchy problem

$$\text{(7)} \qquad \begin{cases} P(x, \frac{d}{dx})u = f, \\ u(t) = u_0, \ldots, u^{(n-1)}(t) = u_{n-1}, \end{cases}$$

where f is a Laurent type series of radious of convergence $r > 0$ and $0 < t < r$. By applying the Mellin transformation $\mathcal{M}$ we transform the problem (7) into a functional equation for the function $G = \mathcal{M}u$

$$\text{(8)} \qquad W(z)G(z) + \sum_{\substack{j=-k \\ j \neq 0}}^{\infty} W_j(z-j)G(z-j) = H(z).$$

Under suitable conditions we shall solve (8) by the method of successive approximations and compute the boundary value $b(G)$ (cf [Ł1]) of its solution. Then $u(x) = \frac{1}{2\pi i} b(G)[x^\cdot]$ is a solution of (7). To be more precise let us introduce

DEFINITION. By a *Laurent series of Gevrey order* $\kappa > 0$ and radious of convergence $r > 0$ we mean a series

$$u(x) = \sum_{\gamma=-\infty}^{\infty} C_\gamma x^\gamma$$

such that for some $M < \infty$ and any $\tilde{r} < r$

$$\text{(9)} \qquad |C_\gamma| \le \begin{cases} C(\tilde{r})/\tilde{r}^\gamma & \text{for } \gamma \in \mathbb{N}_0, \\ C \dfrac{M^{|\gamma|}}{(|\gamma|!)^{1/\kappa}} & \text{for } \gamma \in -\mathbb{N}. \end{cases}$$

The u is a holomorphic function on $B(r) \setminus \{0\}$ satisfying with some $L < \infty$ and any $\tilde{r} < r$

$$\text{(10)} \qquad |u(x)| \le C(\tilde{r}) \exp\{L/|x|^\kappa\} \quad \text{for } 0 < |x| \le \tilde{r}.$$

Conversely, if $u \in \mathcal{O}(B(r) \setminus \{0\})$ satisfies (10) then u is a Laurent series of order Gevrey κ.

Note here that the derivative and the primitive of u if $C_{-1} = 0$ are also Laurent series of the same Gevrey order as u. On the other hand if $u_i, i = 1,2$ are Laurent series of Gevrey orders $\kappa_i, i = 1,2$ then $u_1 \cdot u_2$ is a Laurent series of Gevrey order $\kappa = \max(\kappa_1, \kappa_2)$.

DEFINITION. Let $m \in \mathbb{N}$, $q \in \mathbb{N}_0$, $\rho_1, \ldots, \rho_m \in \mathbb{C}$ with $\rho_\nu \neq \rho_\mu \bmod 1$ for $1 \leq \nu < \mu \leq m$. Let $u^{\mu,p}, \mu \in \{1, \ldots, m\}, p \in \{0, \ldots, q\}$ be Laurent series of Gevrey order κ. Then a sum

$$(11) \qquad u(x) = \sum_{\mu=1}^{m} \sum_{p=0}^{q} A_{\mu,p} \ln^p x \cdot x^{\rho_\mu} \cdot u^{\mu,p}(x)$$

$$= \sum_{\mu=1}^{m} \sum_{p=0}^{q} \sum_{\gamma=-\infty}^{\infty} C_{\mu,p,\gamma} \ln^p x \cdot x^{\rho_\mu + \gamma}$$

is called a *Laurent type series of Gevrey order* κ .

Note here that in the expression (11) ρ_ν can be replaced by $\rho_\nu + l$ with a fixed $l \in \mathbb{Z}$ but the new $C_\gamma = C_{\mu,p,\gamma}$ still satisfies (9).

The main result can be stated as follows.

THEOREM 1. *Let P be a linear differential operator* (1) *with meromorphic coefficients* (2) *at zero. Assume one of the following cases*

Case 1°. *P is of order 1*;

Case 2°. $\mathcal{N}_P \subset \mathbb{N}_0 \times \mathbb{N}_0$;

Case 3°. $\mathcal{N}_P \subset \mathbb{N}_0 \times (-\mathbb{N}_0)$;

Case 4°. *P is a composition of operators satisfying any of the Cases* 1°–3°.

If f is a Laurent type series of Gevrey order κ and radious of convergence r then every solution of $Pu = f$ is a Laurent type series of Gevrey order $\max(\kappa, \kappa_P)$ *and radious of convergence r (κ_P is the Katz invariant of P).*

Before the proof let us observe that Case 4° is an immediate consequence of Cases 1° – 3° since $\kappa_{P_1 \circ P_2} = \max(\kappa_{P_1}, \kappa_{P_2})$. Note also that by the properties of solutions of linear equations we can assume that with C_γ satisfying (9) and some $q \in \mathbb{N}_0$, $\varrho \in \mathbb{C}$

$$(12) \qquad f = \sum_{\gamma=-\infty}^{\infty} C_\gamma f_\gamma \text{ where } f_\gamma(x) := \ln^q x \cdot x^{\rho+\gamma}.$$

Proof of Case 1°. P is an operator of order 1 i.e.

$$P\left(x, \frac{d}{dx}\right) = x\frac{d}{dx} + \sum_{j=j_0}^{\infty} a_j x^j \text{ with some } j_0 \in \mathbb{Z}.$$

Then $k = \kappa_P = \max(0, -j_0), W(z) = z - a_0$ and $W_j(z) = a_j$ for $j \geq -k, j \neq 0$. Considering $\widetilde{u}(x) = x^{-a_0}u(x)$ instead of u we can assume that $a_0 = 0$. Then the general solution u_g of the homogeneous equation given by

$$u_g(x) = C\exp\left\{ -\int \frac{a(x)}{x} dx \right\}$$

is a Laurent series of Gevrey order κ_P if $\kappa_P > 0$ or a Taylor series if $\kappa_P = 0$. On the other hand a special solution u_s of $P(x, \frac{d}{dx})u = f$ is given by

$$u_s(x) = \exp\left\{ -\int \frac{a(x)}{x} dx \right\} \cdot \int \frac{f(x)}{x} \cdot \exp\left\{ \int \frac{a(x)}{x} dx \right\} dx.$$

So for f given by (12),

$$u_s(x) = \sum_{p=0}^{\widetilde{q}} \sum_{\gamma=-\infty}^{\infty} \widetilde{C}_{p,\gamma} \ln^p x \cdot x^{\rho+\gamma} \text{ where } \widetilde{q} = \begin{cases} q & \text{if } \rho \notin \mathbb{Z}, \\ q+1 & \text{if } \rho \in \mathbb{Z} \end{cases}$$

is a Laurent type series of Gevrey order $\max(\kappa_P, \kappa)$ and the same is true for $u_g + u_s$.

For the proof of Cases 2° and 3° we need to recall some facts about Laplace distributions and, the Taylor and Mellin transformations.

Laplace distributions

Let $\nu \in \mathbb{R} \cup \{-\infty\}$ and $\omega \in \mathbb{R} \cup \{\infty\}$. The space $L'_{(\nu,\omega)}(\mathbb{R})$ of *Laplace distributions on* $\mathbb{R}$ is defined ([Z]) as the dual space of

$$L_{(\nu,\omega)}(\mathbb{R}) = \varinjlim_{a>\nu, b<\omega} L_{a,b}(\mathbb{R})$$

where for any $a, b \in \mathbb{R}$

$$L_{a,b}(\mathbb{R}) = \{\varphi \in C^\infty(\mathbb{R}) : \sup_{x\in\mathbb{R}} \sup_{\alpha\leq m} |D^\alpha\varphi(x)|\gamma_{a,b}(x) < \infty \text{ for any } m \in \mathbb{N}_0 \}$$

with

$$\gamma_{a,b}(x) = \begin{cases} e^{-ax} & \text{for } x < 0, \\ e^{-bx} & \text{for } x \geq 0. \end{cases}$$

Let $m \in \mathbb{N}$, $\rho_1, \ldots, \rho_m \in \mathbb{C}$ and $\mathbb{L} = \bigcup_{\mu=1}^m (\rho_\mu + \mathbb{R})$. In an analogous way as above we can define the space $L'_{(\nu,\omega)}(\mathbb{L})$ of *Laplace distributions on* $\mathbb{L}$. If $\operatorname{Im}\rho_\mu \neq \operatorname{Im}\rho_\lambda$ for $\mu \neq \lambda$, $(\mu, \lambda = 1, \ldots, m)$ then

$$L'_{(\nu,\omega)}(\mathbb{L}) = \bigoplus_{\mu=1}^{m} L'_{(\nu,\omega)}(\rho_\mu + \mathbb{R}).$$

The Taylor and Mellin transformations

In this section we always assume that $\nu < \omega$. If $x \in \widetilde{B}(e^\omega; e^\nu)$ – the universal covering space of $B(e^\omega) \setminus B(e^\nu)$ – then the function $\mathbb{L} \ni \alpha \to x^\alpha$ belongs to $L_{(\nu,\omega)}(\mathbb{L})$. So we can define the *Taylor transform* of $S \in L'_{(\nu,\omega)}(\mathbb{L})$ by

$$\mathcal{T}S(x) = S[x^\cdot] \text{ for } x \in \widetilde{B}(e^\omega; e^\nu).$$

Define

$$\begin{aligned}\mathcal{O}^{\{0\}}(\widetilde{B}(e^\omega; e^\nu)) = \{u \in \mathcal{O}(\widetilde{B}(e^\omega; e^\nu)) :& \\ &\text{for any } e^\nu < r < t < e^\omega \text{ there exists } m \in \mathbb{N} \text{ such that} \\ &|u(x)| \le C(1 + |\ln x|)^m \text{ for } r \le |x| \le t\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{O}^{\{\operatorname{Im}\rho_1,\ldots,\operatorname{Im}\rho_m\}}(\widetilde{B}(e^\omega; e^\nu)) = \{u \in \mathcal{O}(\widetilde{B}(e^\omega; e^\nu)) &\text{ such that} \\ u = \sum_{\mu=1}^m x^{\rho_\mu} u_\mu &\text{ with some } u_\mu \in \mathcal{O}^{\{0\}}(\widetilde{B}(e^\omega; e^\nu))\}.\end{aligned}$$

We have

THEOREM 2 ([Ł2], Th. 2). *Let $\nu < \omega$. The Taylor transformation is an isomorphism of $L'_{(\nu,\omega)}(\mathbf{L})$ onto $\mathcal{O}^{\{\operatorname{Im}\rho_1,\ldots,\operatorname{Im}\rho_m\}}(\widetilde{B}(e^\omega; e^\nu))$.*

Fix $e^\nu < t < e^\omega$. Following [Ł2] we define the Mellin transform of $u \in \mathcal{O}^{\{\operatorname{Im}\rho_1,\ldots,\operatorname{Im}\rho_m\}}(\widetilde{B}(e^\omega; e^\nu))$ by

$$\mathcal{M}_t^{\pm} u(z) = \int_{\gamma^\pm(t)} u(x) x^{-z-1} dx \tag{13}$$

where $\gamma^\pm(t) = \{x \in \widetilde{B}(e^\omega; e^\nu) : \mp \arg x \ge 0, |x| = t\}$ and the orientation of $\pm\gamma^\pm(t)$ is positive.

$\mathcal{M}_t^+ u$ (resp. $\mathcal{M}_t^- u$) defined by (13) is holomorphic on $\{\operatorname{Im} z > \max(\operatorname{Im}\rho_1, \ldots, \operatorname{Im}\rho_m)\}$ (resp. on $\{\operatorname{Im} z < \min(\operatorname{Im}\rho_1, \ldots, \operatorname{Im}\rho_m)\}$). Since $u = \sum_{\mu=1}^m x^{\rho_\mu} u_\mu$ with $u_\mu \in \mathcal{O}^{\{0\}}(\widetilde{B}(e^\omega; e^\nu))$ we define

$$\mathcal{M}_t u(z) = \sum \mathcal{M}_t^+ u(z - \rho_\mu) + \sum \mathcal{M}_t^- u(z - \rho_\mu) \quad \text{for } \operatorname{Im} z \notin \{\operatorname{Im}\rho_1, \ldots, \operatorname{Im}\rho_m\}, \tag{13'}$$

where the sum of $\mathcal{M}_t^+ u$ (resp. $\mathcal{M}_t^- u$) is taken over $\mu \in \{1, \ldots, m\}$ with $\operatorname{Im}\rho_\mu < \operatorname{Im} z$ (resp. $\operatorname{Im}\rho_\mu > \operatorname{Im} z$). Theorem 4 of [Ł2] can be restated as follows

THEOREM 3. *Let $u \in \mathcal{O}^{\{\operatorname{Im}\rho_1,\ldots,\operatorname{Im}\rho_m\}}(\widetilde{B}(e^\omega; e^\nu))$ with $\nu < \omega$. Then for any $e^\nu < t < e^\omega$, $G := \mathcal{M}_t u \in \mathcal{O}(\mathbb{C} \setminus \mathbf{L})$ and there exists $N \in \mathbb{N}$ such that*

$$(14)\quad |G(z)| \le \begin{cases} C\dfrac{t^{-\operatorname{Re} z}}{|\operatorname{Im}(z-\rho_\mu)|)^N} & \text{for } \operatorname{Im} z \text{ close to } \operatorname{Im}\rho_\mu,\ \mu=1,\dots,m,\\ C\dfrac{t^{-\operatorname{Re} z}}{|\operatorname{Im} z|} & \text{for } |\operatorname{Im} z| \text{ big enough.}\end{cases}$$

Furthermore, the sum of differences of boundary values

$$(15)\qquad S := \sum_{\mu=1}^{m} \frac{1}{2\pi i}(b^+_{\rho_\mu+\mathbb{R}}G - b^-_{\rho_\mu+\mathbb{R}}G) \in L'_{(\nu,\omega)}(\mathbb{L})$$

is independent of t *and* $\mathcal{T}S = u$.

We can also adapt Theorem 5 of [Ł2] to the situation considered here.

THEOREM 4. *Let* $G \in \mathcal{O}(\mathbb{C}\setminus\mathbb{L})$ *satisfies* (14) *Assume that the sum of differences of boundary values given by* (15) *belongs to* $L'_{(\nu,\omega)}(\mathbb{L})$ *with some* $\nu < \ln t < \omega$. *Then* $G = \mathcal{M}_t u$ *where* $u := \mathcal{T}S \in \mathcal{O}^{\{\operatorname{Im}\rho_1,\dots,\operatorname{Im}\rho_m\}}(\widetilde{B}(e^\omega; e^\nu))$.

In the proof of Theorem 1 we shall need

COROLLARY 1. *Let* $G \in \mathcal{O}(\mathbb{C}\setminus\bigcup_{\mu=1}^m(\rho_\mu+\mathbb{Z}))$ *be such that with some* $q\in\mathbb{N}_0$

$$(16)\quad |G(z)| \le \begin{cases} C\dfrac{t^{-\operatorname{Re} z}}{\left(\operatorname{dist}(z,\rho_\mu+\mathbb{Z})\right)^{q+1}} & \text{for } z \text{ close to } \rho_\mu+\mathbb{Z},\ \mu=1,\dots,m,\\ C\dfrac{t^{-\operatorname{Re} z}}{|\operatorname{Im} z|} & \text{for } |\operatorname{Im} z| \text{ big enough.}\end{cases}$$

Then

$$S := \sum_{\mu=1}^{m} \frac{1}{2\pi i}(b^+_{\rho_\mu+\mathbb{R}}G - b^-_{\rho_\mu+\mathbb{R}}G) = \sum_{\mu=1}^{m}\sum_{p=0}^{q}\sum_{\gamma=-\infty}^{\infty} C_{\mu,p,\gamma}\delta^{(p)}_{(\rho_\mu+\gamma)} \in L'_{(\ln t,\ln t)}(\mathbb{L}).$$

Furthermore, if $C_\gamma = C_{\mu,p,\gamma}$ *satisfy* (9) *then*

1° $S \in L'_{(-\infty,\ln r)}(\mathbb{L})$ $(\mathbb{L} = \bigcup_{\mu=1}^m(\rho_\mu+\mathbb{R}))$,

2° $u = \mathcal{T}S$ *is a Laurent type series of order* κ *and radious of convergence* r,

3° $\mathcal{M}_t u = G$.

EXAMPLE 3. If u is given by a Laurent type series (9) then it is the Taylor transform of

$$S = \sum_{\mu=1}^{m}\sum_{p=0}^{q}\sum_{\gamma=-\infty}^{\infty} C_{\mu,p,\gamma}(-1)^p\delta^{(p)}_{(\rho_\mu+\gamma)} \in L'_{(-\infty,\ln r)}(\mathbb{L}).$$

So $u \in \mathcal{O}^{\{\operatorname{Im}\rho_1,\dots,\operatorname{Im}\rho_m\}}(\widetilde{B}(r;0))$. For any $0<t<r$ we have

$$\mathcal{M}_t u(z) = \sum_{\mu=1}^{m}\sum_{p=0}^{q}\sum_{\gamma=-\infty}^{\infty} C_{\mu,p,\gamma}\sum_{l=0}^{p}\frac{-p!}{(p-l)!}\frac{(\ln t)^{p-l}}{(z-\rho_\mu-\gamma)^{l+1}}t^{-z+\rho_\mu+\gamma}$$

for $z \notin \{\rho_1,\dots,\rho_m\}+\mathbb{Z}$.

In particular, if $t = 1$

(17) $$\mathcal{M}u(z) = \sum_{\mu=1}^{m}\sum_{p=0}^{q}\sum_{\gamma=-\infty}^{\infty} C_{\mu,p,\gamma}\frac{-p!}{(z-\rho_\mu-\gamma)^{p+1}} \text{ for } z \notin \{\rho_1,\ldots,\rho_m\}+\mathbb{Z}.$$

Auxiliary lemmas

In the proof of Cases 2° and 3° we shall also need

LEMMA 1. *Let* $k,l,\nu \in \mathbb{N}$ *and* $i \in \{1,\ldots,k\}^\nu$ *with* $i_1+\ldots+i_\nu = l$. *Then*

$$I(z) := \langle z+i_1\rangle^{i_1}\langle z+i_1+i_2\rangle^{i_2}\cdot\ldots\cdot\langle z+i_1+\ldots+i_\nu\rangle^{i_\nu} \geq \frac{l!}{(k-1)!(4k)^l}$$

for $z \in \mathbb{C}$, *where* $\langle z\rangle := 1+|z|$.

Proof. Since $|z| \geq |\operatorname{Re} z|$ we can assume that $z = x \in \mathbb{R}$. If $x \geq -1$ or $x \leq -l$ we immediately obtain $I(x) \geq l!$. Now let $-p-1 \leq x < -p$ with some $p \in \{1,\ldots,l-1\}$. Let $\mu \in \mathbb{N}_0$ be such that $i_1+\ldots+i_\mu \leq p$ and $i_1+\ldots+i_\mu+i_{\mu+1} \geq p+1$. Then we derive

$$\begin{aligned} I(x) &\geq (1+i_2+\ldots+i_\mu)^{i_1}\cdot\ldots\cdot(1+i_\mu)^{i_{\mu-1}}\cdot 1^{i_\mu} \\ &\quad\times 1^{i_{\mu+1}}(1+i_{\mu+2})^{i_{\mu+2}}\cdot\ldots\cdot(1+i_{\mu+2}+\ldots+i_\nu)^{i_\nu} \\ &\geq \frac{(i_1+\ldots+i_\mu)^{i_1}}{k^{i_1}}\cdot\ldots\cdot\frac{(i_{\mu-1}+i_\mu)^{i_{\mu-1}}}{k^{i_{\mu-1}}}\cdot\frac{i_\mu^{i_\mu}}{k^{i_\mu}} \\ &\quad\times\frac{i_{\mu+1}^{i_{\mu+1}}}{k^{i_{\mu+1}}}\cdot\frac{(i_{\mu+1}+i_{\mu+2})^{i_{\mu+2}}}{k^{i_{\mu+2}}}\cdot\ldots\cdot\frac{(i_{\mu+1}+\ldots+i_\nu)^{i_\nu}}{k^{i_\nu}} \\ &\geq \frac{(i_1+\ldots+i_\mu)!(i_{\mu+1}+\ldots+i_\nu)!}{k^l} \geq \frac{(p+1-k)!(l-p)!}{k^l} \\ &\geq \frac{(l+1-k)!}{2^{l+1-k}k^l} \geq \frac{l!}{(k-1)!(4k)^l}. \end{aligned}$$

In an analogous way we obtain

LEMMA 2. *Let* $\nu \in \mathbb{N}$ *and* $i_1,\ldots,i_\nu \geq 1$. *Then for* $z \in \mathbb{C}$

$$\langle z+i_1\rangle\langle z+i_1+i_2\rangle\cdot\ldots\cdot\langle z+i_1+\ldots+i_\nu\rangle \geq \nu!/2^\nu.$$

Proof of Cases 2° and 3°. Let us consider a Cauchy problem (7) where f is given by (12) and $0 < t < r$. Taking eventually x/t as an independent variable we can assume that $t = 1$ and $r > 1$. We shall solve (7) by the method of the Mellin transformation. We have (with $(z)_i$ defined by (5))

$$\mathcal{M}\big(a^0(x)u\big)(z) = \sum_{j=j_0}^{\infty} a_j^0 \mathcal{M}u(z-j),$$

$$\mathcal{M}\left(a^i(x)\frac{d^i}{dx^i}u\right)(z) = \sum_{j=j_0}^{\infty} a_j^i (z+i-j)_i \mathcal{M}u(z+i-j)$$

$$+\sum_{\nu=0}^{i-1}(-1)^\nu u_{i-1-\nu}\frac{\partial^\nu}{\partial t^\nu}\big(a^i(t)t^{-z-1}\big)_{|t=1} \qquad \text{for } i=1,\dots,n-1,$$

$$\mathcal{M}\left(x^n\frac{d^n}{dx^n}\right)(z) = (z)_n\mathcal{M}u(z) + \sum_{\nu=0}^{n-1}(-1)^\nu u_{n-1-\nu}\frac{\partial^\nu}{\partial t^\nu}\big(t^{n-z-1}\big)_{|t=1}.$$

Thus, applying the Mellin transformation to (7) we obtain

$$\sum_{i=0}^{n-1}\sum_{j=j_0}^{\infty} a_j^i(z+i-j)_i\mathcal{M}u(z+i-j) + (z)_n\mathcal{M}u(z) = H(z),$$

with

$$H(z) = \mathcal{M}f(z) + B(z),$$

where

$$B(z) = -\sum_{i=1}^{n}\sum_{\nu=0}^{i-1}(-1)^\nu u_{i-1-\nu}\frac{\partial^\nu}{\partial t^\nu}\big(a^i(t)t^{-z-1}\big)_{|t=1} \tag{18}$$

is a polynomial of degree $\le n-1$. So with $G(z) = \mathcal{M}u(z)$ and $W_j, j \ge k$, given by (4) and (6), we obtain a functional equation (8). To solve (8) we shall apply the method of successive approximations according to the scheme

$$\begin{cases} G_0(z) = \dfrac{H(z)}{W(z)}, \\ G_\nu(z) = \dfrac{-1}{W(z)}\displaystyle\sum_{\substack{j=-k\\ j\neq 0}}^{\infty} W_j(z-j)G_{\nu-1}(z-j) + \dfrac{H(z)}{W(z)} \text{ for } \nu\in\mathbb{N}. \end{cases}$$

Put $\widetilde{G}_0 = G_0$ and $\widetilde{G}_\nu = G_\nu - G_{\nu-1}$ for $\nu \in \mathbb{N}$.

By induction we derive for $v \in \mathbb{N}$

$$\widetilde{G}_\nu(z) = (-1)^\nu\sum_{l=0}^{\infty} A(\nu,-\nu k+l;z)H(z+\nu k-l), \tag{19}$$

where for $l\in\mathbb{N}_0$ and $\nu\in\mathbb{N}$

(20) $A(\nu,-\nu k+l;z) =$

$$\sum \frac{W_{i_1-k}(z+k-i_1)W_{i_2-k}(z+2k-i_1-i_2)\cdot\ldots\cdot W_{i_\nu-k}(z+\nu k-i_1-\ldots-i_\nu)}{W(z)W(z+k-i_1)W(z+2k-i_1-i_2)\cdot\ldots\cdot W(z+\nu k-i_1-\ldots-i_\nu)}$$

with the sum taken over $i \in \mathbb{N}_0^\nu$ with $i_1 + \ldots + i_\nu = l$ and $i_\mu \neq k$ for $\mu = 1, \ldots, \nu$.

We shall prove the convergence of the series $\sum_{\nu=0}^{\infty} \widetilde{G}_\nu$ and compute the boundary value of the limit $G := \sum_{\nu=0}^{\infty} \widetilde{G}_\nu$.

Case 2°. $\mathcal{N}_P$ lies above the characteristic level which is equivalent to the demand that $k = \kappa_P = 0$. Then (19) and (20) take the form

$$\widetilde{G}_\nu(z) = (-1)^\nu \sum_{l=\nu}^{\infty} A(\nu, l; z) H(z - l), \qquad \nu \in \mathbb{N} \tag{21}$$

where for $l \in \mathbb{N}$ and $\nu \in \mathbb{N}$

$$\begin{aligned} &A(\nu, l; z) \\ &= \sum_{\substack{i_1+\ldots+i_\nu=l \\ 1 \leq i_\mu, \mu=1,\ldots,\nu}} \frac{W_{i_1}(z - i_1) W_{i_2}(z - i_1 - i_2) \cdot \ldots \cdot W_{i_\nu}(z - i_1 - \ldots - i_\nu)}{W(z) W(z - i_1) W(z - i_1 - i_2) \cdot \ldots \cdot W(z - i_1 - \ldots - i_\nu)}. \end{aligned} \tag{22}$$

Denote by $\{\widetilde{\rho}_1, \ldots, \widetilde{\rho}_s\}$ the zeroes of P with multiplicity $\widetilde{q}_1, \ldots, \widetilde{q}_s$ respectively (so $\sum_{\sigma=1}^{s} \widetilde{q}_\sigma = n$). Devide $\{\widetilde{\rho}_1, \ldots, \widetilde{\rho}_s\}$ into sets $B_1, \ldots, B_m$ in such a way that $\widetilde{\rho}_\sigma$ and $\widetilde{\rho}_\tau$ belong to the same set for $\sigma, \tau \in \{1, \ldots, s\}$ if and only if $\widetilde{\rho}_\sigma - \widetilde{\rho}_\tau \in \mathbb{Z}$. Let $B_\mu = \{\rho_\mu := \rho_{\mu,1}, \ldots, \rho_{\mu,m_\mu}\}$ where $\rho_{\mu,\lambda} = \rho_{\mu,1} + \gamma_{\mu,\lambda}$ for $\lambda = 1, \ldots, m_\mu$ with $0 = \gamma_{\mu,1} < \gamma_{\mu,2} < \ldots < \gamma_{\mu,m_\mu}$ ($\mu = 1, \ldots, m$); $q_{\mu,\lambda} = \widetilde{q}_{\mu,1} + \ldots + \widetilde{q}_{\mu,\lambda}$ for $\mu \in \{1, \ldots, m\}, \lambda \in \{1, \ldots, m_\mu\}$; $q_\mu = \widetilde{q}_{\mu,m_\mu}$ ($\mu = 1, \ldots, m$). Then each summand of $A(\nu, l; z)$ has poles at most at $\rho_1 + \mathbb{N}_0, \ldots, \rho_m + \mathbb{N}_0$ and the order of a pole at $\rho_\mu + \gamma$ for $\gamma_{\mu,\lambda} \leq \gamma < \gamma_{\mu,\lambda+1}$ with $\lambda = 1, \ldots, m_\mu$ (here $\gamma_{\mu,m_\mu+1} = \infty$) is at most equal to $q_{\mu,\lambda}$ ($\mu = 1, \ldots, m$). Put

$$\delta = 1/2 \cdot \min\ \min\big(1, \operatorname{dist}(\rho_\sigma + \mathbb{N}_0, \rho_\mu + \mathbb{N}_0)\big) > 0 \text{ where } \rho_0 = \rho \text{ from (12)},$$

and the first minimum is taken over $0 \leq \sigma < \mu \leq m$ if $\operatorname{dist}(\rho_0 + \mathbb{Z}, \{\rho_1, \ldots, \rho_m\}) > 0$, and over $1 \leq \sigma < \mu \leq m$ otherwise.

By the convergence of the series (2) and the definition of $W_j, j \in \mathbb{N}$ (see (6)) we find that for any $\widetilde{r} < r$ there exists $M_1 = M_1(\widetilde{r}) < \infty$ such that

$$|W_j(z)| \leq \frac{M_1}{\widetilde{r}^j} \langle z \rangle^{n-1} \text{ for } z \in \mathbb{C},\ j \in \mathbb{N}.$$

We can also find $C < \infty$ such that for any $0 < \varepsilon < \delta$ and $\sigma \in \{1, \ldots, s\}$

$$\left| \frac{1}{W(z)} \right| \leq \frac{C}{\varepsilon^{\widetilde{q}_\sigma} \langle z \rangle^n}$$
$$\text{for } |z - \widetilde{\rho}_\sigma| \geq \varepsilon \text{ and } |z - \widetilde{\rho}_\tau| \geq \delta \text{ for } \tau \in \{1, \ldots, s\}, \quad \tau \neq \sigma.$$

Thus, for any $0 < \varepsilon < \delta$ the summands of $A(\nu, l; z)$ are estimated by (with $M_2 = CM_1$)

$$C\frac{M_2^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l\langle z\rangle^n\langle z-i_1\rangle\cdot\ldots\cdot\langle z-i_1-\ldots-i_\nu\rangle} \tag{23}$$

for $z \in \mathbb{C}$ such that $\operatorname{dist}(z, \rho_\mu + \mathbb{N}_0) \geq \varepsilon$ and $\operatorname{dist}(z, \rho_\lambda + \mathbb{N}_0) \geq \delta$ for $\lambda \in \{1,\ldots,m\}, \lambda \neq \mu$ $(\mu = 1,\ldots,m)$. Now by Lemma 2, (23) is bounded by

$$C\frac{(2M_2)^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l\nu!\langle z\rangle^n} \text{ for } z \in \mathbb{C} \text{ as above,}$$

and since there are $\binom{l-1}{\nu-1} \leq \frac{l^\nu}{\nu!}$ summands in $A(\nu, l; z)$ we get

$$|A(\nu, l; z)| \leq C\frac{(2M_2 l)^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l(\nu!)^2\langle z\rangle^n} \text{ for } z \in \mathbb{C} \text{ as in (23).} \tag{24}$$

Now we shall study the convergence of $\sum \widetilde{G}_\nu$. To this end observe that for f given by (12), $\widetilde{G}_\nu = \sum_{j\in\mathbb{Z}} C_j\widetilde{G}_{j,\nu}$ with $\widetilde{G}_{j,\nu}$ obtained by the approximation scheme for

$$\begin{cases} P\left(x, \frac{d}{dx}\right)u_j & = f_j \text{ for } j \in \mathbb{Z}; \\ u_j^{(i)}(1) & = \begin{cases} u_i & \text{for } j = 0,\ i = 0,1,\ldots,n-1, \\ 0 & \text{for } j \neq 0,\ i = 0,1,\ldots,n-1. \end{cases} \end{cases}$$

Put

$$H_j(z) = \begin{cases} B(z) - \dfrac{q!}{(z-\rho)^{q+1}} & \text{for } j = 0, \\ \dfrac{-q!}{(z-\rho-j)^{q+1}} & \text{for } j \neq 0 \end{cases}$$

with $B(z)$ given by (18). Since B is a polynomial of degree $\leq n-1$ we get for $l \in \mathbb{N}_0$

$$|H_j(z-l)| \leq \begin{cases} C\langle z\rangle^{n-1}\langle l\rangle^{n-1} + 1/|z-l-\rho|^{q+1} & \text{for } j = 0,\ z \neq \rho + l, \\ C/|z-l-\rho-j|^{q+1} & \text{for } j \neq 0,\ z \neq \rho + j + l. \end{cases}$$

Observe that for any $\widetilde{r} > 1$

$$\begin{aligned} \frac{1}{\nu!}\sum_{l=\nu}^{\infty}\frac{l^\nu\langle l\rangle^{n-1}}{\widetilde{r}^l} &\leq \frac{1}{\nu!}\sum_{l=0}^{\infty}\frac{(l+\nu+n-1)_{\nu+n-1}}{\widetilde{r}^l} \\ = \frac{1}{\nu!}\frac{(\nu+n-1)!}{(1-1/\widetilde{r})^{\nu+n}} &\leq (n-1)!\left(\frac{2}{1-1/\widetilde{r}}\right)^{\nu+n}. \end{aligned}$$

So in the case when $(\rho+\mathbb{Z})\cap\{\rho_1,\ldots,\rho_m\} = \emptyset$, $\widetilde{G}_{j,\nu}$ is a holomorphic function outside $(\{\rho_1,\ldots,\rho_m\}+\mathbb{N}_0)\cup(\rho+j+\mathbb{N}_0)$ satisfying with any $0<\varepsilon<\delta$

$$\begin{aligned}
|\widetilde{G}_{j,\nu}(z)| &\le \sum_{l=\nu}^{\infty}|A(\nu,l;z)|\cdot|H_j(z-l)| \\
&\le C\sum_{l=\nu}^{\infty}\frac{(2M_2l)^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l(\nu!)^2\langle z\rangle^n}\langle z\rangle^{n-1}\langle l\rangle^{n-1} \\
&\le C_1\frac{M_3^\nu}{\varepsilon^{q_\mu}\nu!\langle z\rangle}\quad\Big(C_1=C(n-1)!\Big(\frac{2}{1-1/\widetilde{r}}\Big)^n,\ M_3=\frac{4M_2}{1-1/\widetilde{r}}\Big)
\end{aligned}$$

for $z\in\mathbb{C}$ such that $\operatorname{dist}(z,\rho_\mu+\mathbb{N}_0)\ge\varepsilon$ and $\operatorname{dist}(z,\rho_\lambda+\mathbb{N}_0)\ge\delta$ for $\lambda\in\{0,\ldots,m\},\lambda\ne\mu,\mu=0,\ldots,m$ where $\rho_0=\rho+j,q_0=q+1$, Consequently $G_j=\sum_{\nu=0}^{\infty}\widetilde{G}_{j,\nu}$ is holomorphic outside $(\{\rho_1,\ldots,\rho_m\}+\mathbb{N}_0)\cup\{\rho+j+\mathbb{N}_0\}$ and satisfies with any $0<\varepsilon<\delta$

$$|G_j(z)|\le\frac{C}{\varepsilon^{q_\mu}\langle z\rangle}\text{ for }z\in\mathbb{C}\text{ as above.}$$

Finally $G=\sum_{j\in\mathbb{Z}}C_jG_j$ is holomorphic outside $(\{\rho_1,\ldots,\rho_m\}+\mathbb{N}_0)\cup\{\rho+\mathbb{Z}\}$ and satisfies with any $0<\varepsilon<\delta$

$$\begin{gathered}
|G(z)|\le\frac{C}{\varepsilon^{q_\mu}\langle z\rangle}\text{ for }z\in\mathbb{C}\text{ such that} \qquad (25)\\
\operatorname{dist}(z,\rho+\mathbb{Z})\ge\varepsilon\text{ and }\operatorname{dist}(z,\{\rho_1,\ldots,\rho_m\}+\mathbb{N}_0)\ge\delta,\text{ if }\mu=0,\\
\operatorname{dist}(z,\rho_\mu+\mathbb{N}_0)\ge\varepsilon\text{ and }\operatorname{dist}\Big(z,(\rho+\mathbb{Z})\cup\bigcup_{0<\lambda\ne\mu\le m}(\rho_\lambda+\mathbb{N}_0)\Big)\ge\delta,\\
\text{if }\mu\in\{1,\ldots,m\}.
\end{gathered}$$

This implies that

$$b(G)=\sum_{p=0}^{q}\sum_{\gamma=-\infty}^{\infty}C_{0,p,\gamma}\delta^{(p)}_{(\rho+\gamma)}+\sum_{\mu=1}^{m}\sum_{p=0}^{q_\mu-1}\sum_{\gamma=0}^{\infty}C_{\mu,p,\gamma}\delta^{(p)}_{(\rho_\mu+\gamma)}.$$

Now we shall estimate $C_{\mu,p,\gamma}$. Let $\mu=0$ and fix $\gamma\in\mathbb{Z}$. Since $A(\nu,l;z)$ is holomorphic at $\rho+\gamma$ we have

$$\begin{aligned}
b(\widetilde{G}_{j,\nu})_{|z=\rho+\gamma} &= (-1)^{\nu+1}q!\,b\Big(\frac{A(\nu,\gamma-j;z)}{(z-\rho-\gamma)^{q+1}}\Big)_{|z=\rho+\gamma} \\
&= \begin{cases}\sum_{p=0}^{q}C^{j,\nu}_{0,p,\gamma}\delta^{(p)}_{(\rho+\gamma)} & \text{for }\nu\le\gamma-j,\\ 0 & \text{otherwise}\end{cases}
\end{aligned}$$

with $C^{j,\nu}_{0,p,\gamma}=2\pi i(-1)^{p+\nu}\binom{q}{p}A^{(q-p)}(\nu,\gamma-j;\rho+\gamma)$. Note here that by (24),

$$|A(\nu,l;z)|\le C\frac{(2M_2l)^\nu}{\widetilde{r}^l(\nu!)^2}\text{ for }\operatorname{dist}(z,\rho+\mathbb{Z})\le\delta.$$

Thus, by the Cauchy inequality and the duplication formula for factorials, we estimate $C^j_{0,p,\gamma} = \sum_{\nu=0}^{\infty} C^{j,\nu}_{0,p,\gamma}$,

$$|C^j_{0,p,\gamma}| \le C \sum_{\nu=0}^{\infty} \frac{(2M_2)^\nu(\gamma-j)^\nu}{\widetilde{r}^{\gamma-j}(\nu!)^2} \le \frac{C\exp\{\sqrt{8M_2(\gamma-j)}\}}{\widetilde{r}^{\gamma-j}} \text{ for } \gamma \ge j,$$

and $C^j_{0,p,\gamma} = 0$ for $\gamma < j$.

Finally, by (9) we derive for any $\widetilde{\widetilde{r}} < \widetilde{r}$

$$|C_{0,p,\gamma}| \le \sum_{j\le\gamma} |C^j_{0,p,\gamma}||C_j| \le \begin{cases} C/\widetilde{\widetilde{r}}^{\gamma} & \text{for } \gamma \ge 0, \\ C\dfrac{M^{-\gamma}}{(-\gamma)!^{1/\kappa}} & \text{for } \gamma < 0 \end{cases}$$

with some $M < \infty$.

Let now $\mu \in \{1, \dots, m\}$. Then

$$b(\widetilde{G}_\nu)_{|z=\rho_\mu+\gamma} = \begin{cases} \sum\limits_{p=0}^{q_\mu-1} C^\nu_{\mu,p,\gamma}\delta^{(p)}_{(\rho_\mu+\gamma)} & \text{for } \gamma \ge 0, \\ 0 & \text{otherwise} \end{cases}$$

with

$$C^\nu_{\mu,p,\gamma} = \frac{2\pi i}{(q_\mu-1)!} \times \sum_{p=0}^{q_\mu-1} (-1)^{p+\nu+1} \binom{q_\mu-1}{p} \frac{d^{q_\mu-1-p}}{dz^{q_\mu-1-p}}[(z-\rho-\gamma)^{q_\mu}\widetilde{G}_\nu(z)]_{|z=\rho_\mu+\gamma}.$$

Note that $H(z-l)$ is regular at $\rho_\mu+\mathbb{N}_0$ and $A(\nu,l;z)$ is regular at $z=\rho_\mu+\gamma$ if $l<\gamma$. Since

$$|A(\nu,l;z)| \le C\frac{(2M_2l)^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l(\nu!)^2\langle z\rangle^n} \text{ for } \varepsilon \le \operatorname{dist}(z,\rho_\mu+\mathbb{N}_0) \le \delta$$

and

$$|H(z-l)| \le C\langle z\rangle^{n-1}\langle l\rangle^{n-1} \text{ for } \operatorname{dist}(z,\rho_\mu+\mathbb{Z}) \le \delta$$

by the Cauchy formula we derive

$$|C^\nu_{\mu,p,\gamma}| \le C\sum_{l=\gamma}^{\infty} \frac{(2M_2l)^\nu\cdot\langle l\rangle^{n-1}}{(\nu!)^2\widetilde{r}^l}$$

$$\le C\frac{M_3^\nu}{(\nu!)^2\widetilde{r}^\gamma}\sum_{j=0}^{\infty}\frac{(\gamma+j)^\nu\langle\gamma+j\rangle^{n-1}}{\widetilde{r}^j} \le C\frac{L^\nu}{\nu!(\widetilde{r}^*)^\gamma}$$

with some $L<\infty$ and $\widetilde{r}^* := \widetilde{r}\exp\{1/\widetilde{r}-1\}$. Finally

$$|C_{\mu,p,\gamma}| \le C/(\widetilde{r}^*)^\gamma \text{ for } \gamma \in \mathbb{N}_0.$$

Now let $\operatorname{dist}(\rho+\mathbb{Z},\{\rho_1,\dots,\rho_m\}) = 0$. We can assume that $\rho=\rho_1$. Then at points of $\rho+\mathbb{Z}$ the poles of $A(\nu,l;z)$ glue with that of $H(z-l)$. Thus G

is holomorphic outside $(\{\rho_2,\ldots,\rho_m\}+\mathbb{N}_0)\cup(\rho_1+\mathbb{Z})$ satisfying (25) with $\rho_0=\rho_1=\rho$ and q_0,q_1 replaced by q_0+q_1. this implies that

$$b(G)=\sum_{p=0}^{q_1+q}\sum_{\gamma=-\infty}^{\infty}C_{1,p,\gamma}\delta^{(p)}_{(\rho_1+\gamma)}+\sum_{\mu=2}^{m}\sum_{p=0}^{q_\mu-1}\sum_{\gamma=0}^{\infty}C_{\mu,p,\gamma}\delta^{(p)}_{(\rho_\mu+\gamma)}.$$

As in the previous case we get

$$|C_{\mu,p,\gamma}|\le\begin{cases}C/(\widetilde{r}^*)^\gamma & \text{for } \mu\in\{1,\ldots,m\}\ \gamma\in\mathbb{N}_0,\\ C\dfrac{M^{-\gamma}}{(-\gamma)!^{1/\kappa}} & \text{for } \mu=1,\ \gamma\in-\mathbb{N}_0.\end{cases}$$

REMARK 1. The estimation of $C_{\mu,p,\gamma}$ implies that the radious of convergence of the solution u is $\ge r^*:=r\exp\{1/r-1\}$. Iterating the procedure we conclude that in fact it is at least equal to r.

Case 3°. $\mathcal{N}_P$ lies below the characteristic level (and $k\ge1$, so $\kappa_P>0$).

The proof of Case 3° is similar to that of Case 2°. So we only give the main points, leaving detailes to the reader. Since $W_j=0$ for $j\ge1$ and for $j<-k$ we have

$$\widetilde{G}_\nu(z)=(-1)^\nu\sum_{l=\nu}^{\nu k}A(\nu,-l;z)H(z+l),\ \nu\in\mathbb{N}\tag{26}$$

where for $l\in\mathbb{N}$ and $\nu\in\mathbb{N}$

$$A(\nu,-l;z)\tag{27}$$
$$=\sum_{\substack{i_1+\ldots+i_\nu=-l\\ -k\le i_\mu\le-1,\mu=1,\ldots,\nu}}\frac{W_{i_1}(z-i_1)W_{i_2}(z-i_1-i_2)\cdot\ldots\cdot W_{i_\nu}(z-i_1-\ldots-i_\nu)}{W(z)W(z-i_1)W(z-i_1-i_2)\cdot\ldots\cdot W(z-i_1-\ldots-i_\nu)}.$$

As in the Case 2° we devide the zeroes $\{\widetilde{\rho}_1,\ldots,\widetilde{\rho}_s\}$ of P into sets $B_1,\ldots,B_m$, but this time it is more convenient to take as $\rho_\mu(\mu=1,\ldots,m)$ the element of B_μ with the biggest real part. Since only a finite number of W_i is different from zero for any $\widetilde{r}<\infty$ we can find $M=M(\widetilde{r})<\infty$ and $C<\infty$ such that for any $0<\varepsilon<\delta$ and $\sigma\in\{1,\ldots,s\}$

$$\left|\frac{W_i(z)}{W(z)}\right|\le C\frac{M\widetilde{r}^i\langle z\rangle^{i/\kappa_P}}{\varepsilon^{\widetilde{q}_\sigma}}$$

for $z\in\mathbb{C}$, with $|z-\widetilde{\rho}_\sigma|\ge\varepsilon$ and $|z-\widetilde{\rho}_\tau|\ge\delta$ for $\tau\in\{1,\ldots,s\}$, $\tau\ne\sigma$, where κ_P is the Katz invariant of P. So we can also find $C<\infty$ such that for any $0<\varepsilon<\delta$ the summands of $A(\nu,-l;z)$ are estimated by (with $M_1=CM$)

$$C\frac{M_1^\nu}{\varepsilon^{q_\mu}\widetilde{r}^l\langle z\rangle^n\langle z-i_1\rangle^{-i_1/\kappa_P}\cdot\ldots\cdot\langle z-i_1-\ldots-i_\nu\rangle^{-i_\nu/\kappa_P}}$$

for $z\in\mathbb{C}$ such that $\operatorname{dist}(z,\rho_\mu-\mathbb{N}_0)\ge\varepsilon$ and $\operatorname{dist}(z,\rho_\lambda-\mathbb{N}_0)\ge\delta$ for $\lambda\in\{1,\ldots,m\},\lambda\ne\mu(\mu=1,\ldots,m)$. Consequently, since there are at most

$l^\nu/\nu!$ summands, by Lemma 1 we obtain for $\nu \le l \le \nu k$

$$|A(\nu, -l; z)| \le C \frac{M_2^\nu}{\varepsilon^{q_\mu} \widetilde{r}^l (l!)^{1/\kappa_P} \langle z \rangle^n} \tag{28}$$

for $z \in \mathbb{C}$ as above ($M_2 = M_1 k e (4k)^{k/\kappa_P}$).

Assume that $(\rho + \mathbb{Z}) \cap \{\rho_1, \dots, \rho_m\} = \emptyset$. Then as in the Case 2° we find that G is holomorphic outside $(\{\rho_1, \dots, \rho_m\} - \mathbb{N}_0) \cup (\rho + \mathbb{Z})$ and there exists C such that for any $0 < \varepsilon < \delta$, (25) holds for $z \in \mathbb{C}$ such that

$$\operatorname{dist}(z, \rho + \mathbb{Z}) \ge \varepsilon \text{ and } \operatorname{dist}(z, \{\rho_1, \dots, \rho_m\} - \mathbb{N}_0) \ge \delta, \quad \text{if } \mu = 0,$$
$$\operatorname{dist}(z, \rho_\mu - \mathbb{N}_0) \ge \varepsilon \text{ and } \operatorname{dist}\Big(z, (\rho + \mathbb{Z}) \cup \bigcup_{0 < \lambda \ne \mu \le m} (\rho_\lambda - \mathbb{N}_0)\Big) \ge \delta,$$
$$\text{if } \mu \in \{1, \dots, m\}.$$

So

$$b(G) = \sum_{p=0}^{q} \sum_{\gamma=-\infty}^{\infty} C_{0,p,\gamma} \delta^{(p)}_{(\rho+\gamma)} + \sum_{\mu=1}^{m} \sum_{p=0}^{q_\mu - 1} \sum_{\gamma=-\infty}^{0} C_{\mu,p,\gamma} \delta^{(p)}_{(\rho_\mu + \gamma)}.$$

This time however we need a more refined estimation of $C_{\mu,p,\gamma}$. Let $\mu = 0$. Then $C_{0,p,\gamma} = \sum_{j \ge \gamma} C^j_{0,p,\gamma} C_j$ with C_j satisfying (9) and $C^j_{0,p,\gamma} = \sum_{\nu=0}^{\infty} C^{j,\nu}_{0,p,\gamma}$, where

$$C^{j,\nu}_{0,p,\gamma} = \begin{cases} 2\pi i (-1)^{p+\nu} \binom{q}{p} A^{(q-p)}(\nu, j-\gamma); \rho + \gamma) & \text{if } \nu \le j - \gamma \le \nu k, \\ 0 & \text{otherwise.} \end{cases}$$

Since by (28)

$$|A(\nu, -l; z)| \le C \frac{M_2^\nu}{\widetilde{r}^l (l!)^{1/\kappa_P}} \text{ for } \operatorname{dist}(z, \rho + \mathbb{Z}) \le \delta$$

by the Cauchy inequality we derive

$$|C^j_{0,p,\gamma}| \le \begin{cases} C \sum_{\nu=0}^{\infty} \frac{M_2^\nu}{\widetilde{r}^{j-\gamma} (\nu!)^{1/\kappa_P}} \le \frac{C}{\widetilde{r}^{j-\gamma}} & \text{for } j \ge \gamma \ge 0, \\ C \sum_{\substack{j-\gamma \ge \nu \\ \ge (j-\gamma)/k}} \frac{M_2^\nu}{\widetilde{r}^{j-\gamma} ((j-\gamma)!)^{1/\kappa_P}} \le C \frac{M_3^{j-\gamma}}{\widetilde{r}^{j-\gamma} \big((j-\gamma)!\big)^{1/\kappa_P}} & \\ & \text{for } j \ge \gamma, \gamma < 0, \\ 0 & \text{for } j < \gamma. \end{cases}$$

and

$$|C_{0,p,\gamma}| \le \begin{cases} \sum_{j \ge \gamma} \frac{C}{\widetilde{r}^{j-\gamma}} \cdot \frac{C}{\widetilde{r}^j} \le C/\widetilde{r}^\gamma & \text{for } \gamma \ge 0, \\ C \frac{M^{-\gamma}}{((-\gamma)!)^\beta} \text{ with } \beta = \min\big(1/\kappa_P, 1/\kappa\big),\ M < \infty & \text{for } \gamma < 0. \end{cases}$$

Similarily for $\mu \in \{1, \ldots, m\}$ we obtain

$$|C_{\mu,p,\gamma}| \leq C \frac{M^{-\gamma}}{((-\gamma)!)^{1/\kappa_P}} \text{ for } \gamma \leq 0.$$

In the case when $\operatorname{dist}(\rho + \mathbb{Z}, \{\rho_1, \ldots, \rho_m\}) = 0$ assuming that $\rho = \rho_1$ we obtain

$$b(G) = \sum_{p=0}^{q_1+q} \sum_{\gamma=-\infty}^{\infty} C_{1,p,\gamma} \delta^{(p)}_{(\rho_1+\gamma)} + \sum_{\mu=2}^{m} \sum_{p=0}^{q_i-1} \sum_{\gamma=-\infty}^{0} C_{\mu,p,\gamma} \delta^{(p)}_{(\rho_i+\gamma)}$$

where

$$|C_{\mu,p,\gamma}| \leq \begin{cases} C/\widetilde{r}^{\gamma} & \text{for } \mu = 1,\ \gamma \in \mathbb{N}_0, \\ C \dfrac{M^{-\gamma}}{((-\gamma)!)^{\beta}} \text{ with } \beta = \min\left(1/\kappa_P, 1/\kappa\right) & \text{for } \mu = 1,\ \gamma \in -\mathbb{N}, \\ C \dfrac{M^{-\gamma}}{((-\gamma)!)^{1/\kappa_P}} & \text{for } \mu \in \{2, \ldots, m\}, \gamma \in -\mathbb{N}_0. \end{cases}$$

Final remarks

EXAMPLE 4. Consider the problem

$$P\left(x, \frac{d}{dx}\right) u = x^2 \frac{d^2 u}{dx^2} - \left(x + \frac{1}{x}\right) u = 0,\ u(1) = 0,\ u'(1) = -1.$$

Then $k = 1, \kappa_P = 1/2$, $W(z) = z(z-1), W_{-1} = W_1 = -1$. Since $\kappa_P \notin \mathbb{N}_0$, P is not a composition of operators of order 1. So P does not satisfy assumptions of Theorem 1. Taking the Mellin transform we arrive at the equation

$$W(z)G(z) - G(z+1) - G(z-1) = 1.$$

The approximations $\widetilde{G}_\nu$ are given by $\widetilde{G}_\nu(z) = \sum_{l=0}^{\nu} A(\nu, -\nu + 2l; z)$ with

$$A(\nu, -\nu + 2l; z) = \sum_{\substack{i_1 + \ldots + i_\nu = l \\ i_\mu \in \{0,1\}, \mu = 1, \ldots, \nu}} \frac{1}{W(z)W(z+1-2i_1) \cdot \ldots \cdot W(z+\nu-2i_1-\ldots-2i_\nu)},$$

$l \in \{0, 1, \ldots, \nu\}$. Since $\widetilde{G}_\nu$ has a pole at zero of order $1 + \nu$ (for $l = \lfloor \nu/2 \rfloor$) the sum $\sum \widetilde{G}_\nu$ does not converge to any function $G \in \mathcal{O}(\mathbb{C} \setminus \mathbb{Z})$ having polynomial growth near $\mathbb{Z}$.

REMARK 2. The method of the Mellin transformation can be applied to the Case 1° of Theorem 1. In this case to prove the convergence of the approximation scheme for the functional equation (8) one requires sophisticated combinatorial tricks to show that $A(\nu, -\nu k + l; z)$ have simple poles at integers (if $a_0 = 0$), however its summands can have poles of arbitrary order.

REMARK 3. The method of the Mellin transformation can be also applied to the equation $Pu = f$, where P satisfies assumptions of Theorem 1 and $f \in \mathcal{O}^{\{\tau_1,\ldots,\tau_\mu\}}(\widetilde{B}(r;0))$ with some $\tau_1,\ldots,\tau_\mu \in \mathbb{R}$. In that case any special solution of $Pu = f$ belongs to $\mathcal{O}^{\{\tau_1,\ldots,\tau_\mu,\operatorname{Im}\rho_1,\ldots\operatorname{Im}\rho_m\}}(\widetilde{B}(r;0))$ where $\rho_1,\ldots,\rho_m$ are the roots of the characteristic polynomial for P.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Śniadeckich 8
00-950 WARSAW, POLAND
E-mail: lysik@impan.gov.pl

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