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ON SOME FURTHER WIRTINGER-BEESACK INTEGRAL INEQUALITIES

1. Introduction

In the previous paper [5] the uniform method of obtaining and investigating various types of integral inequalities involving a function and its first derivative (see [4],[2], [7] and [3]) has been extended to the integral inequalities involving a function and its second derivative. Some quadratic integral inequalities of the second order of the form

$$\int_I sh^2 dt \leq \int_I rh''^2 dt, \quad h \in H, \tag{1}$$

where $I = (\alpha, \beta), -\infty \leq \alpha < \beta \leq \infty$, r and s are real functions of the variable t, H is a class of functions absolutely continuous on I has been derived.

In this paper we derive some new integral inequalities of the form (1). The method we use consists in determining the function s and the auxiliary functions w_0, w_1 and w_2 depending on the given function r and the auxiliary function φ and next using these functions to determine the class H of functions h for which the inequality (1) holds. The class of functions h for which the inequality (1) holds determined in this paper doesn't cover with the class of functions h obtained in the paper [5].

We also derive some new integral inequalities of the form (1) with the Chebyshev weight functions $r = (1-t^2)^a$ and $s = \gamma(1-t^2)^b$ on $I = (-1, 1)$, where a, b, $\gamma > 0$ are certain constants.

The integral inequalities of the form (1) have been also obtained before, but another methods were applied (for references see e.g. [5], [6], [8]). In the particular case when the interval I is compact and the weight function $r = 1$ on I these inequalities were thoroughly studied by Beesack in the

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classic paper [1] and therefore we called the inequalities of the form (1) the Wirtinger-Beesack inequalities.

2. Main results

Let $I = (\alpha, \beta), -\infty \leq \alpha < \beta \leq \infty$, be an arbitrary open interval. We denote by $AC(I)$ the class of real functions defined and absolutely continuous on the interval I, and by $AC^1(I)$ the class of functions $f \in AC(I)$ such that $f' \in AC(I)$. Let $r \in AC(I)$ and $\varphi \in AC^1(I)$ be given functions such that $r > 0$, $\varphi > 0$ on the interval I and $r\varphi'' \in AC^1(I)$. Let us put

$$s = (r\varphi'')''\varphi^{-1}. \tag{2}$$

Let us denote by H the class of functions $h \in AC^1(I)$ satisfying the following integral conditions

$$\int_I rh''^2 dt < \infty, \qquad \int_I sh^2 dt > -\infty \tag{3}$$

and the limit conditions

$$\begin{aligned} &\liminf_{t\to\alpha}(w_0h^2 + 2w_1hh' + w_2h'^2) < \infty, \\ &\limsup_{t\to\beta}(w_0h^2 + 2w_1hh' + w_2h'^2) > -\infty \end{aligned} \tag{4}$$

and

$$\liminf_{t\to\alpha}(w_0h^2 + 2w_1hh' + w_2h'^2) \leq \limsup_{t\to\beta}(w_0h^2 + 2w_1hh' + w_2h'^2), \tag{5}$$

where

$$w_0 = r(\varphi^{-1}\varphi')^3 + r\varphi''(\varphi^{-1})' - (r\varphi'')'\varphi^{-1}, \tag{6}$$

$$w_1 = r(\varphi^{-1}\varphi')', \tag{7}$$

$$w_2 = r\varphi^{-1}\varphi'. \tag{8}$$

Assumptions apply that $w_0 \in AC(I)$ and $w_1, w_2 \in AC^1(I)$.

The following theorem is the main result of this paper:

THEOREM 1. *Let*

$$w = (r\varphi')'\varphi + 2r\varphi\varphi'' - 2r\varphi'^2 \leq 0 \tag{9}$$

almost everywhere on the interval I.

Then for every function $h \in H$ the inequality

$$\int_I sh^2 dt \leq \int_I rh''^2 dt \tag{10}$$

holds.

If $w \not\equiv 0$ and $h \not\equiv 0$ then the inequality (10) *becomes an equality if and only if $h = c\varphi$ with $c =$ const $\neq 0$ and the additional conditions*

$$\varphi \in H, \tag{11}$$
$$\lim_{t\to\alpha}(w_0\varphi^2 + 2w_1\varphi\varphi' + w_2\varphi'^2) = \lim_{t\to\beta}(w_0\varphi^2 + 2w_1\varphi\varphi' + w_2\varphi'^2)$$

are satisfied.

Proof. This proof is a modification of the proof of Theorem 1 in [5]. Let $h \in AC^1(I)$. By virtue of (6), (7), (8) and from assumptions we have $w_0h^2 + 2w_1hh' + w_2h'^2 \in AC(I)$ and $h\varphi^{-1} \in AC^1(I)$. It has been shown in [5] that the identity

$$rh''^2 - sh^2 = (v_0h^2 + 2v_1hh')' + g, \tag{12}$$

where $v_0 = r\varphi''(\varphi^{-1})' - (r\varphi'')'\varphi^{-1}$, $v_1 = r\varphi''\varphi^{-1}$ and

$$g = r[\varphi(\varphi^{-1}h)'' + 2\varphi'(\varphi^{-1}h)']^2 - 2r\varphi\varphi''[(\varphi^{-1}h)']^2 \tag{13}$$

is valid almost everywhere on I.

Now we transform the right side of the identity (12) and, after simple calculations, obtain the identity

$$rh''^2 - sh^2 = (w_0h^2 + 2w_1hh' + w_2h'^2)' + g_1 \tag{14}$$

where

$$g_1 = r[\varphi(\varphi^{-1}h)'' + \varphi'(\varphi^{-1}h)']^2 - w[(\varphi^{-1}h)']^2 \tag{15}$$

which is valid a.e. on I. We note that according to the assumptions $g_1 \geq 0$ a.e on I.

Now let $h \in H$. The first condition of (3) implies that the function rh''^2 is summable on I since $rh''^2 \geq 0$ on I. It follows from assumptions that the functions sh^2 and $w_0h^2 + 2w_1hh' + w_2h'^2$ are summable on each compact interval $[a,b] \subset I$. Thus by (14) we get the summability of the function g_1 on $[a,b] \subset I$ and we obtain the equality

$$\int_a^b rh''^2dt = \int_a^b sh^2dt + (w_0h^2 + 2w_1hh' + w_2h'^2)|_a^b + \int_a^b g_1dt \tag{16}$$

for arbitrary $\alpha < a < b < \beta$. In view of (4) there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty}(w_0h^2 + 2w_1hh' + w_2h'^2)|_{a_n} < \infty\ ,$$
$$\lim_{n\to\infty}[-(w_0h^2 + 2w_1hh' + w_2h'^2)]|_{b_n} < \infty.$$

Thus there is a constant C such that

$$-(w_0h^2 + 2w_1hh' + w_2h'^2)|_{a_n}^{b_n} \leq C < \infty.$$

By virtue of the condition $g_1 \geq 0$ a.e. on I and from the equality (16) we infer that

$$\int_{a_n}^{b_n} sh^2 dt \leq \int_{a_n}^{b_n} rh''^2 dt + C \leq \int_I rh''^2 dt + C$$

and from this letting $n \to \infty$ we obtain

$$\int_I sh^2 dt \leq \int_I rh''^2 dt + C < \infty.$$

By this estimate and by the second condition of (3) we conclude that sh^2 is summable on I. Next in a similar way using (16) and the summability of the function sh^2 on I we prove that the function g_1 is summable on I. Thus all the integrals in the equality (16) have finite limits as $a \to \alpha$ or $b \to \beta$, and hence both of the limits in (4) are proper and finite. Therefore the conditions (4) and (5) may be written in the equivalent form

$$(17) \quad -\infty < \lim_{t\to\alpha}(w_0h^2 + 2w_1hh' + w_2h'^2) \leq \lim_{t\to\beta}(w_0h^2 + 2w_1hh' + w_2h'^2) < \infty.$$

Now by (16) as $a \to \alpha$ and $b \to \beta$ we obtain the equality

$$(18) \quad \int_I rh''^2 dt - \int_I sh^2 dt = \lim_{t\to\beta}(w_0h^2 + 2w_1hh' + w_2h'^2) - \lim_{t\to\alpha}(w_0h^2 + 2w_1hh' + w_2h'^2) + \int_I g dt,$$

whence, in view of (), the inequality (10) follows since $g_1 \geq 0$ a.e. on I.

If the inequality (10) becomes an equality for a non-vanishing function $h \in H$, then by () and (18) we have

$$(19) \quad \lim_{t\to\alpha}(w_0h^2 + 2w_1hh' + w_2h'^2) = \lim_{t\to\beta}(w_0h^2 + 2w_1hh' + w_2h'^2),$$

$$\int_I g_1 dt = 0.$$

As $g_1 \geq 0$ a.e. on I we obtain $g_1 = 0$ a.e. on I. Hence

$$(20) \quad \varphi(\varphi^{-1}h)'' + \varphi'(\varphi^{-1}h)' = 0$$

a.e. on I and $w[(\varphi f^2)']^2 = 0$ a.e. on I. According to the assumptions $w \not\equiv 0$ and whence there exists $t_0 \in I$ such that $(\varphi^{-1}h)'(t_0) = 0$. The function $(\varphi^{-1}h)' \in AC(I)$ satisfies the homogeneous linear differential equation (20) with the initial value $(\varphi^{-1}h)'(t_0) = 0$, and whence $(\varphi^{-1}h)' = 0$ on I. This implies that $h = c\varphi$, where $c = \text{const} \neq 0$, since $\varphi^{-1}h \in AC^1(I)$. Thus $\varphi \in H$ so that we obtain the first condition of (11) and from the second condition of (19) we get the second condition of (11).

Now let (11) be satisfied and let $h = c\varphi$, where $c = \text{const} \neq 0$. That implies $g_1 = 0$ a.e. on I so that $\int_I g_1 dt = 0$ and, in view of (18), the inequality (10) becomes an equality which completes the proof. ■

REMARK 1. In case of $r = 1$ on I we have $w = 3\varphi\varphi'' - 2\varphi'^2$. Moreover if $-\infty < \alpha < \beta < \infty$, $\varphi(\alpha) = \varphi'(\alpha) = \varphi(\beta) = \varphi'(\beta) = 0$ and there exist finite limit values $\varphi''(\alpha)$ and $\varphi''(\beta)$ and $s > 0$ on I, then it follows from Lemma 2.3 in [1], that $w \leq 0$ on I. Therefore under these assumptions Theorem 1 is valid. So that Theorem 2.2 in [1] is a particular case of Theorem 1.

EXAMPLE. Let us take $I = (-1,1)$ and the functions $r = (1-t^2)^a$ and $\varphi = (1-t^2)^{2-a}$ on I, where a is an arbitrary constant such that $a < 1$.Then we have

$$s = 4(a-2)(2a-3)(1-t^2)^{a-2} > 0$$

and

$$w = 2(a-2)(3-t^2)(1-t^2)^{2-a} < 0$$

on I.

From Theorem 1 we obtain that the inequality holds for every function $h \in H$, where H is the class of functions $h \in AC^1((-1,1))$ satisfying the integral condition

$$\int_{-1}^{1} (1-t^2)^a h''^2 dt < \infty \tag{21}$$

and the limit condition

$$-\infty < \lim_{t\to -1} S(t,h,h') \leq \lim_{t\to 1} S(t,h,h') < \infty, \tag{22}$$

where

$$\begin{aligned} S(t,h,h') = {} & \omega_0(t)(1-t^2)^{a-3} h^2 + \omega_1(t)(1-t^2)^{a-2} hh' \\ & + \omega_2(t)(1-t^2)^{a-1} h'^2 \end{aligned} \tag{23}$$

and

$$\begin{aligned} \omega_0(t) &= 4(a-2)\, t\,[(5-3a) + (a-1)t^2], \\ \omega_1(t) &= 4(a-2)(1+t^2), \\ \omega_2(t) &= 2(a-2)t, \end{aligned}$$

since the second condition of (3) is satisfied trivially.

Now we show that a function $h \in AC^1((-1,1))$ satisfying the integral condition (21) and the limit conditions

$$h(-1) = h'(-1) = h(1) = h'(1) = 0 \tag{24}$$

belongs to the class H.

At first we show that if $h(1) = h'(1) = 0$ and (21) holds, then the limit condition

$$\lim_{t\to 1} S(t,h,h') = 0$$

is satisfied. Let us consider the left-hand neighbourhood U of the point 1. Using the Schwarz inequality we obtain the estimate

$$\begin{aligned} |h'(t)|^2 &= \Big(\int_t^1 h'' d\tau\Big)^2 \le \int_t^1 (1-\tau^2)^{-a}\, d\tau \int_t^1 (1-\tau^2)^a\, h''^2 d\tau \\ &\le A\int_t^1 (1-\tau)^{-a}\, d\tau \int_t^1 (1-\tau^2)^a\, h''^2 d\tau \\ &= \frac{A}{1-a}(1-t)^{1-a}\int_t^1 (1-\tau^2)^a\, h''^2 d\tau \end{aligned}$$

for $t \in U$, where $A > 0$ is a constant such that $(1+t)^{-a} \le A$ on U. Whence we have

$$|h'(t)| \le k(t)(1-t)^{\frac{1-a}{2}} \tag{25}$$

for $t \in U$, where

$$k(t) = \Big\{\frac{A}{1-a}\int_t^1 (1-\tau^2)^a\, h''^2\, d\tau\Big\}^{\frac{1}{2}} > 0, \quad t \in U$$

is a continuous function on I such that $lim_{t\to 1} k(t) \equiv k(1) = 0$. Next using (25) and the Schwartz inequality again we get the estimate

$$\begin{aligned} |h(t)|^2 &= \Big|\int_t^1 h'\, d\tau\Big|^2 \le \Big(\int_t^1 |h'| d\tau\Big)^2 \le \Big(\int_t^1 k(\tau)(1-\tau)^{\frac{1-a}{2}}\, d\tau\Big)^2 \\ &\le \int_t^1 k^2(\tau)d\tau \int_t^1 (1-\tau)^{1-a} d\tau = \frac{1}{2-a}k^2(\theta)\,(1-t)^{3-a} \end{aligned}$$

for $t \in U$, where $t < \theta < 1$ and, in the last step, we applied Mean Value Theorem for the integral $\int_t^1 k^2(\tau)d\tau$. From the above estimate we get

$$|h(t)| \le B\, k(\theta)\,(1-t)^{\frac{3-a}{2}} \tag{26}$$

for $t \in U$, where $B = \frac{1}{\sqrt{2-a}}$ and $\lim_{t\to 1} k(\theta) = k(1) = 0$.

Next based on the estimates (25) and (26) we obtain from (23) that

$$\begin{aligned} |S(t,h,h')| &\le B^2|\omega_0(t)|k^2(\theta)(1+t)^{a-3} + B|\omega_1(t)|k(\theta)k(t)(1+t)^{a-2} \\ &\quad + |\omega_2(t)|k^2(t)(1+t)^{a-1} \equiv m(t) \end{aligned}$$

for $t \in U$ with $\lim_{t\to 1} m(t) = 0$. Whence it follows that $\lim_{t\to 1} S(t,h,h') = 0$.

In analogous way we show that if $h(-1) = h'(-1) = 0$ and (21) holds, then

$$\lim_{t \to -1} S(t, h, h') = 0.$$

Based on the above considerations we obtain that from the conditions (21) and (24) follows the condition (22).

Therefore we get the following:

If $a < 1$ and the function $h \in AC^1((-1, 1))$ satisfies the integral condition

$$\int_{-1}^{1} (1 - t^2)^a h''^2 dt < \infty$$

and the limit conditions

$$h(-1) = h'(-1) = h(1) = h'(1) = 0$$

then the inequality

$$4(a-2)(2a-3) \int_{-1}^{1} (1 - t^2)^{a-2} h^2 dt \le \int_{-1}^{1} (1 - t^2)^a h''^2 dt \tag{27}$$

holds.

The inequality (27) becomes an equality if and only if $h = c(1 - t^2)^{2-a}$, where c =const.

In the particular case for $a = 0$ we obtain the inequality otherwise deduced in [1].

REMARK 2. In the paper [5] the inequality (10) has been derived under the assumption $\varphi'' \le 0$ on I. In this paper the assumption has been put on both functions r and φ, and this involves also the cases in which the condition $\varphi'' \le 0$ on I is not satisfied. Indeed, the inequality (27) with $a < 1$ cannot be obtained from Theorem 1 in [5], as for $a < 1$ the function $\varphi = (1 - t^2)^{2-a}$ doesn't satisfy the condition $\varphi'' \le 0$ on $(-1, 1)$. Therefore the classes of functions h for which the inequality (1) holds obtained in [5] and in this paper are not equal.

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