DEMONSTRATIO MATHEMATICA
Vol. XXXII No 3 1999

Y. J. Cho, S. S. Dragomir, A. White, S. S. Kim

SOME INEQUALITIES IN 2-INNER PRODUCT SPACES

Abstract. In this paper we extend some results on the refinement of Cauchy-Buniakowski-Schwarz's inequality and Aćzel's inequality in inner product spaces to 2−inner product spaces.

1. Introduction

Let X be a real linear space of dimension greater than 1 and let $\|\cdot,\cdot\|$ be a real-valued function on $X \times X$ satisfying the following conditions:

(N_1) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(N_2) $\|x,y\| = \|y,x\|$;
(N_3) $\|\alpha x,y\| = |\alpha|\|x,y\|$ for any real number α;
(N_4) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

$\|\cdot,\cdot\|$ is called a $2-$*norm* on X and $(X,\|\cdot,\cdot\|)$ *a linear 2-normed space* (cf. [10]). Some of the basic properties of the 2-norms are that they are nonnegative, and $\|x,y+\alpha x\| = \|x,y\|$ for every x,y in X and every real number α.

For any non-zero $x_1,x_2,\ldots,x_n$ in X, let $V(x_1,x_2,\ldots,x_n)$ denote the subspace of X generated by $x_1,x_2,\ldots,x_n$. Whenever the notation $V(x_1,x_2,\ldots$ $\ldots,x_n)$ is used, we will understand that $x_1,x_2,\ldots,x_n$ are linearly independent.

A concept which is closely related to linear 2-normed space is that of 2-inner product spaces. For a linear space X of dimension greater than 1 let $(\cdot,\cdot\mid\cdot)$ be a real-valued function on $X\times X\times X$ which satisfies the following conditions:

1991 *Mathematics Subject Classification*: Primary 26D99; Secondary 46Cxx.
Key words and phrases: 2-Inner Product, Cauchy-Buniakowski-Schwarz's Inequality, Triangle Inequality, Aćzel's Inequality.
This paper has been supported in part by NON-DIRECTED RESEARCH FUND, Korea Research Foundation, 1996.

(I_1) $(x, x \mid z) \geq 0; (x, x \mid z) = 0$ if and only if x and z are linearly dependent;

(I_2) $(x, x \mid z) = (z, z \mid x)$;

(I_3) $(x, y \mid z) = (y, x \mid z)$;

(I_4) $(\alpha x, y \mid z) = \alpha(x, y \mid z)$ for any real number α;

(I_5) $(x + x', y \mid z) = (x, y \mid z) + (x', y \mid z)$.

$(\cdot, \cdot \mid \cdot)$ is called a 2−*inner* product and $(X, (\cdot, \cdot \mid \cdot))$ a 2−*inner product space* ([3]).

These spaces are studied extensively in [1], [2], [4]-[6] and [11]. In [3] it is shown that $\|x, z\| = (x, x \mid z)^{\frac{1}{2}}$ is a 2−norm on $(X, \|\cdot, \cdot\|)$. Every 2−inner product space will be considered to be a linear 2−normed space with the 2−norm $\|x, z\| = (x, x \mid z)^{\frac{1}{2}}$. R. Ehret [9] has shown that for any 2−inner product space $(X, (\cdot, \cdot \mid \cdot)), \|x, z\| = (x, x \mid z)^{\frac{1}{2}}$ defines a 2-norm for which

$$(x, y \mid z) = \tfrac{1}{4}(\|x + y, z\|^2 - \|x - y, z\|^2), \tag{1}$$

$$\|x + y, z\|^2 + \|x - y, z\|^2 = 2(\|x, z\|^2 + \|y, z\|^2). \tag{2}$$

Besides, if $(X, \|\cdot, \cdot\|)$ is a linear 2−normed space in which condition (2), being a 2-dimensional analogue of the parallelogram law, is satisfied for every $x, y, z \in X$, then a 2-inner product on X is defined on by (1).

For a 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$ Cauchy-Schwarz's inequality

$$|(x, y \mid z)| \leq (x, x \mid z)^{\frac{1}{2}}(y, y \mid z)^{\frac{1}{2}} = \|x, z\|\|y, z\|,$$

a 2−dimensional analogue of Cauchy-Buniakowski-Schwarz's inequality, holds (cf. [3]).

2. Refinements of Cauchy-Schwarz's Inequality

Throughout this paper, let $(X, (\cdot, \cdot \mid \cdot))$ denote a 2−inner product space with $\|x, z\| = (x, x \mid z)^{\frac{1}{2}}$, $\mathbf{R}$ the set of real numbers and $\mathbf{N}$ the set of natural numbers.

THEOREM 2.1. *Let $x, y, z, u, v \in X$ with $z \notin V(x, y, u, v)$ be such that*

$$\|u, z\|^2 \leq 2(x, u \mid z), \qquad \|v, z\|^2 \leq 2(y, v \mid z). \tag{3}$$

Then, we have the inequality

$$\begin{aligned}(2(x, u \mid z) - \|u, z\|^2)^{\frac{1}{2}}(2(y, v \mid z) - \|v, z\|^2)^{\frac{1}{2}} \\ + |(x, y \mid z) - (x, v \mid z) - (u, y \mid z) + (u, v \mid z)| \leq \|x, z\|\|y, z\|.\end{aligned} \tag{4}$$

Proof. Note that

$$(m^2 - n^2)(p^2 - q^2) \leq (mp - nq)^2 \tag{5}$$

for every $m, n, p, q \in \mathbf{R}$. Since

$$|(x,y \mid z)-(x,v \mid z)-(u,y \mid z)+(u,v \mid z)|^2$$
$$= |(x-u, y-v \mid z)|^2 \leq \|x-u,z\|^2\|y-v,z\|^2$$
$$= (\|x,z\|^2+\|u,z\|^2-2(x,u \mid z))(\|y,z\|^2+\|v,z\|^2-2(y,v \mid z)),$$

by (5), we have

$$|(x,y \mid z)-(x,v \mid z)-(u,y \mid z)+(u,v \mid z)|^2 \tag{6}$$
$$\leq \{\|x,z\|\|y,z\|-(2(x,u \mid z)-\|u,z\|^2)^{\frac{1}{2}}(2(y,v \mid z)-\|v,z\|^2)^{\frac{1}{2}}\}^2.$$

On the other hand

$$0 \leq (2(x,u \mid z)-\|u,z\|^2)^{\frac{1}{2}} \leq \|x,z\|,$$
$$0 \leq (2(y,v \mid z)-\|v,z\|^2)^{\frac{1}{2}} \leq \|y,z\|,$$

which imply

$$(2(x,u \mid z)-\|u,z\|^2)^{\frac{1}{2}}(2(y,v \mid z)-\|v,z\|^2)^{\frac{1}{2}} \leq \|x,z\|\|y,z\|.$$

Therefore, from (6), we have the inequality (4). This completes the proof. ∎

COROLLARY 2.2. *Let $x, y, z, e \in X$ be such that $\|e,z\| = 1$ and $z \notin V(x,y,e)$. Then*

$$|(x,y \mid z)| \leq |(x,y \mid z)-(x,e \mid z)(e,y \mid z)| \tag{7}$$
$$+|(x,e \mid z)(e,y \mid z)| \leq \|x,z\|\|y,z\|.$$

Proof. If we put $u = (x,e \mid z)e$ and $v = (y,e \mid z)e$, then the conditions (3) hold. In fact,

$$2(x,u \mid z)-\|u,z\|^2 = 2(x,(x,e \mid z)e \mid z)-\|(x,e \mid z)e,z\|^2$$
$$= 2(x,e \mid z)(x,e \mid z)-(x,e \mid z)^2 = (x,e \mid z)(x,e \mid z) \geq 0.$$

And similarly for the second condition in (3).

Moreover,

$$|(x,y \mid z)-(x,v \mid z)-(u,y \mid z)+(u,v \mid z)|4$$
$$= |(x,y \mid z)-(x,e \mid z)(y,e \mid z)-(x,e \mid z)(e,y \mid z)+(x,e \mid z)(y,e \mid z)|$$
$$= |(x,y \mid z)-(x,e \mid z)(e,y \mid z)|,$$

so, by Theorem 2.1, we have (7). ∎

COROLLARY 2.3. *Let $x, y, z \in X$ be such that $\|x,z\|^2 \leq 2, \|y,z\|^2 \leq 2$ and $z \notin V(x,y)$. Then*

$$|(x,y \mid z)|^2(2-\|x,z\|^2)^{\frac{1}{2}}(2-\|y,z\|^2)^{\frac{1}{2}} \tag{8}$$
$$+|(x,y \mid z)||1-\|x,z\|^2-\|y,z\|^2+(x,y \mid z)|^2 \leq \|x,z\|\|y,z\|.$$

Proof. If we put $u = (x, y \mid z)y$ and $v = (y, x \mid z)x$, then the inequality (5) holds. Moreover, we have

$$\begin{aligned}&(2(x, u \mid z) - \|u, z\|^2)^{\frac{1}{2}}(2(y, v \mid z) - \|v, z\|^2)^{\frac{1}{2}} \\ &= |(x, y \mid z)|^2(2 - \|x, z\|^2)^{\frac{1}{2}}(2 - \|y, z\|^2)^{\frac{1}{2}}, \\ &|(x, y \mid z) - (x, v \mid z) - (u, y \mid z) + (u, v \mid z)| \\ &= |(x, y \mid z)||1 - \|x, z\|^2 - \|y, z\|^2 + |(x, y \mid z)|^2|.\end{aligned}$$

Therefore, by Theorem 2.1, we have the inequality (8). ∎

THEOREM 2.4. *Let $x, y, z, e \in X$ be such that $\|e, z\| = 1$ and $z \notin V(x, y, e)$. Then*

$$\begin{aligned}&|(x, y \mid z) - (x, e \mid z)(e, y \mid z)|^2 \\ &\leq (\|x, z\|^2 - |(x, e \mid z)|^2)(\|y, z\|^2 - |(y, e \mid z)|^2).\end{aligned} \tag{9}$$

Proof. Consider a mapping $P : X \times X \times X \to \mathbf{R}$ defined by $P(x, y, z) = (x, y \mid z) - (x, e \mid z)(e, y \mid z)$ for every $x, y, z, e \in X$, having the properties:

(i) $P(x, x, z) \geq 0$,
(ii) $P(\alpha x + \beta x', y, z) = P(x, y, z) + \beta P(x', y, z)$,
(iii) $P(x, y, z) = P(y, x, z)$.

Then Cauchy-Schwarz's inequality

$$|P(x, y, z)|^2 \leq P(x, x, z)P(y, y, z) \tag{10}$$

holds. Indeed, we observe that

$$\begin{aligned}&0 \leq P(x + \alpha P(x, y, z)y, x + \alpha P(x, y, z)y, z) \\ &= P(x, x, z) + 2\alpha P(x, y, z)^2 + \alpha^2 P(x, y, z)^2 P(y, y, z), \quad \forall \alpha \in \mathbf{R}.\end{aligned}$$

It is well known that if $a \geq 0$ and $a\alpha^2 + b\alpha + c \geq 0$ for all $\alpha \in \mathbf{R}$ then $\Delta = b^2 - 4ac \leq 0$. By the above inequality we deduce

$$P(x, y, z)^4 \leq P(x, x, z)P(y, y, z)P(x, y, z)^2. \tag{11}$$

If $P(x, y, z) = 0$ then (10) holds. If $P(x, y, z) \neq 0$ then we can devide in (11) by $P(x, y, z)^2$ and obtain (10).

The theorem is thus proved. ∎

REMARK 2.1. By the inequalities (5) and (9), we have

$$\begin{aligned}&|(x, y \mid z) - (x, e \mid z)(e, y \mid z)|^2 \\ &\leq (\|x, z\|^2 - |(x, e \mid z)|^2)(\|y, z\|^2 - |(y, e \mid z)|^2) \\ &\leq (\|x, z\|\|y, z\| - |(x, e \mid z)(e, y \mid z)|)^2.\end{aligned}$$

Since $\|x,z\|\|y,z\| \geq |(x,e \mid z)(e,y \mid z)|$, we get

$$|(x,y \mid z) - (x,e \mid z)(e,y \mid z)| \leq \|x,z\|\|y,z\| - |(x,e \mid z)(e,y \mid z)|$$

which yields the inequality (7).

COROLLARY 2.5. *Let $x,y,z,e \in X$ be such that $\|e,z\| = 1$ and $z \notin V(x,y,e)$. Then*

$$(12)\qquad (\|x+y,z\|^2 - |(x+y,e \mid z)|^2)^{\frac{1}{2}} \leq (\|x,z\|^2 - |(x,e \mid z)|^2)^{\frac{1}{2}} + (\|y,z\|^2 - |(y,e \mid z)|^2)^{\frac{1}{2}}.$$

Proof. If we define $S : X \times X \to \mathbf{R}$ by $S(x,z) = P(x,x,z)^{\frac{1}{2}}$ for every $x,y \in X$ and use the triangle inequality for $S(x,z)$, then we have (12). ∎

COROLLARY 2.6. *For every non-zero $x,y,z,u \in X$, with $z \notin V(x,y,u)$, we have*

$$(13)\qquad \left|\frac{(x,y \mid z)}{\|x,z\|\|y,z\|}\right|^2 + \left|\frac{(y,u \mid z)}{\|y,z\|\|u,z\|}\right|^2 + \left|\frac{(u,x \mid z)}{\|u,z\|\|x,z\|}\right|^2 \leq 1 + 2\left|\frac{(x,y \mid z)(y,u \mid z)(u,x \mid z)}{\|x,z\|^2\|y,z\|^2\|u,z\|^2}\right|.$$

For the proof of next theorem, we need the following lemma.

LEMMA 2.7. *For every non-zero $x,y,z \in X$ with $z \notin V(x,y)$, we have*

$$(14)\qquad (\|x,z\| + \|y,z\|)\left\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\right\| \leq 2\|x-y,z\|.$$

Proof. Since

$$\frac{\|x,z\|}{\|y,z\|} + \frac{\|y,z\|}{\|x,z\|} \geq 2,$$

we have the inequality

$$(\|x,z\| + \|y,z\|)^2 - (x,y \mid z)\left(\frac{\|x,z\|}{\|y,z\|} + \frac{\|y,z\|}{\|x,z\|}\right) - 2(x,y \mid z) \leq 2\|x,z\|^2 + \|y,z\|^2 - 4(x,y \mid z)$$

which implies (14). ∎

THEOREM 2.8. *For every non-zero $x,y,z \in X$ with $z \notin V(x,y)$ we have*

$$(15)\qquad (\|x,z\| + \|y,z\|)^2\left(\left\|\frac{x}{\|x,z\|} - \frac{y}{\|y,z\|}, z\right\|^2 + \left\|\frac{x}{\|x,z\|} + \frac{y}{\|y,z\|}, z\right\|^2\right) \leq 8(\|x,z\|^2 + \|y,z\|^2).$$

Proof. By (14) we have

$$(\|x,z\|+\|y,z\|)^2\left(\left\|\frac{x}{\|x,z\|}-\frac{y}{\|y,z\|},z\right\|^2+\left\|\frac{x}{\|x,z\|}+\frac{y}{\|y,z\|},z\right\|^2\right)$$
$$\leq 4(\|x-y,z\|^2+\|x+y,z\|^2)$$

and, by a 2−dimensional analogue of the parallelogram law, we get (15). ∎

REMARK 2.2. For some similar results in inner product spaces, see [7].

3. Aćzel's Inequality

In this section, we shall point out some results in 2−inner product spaces in connection to Aćzel's inequality [12]. For some other similar results in inner products, see [8]. We note that the results obtained here, in 2−inner product spaces used different techniques as those in [8].

THEOREM 3.1. *Let* $(X,(\cdot,\cdot\mid\cdot))$ *be a 2−inner product space,* $M_1, M_2 \in \mathbf{R}$ *and* $x,y,z \in X$ *such that* $\|x,z\| \leq |M_1|$, $\|y,z\| \leq |M_2|$, *then*

$$(16)\qquad (M_1^2-\|x,z\|^2)(M_2^2-\|y,z\|^2) \leq (|M_1M_2|-(x,y\mid z))^2.$$

Proof. Using the elementary inequality (5), we get

$$0 \leq (M_1^2-\|x,z\|^2)(M_2^2-\|y,z\|^2) \leq (|M_1M_2|-\|x,z\|\|y,z\|)^2$$

and, by Cauchy-Schwarz's inequality,

$$0 \leq |M_1M_2|-\|x,z\|\|y,z\| \leq |M_1M_2|-(x,y\mid z)$$

implying (16). ∎

COROLLARY 3.2. *If* $x,y,z \in X$ *and* $M>0$ *are such that* $\|x,z\|, \|y,z\| \leq M$, *then we have the inequality*

$$(17)\qquad 0 \leq \|x,z\|^2\|y,z\|^2-(x,y\mid z)^2 \leq M^2\|x-y,z\|^2$$

which is a counterpart of Cauchy-Schwarz's inequality.

Another result similar to the generalization (16) of Aćzel's inequality, is the following one

THEOREM 3.3. *Let* $(X,(\cdot,\cdot\mid\cdot))$ *be a 2−inner product space,* $M_1, M_2 \in \mathbf{R}$ *and* $x,y,z \in X$ *such that* $\|x,z\| \leq |M_1|$, $\|y,z\| \leq |M_2|$. *Then*

$$(18)\qquad (|M_1|-\|x,z\|)^{\frac{1}{2}}(|M_2|-\|y,z\|)^{\frac{1}{2}} \leq |M_1M_2|^{\frac{1}{2}}-|(x,y\mid z)|^{\frac{1}{2}}.$$

Proof. Applying (5) for $m=\sqrt{|M_1|}$, $p=\sqrt{|M_2|}$, $n=\sqrt{\|x,z\|}$, $q=\sqrt{\|y,z\|}$ and using Cauchy-Schwarz's inequality for 2−inner products we deduce, (18). ∎

COROLLARY 3.4. *Suppose that $x, y, z \in X$ and $M > 0$ are such that $\|x, z\|$, $\|y, z\| \leq M$. Then we have the following converse of Cauchy-Schwarz's inequality*

$$\begin{aligned} (19) \qquad 0 &\leq \|x, z\|\|y, z\| - |(x, y \mid z)| \\ &\leq M(\|x, z\| + \|y, z\| - 2|(x, y \mid z)|^{1/2}). \end{aligned}$$

THEOREM 3.5. *Let $(\cdot, \cdot \mid \cdot)$ be a 2-inner product and $\{(\cdot, \cdot \mid \cdot)_i\}_{i \in \mathbf{N}}$ a sequence of 2-inner products satisfying*

$$(20) \qquad \|x, z\|^2 > \sum_{i=0}^{\infty} \|x, z\|_i^2$$

for all x, z, being linearly independent. Then we have the following refinemenet of Cauchy-Schwarz's inequality

$$(21) \ \|x, z\|\|y, z\| - |(x, y \mid z)| \geq \Big[\sum_{i=0}^{\infty} \|x, z\|_i \sum_{i=0}^{\infty} \|y, z\|_i - |(x, y \mid z)|\Big] \geq 0$$

for all $x, y, z \in X$.

Proof. Let $n \in \mathbf{N}$ and $n \geq 1$. Define the mapping

$$(x, y \mid z)_n = (x, y \mid z) - \sum_{i=0}^{n} (x, y \mid z)_i, \qquad x, y, z \in X.$$

We observe, by (20), that the mapping $(\cdot, \cdot \mid \cdot)_n$ satisfies the properties

(i) $(x, x \mid z)_n \geq 0$,
(ii) $(\alpha x + \beta x', y \mid z)_n = \alpha(x, y \mid z)_n + \beta(x', y \mid z)_n$,
(iii) $(x, y \mid z)_n = (y, x \mid z)_n$

for every $x, x', y, z \in X$ and $\alpha, \alpha' \in \mathbf{R}$.

By a similar proof to that in Theorem 2.4, we can state Cauchy-Schwarz's inequality

$$(x, x \mid z)_n (y, y \mid z)_n \geq |(x, y \mid z)_n|^2, \quad x, y, z \in X,$$

that is

$$\begin{aligned} (22) \qquad &\Big(\|x, z\|^2 - \sum_{i=0}^{n} \|x, z\|_i^2\Big)\Big(\|y, z\|^2 - \sum_{i=0}^{n} \|y, z\|_i^2\Big) \\ &\geq \Big((x, y \mid z) - \sum_{i=0}^{n} (x, y \mid z)_i\Big)^2. \end{aligned}$$

Using Aćzel's inequality [12]

$$\Big(a^2 - \sum_{i=0}^{m} a_i^2\Big)\Big(b^2 - \sum_{i=0}^{m} b_i^2\Big) \leq \Big(ab - \sum_{i=0}^{m} a_i b_i\Big)^2,$$

where $a, b, a_i, b_i \in \mathbf{R}$ for $i = 0, \ldots, m$; we can prove that

$$(23) \qquad \Big(\|x,z\|\|y,z\| - \sum_{i=0}^{n} \|x,z\|_i\|y,z\|_i\Big)^2 \geq \Big(\|x,z\|^2 - \sum_{i=0}^{n} \|x,z\|_i^2\Big)\Big(\|y,z\|^2 - \sum_{i=0}^{n} \|y,z\|_i^2\Big)$$

for all $x, y, z \in X$. Since, by Cauchy-Buniakowski-Schwarz's inequality,

$$\|x,z\|\|y,z\| \geq \Big(\sum_{i=0}^{n} \|x,z\|_i^2 \sum_{i=0}^{n} \|y,z\|_i^2\Big)^{1/2} \geq \sum_{i=0}^{n} \|x,z\|_i\|y,z\|_i,$$

then, by (22) and (23), we deduce

$$\|x,z\|\|y,z\| - \sum_{i=0}^{n} \|x,z\|_i\|y,z\|_i = \Big|\|x,z\|\|y,z\| - \sum_{i=0}^{n} \|x,z\|_i\|y,z\|_i\Big| \geq |(x,y \mid z)| - \sum_{i=0}^{n} |(x,y \mid z)_i|$$

which implies (21), by using the inequality

$$\|x,z\|_i\|y,z\|_i - |(x,y \mid z)_i| \geq 0.$$

The theorem is thus proved. ∎

The following corollaries are interesting as refinements of the triangle inequality for 2-norms generated by 2-inner products.

COROLLARY 3.6. *With the assumptions from Theorem, we have the following refinement of the triangle inequality*

$$(\|x,z\| + \|y,z\|)^2 - \|x+y,z\|^2 \geq \sum_{i=0}^{\infty}[(\|x,z\|_i + \|y,z\|_i)^2 - \|x+y,z\|_i^2] \geq 0, \qquad x,y,z \in X.$$

COROLLARY 3.7. *Let $(.,. \mid .)_1, (.,. \mid .)_2$ be two 2-inner products such that*

$$\|x,z\|_2 > \|x,z\|_1$$

for all x, z being linearly independent in X. Then

$$\|x,z\|_2\|y,z\|_2 - |(x,y \mid z)_2| \geq \|x,z\|_1\|y,z\|_1 - |(x,y \mid z)_1| \geq 0, \qquad x,y,z \in X.$$

COROLLARY 3.8. *Let $(.,. \mid .)_1, (.,. \mid .)_2$ be as above. Then*

$$(\|x,z\|_2 + \|y,z\|_2)^2 - \|x+y,z\|_2^2 \geq (\|x,z\|_1 + \|y,z\|_1)^2 - \|x+y,z\|_1^2 \geq 0, x,y,z \in X.$$

References

[1] Y. J. Cho, S. S. Kim, *Gâteaux derivatives and 2-inner product spaces*, Glašnik Mat. 27 (1992), 271–282.
[2] Y. J. Cho, C. S. Lin, A. Misiak, *Theory of 2-Inner Product Spaces*, unpublished book.
[3] C. Diminnie, S. Gähler, A. White, *2-inner product spaces*, Demonstratio Math. 6 (1973), 525–536.
[4] C. Diminnie, S. Gähler, A. White, *Remarks on generalizations of 2-inner products*, Math. Nachr. 74 (1974), 269–278.
[5] C. Diminnie, S. Gähler, A. White, *2-inner product spaces II*, Demonstratio Math. 10 (1977), 169–188.
[6] C. Diminnie, A. White, *2-inner product spaces and Gâteaux partial derivatives*, Comment. Math. Univ. Carolinae, 16 (1975), 115–119.
[7] S. S. Dragomir, I. Sándor, *Some inequalities in prehilbertian spaces*, Studia Univ. Babes-Bolyai, Mathematica, 32 (1987), 71–78.
[8] S. S. Dragomir, *A generalization of Aćzel's inequality in inner product spaces*, Acta Math. Hungar. 65 (1994), 141–148.
[9] R. Ehret, *Linear 2-normed spaces*, Doctoral Diss., Saint Louis Univ., 1969.
[10] S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. 28 (1965), 1–43.
[11] S. Gähler, A. Misiak, *Remarks on 2-inner products*, Demonstratio Math. 17 (1984), 655–670.
[12] D. S. Mitrinović, *Analytic Inequalities*, Springer Verlag, 1970.

Y. J. Cho
DEPARTMENT OF MATHEMATICS
GYEONSANG NATIONAL UNIVERSITY
CHINJU 660-701, KOREA
E-mail:yjcho@nongae.gsnu.ac.kr

S. S. Dragomir
SCHOOL OF COMMUNICATIONS AND INFORMATICS
VICTORIA UNIVERSITY OF TECHNOLOGY
PO Box 14428
MELBOURNE CITY MC
VICTORIA 8001, AUSTRALIA
E-mail: sever@matilda.vut.edu.au

A. White
DEPARTMENT OF MATHEMATICS
ST. BONAVENTURE UNIVERSITY
ST. BONAVENTURE, NY
14778, U.S.A.

S. S. Kim
DEPARTMENT OF MATHEMATICS
DONGEUI UNIVERSITY
PUSAN 614-714

Received January 19, 1998; revised version February 2nd, 1999.