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A GEOMETRICAL CHARACTERIZATION OF MINKOWSKI PLANES OF ORDER 3 AND 4

Abstract. In [5] H. A. Wilbrink proved that a certain class of Minkowski planes induce nearaffine planes. Of course, this class contains all Minkowski planes over fields. But only Minkowski planes of order 3 and 4 induce nearaffine planes which are also Minkowski planes (moreover they are affine planes, too).

1. Basic concept

Let Π be a set of points provided with a family Λ of subsets called circles, and two other families Σ_+, Σ_- of subsets called (+)generators and (−)generators, respectively. Elements of $\Sigma_+ \cup \Sigma_-$ will just be called generators.

Definition 1. The quadruple $M = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ is a Minkowski plane if the following axioms hold ([3, p. 269]):

(M1) For every point P there exists a unique (+)generator, denoted by $[P]_+$, and a unique (−)generator, denoted by $[P]_-$, containing P.
(M2) Every (+)generator meets every (−)generator in a unique point.
(M3) There is a circle containing at least three points.
(M4) Through three distinct points A, B, C, no two of which are on a common generator there is a unique circle, denoted by (A, B, C).
(M5) Every circle intersects every generator in a unique point.
(T) Given a circle λ, a point $P \in \lambda$ and a point $Q \notin \lambda$ with P and Q not on a generator, there is one and only one circle μ through Q such that $\lambda \cap \mu = \{P\}$.

Corollary 1. *In the case of finite Minkowski planes (i.e. if* Π *is finite) the axiom* (T) *follows immediately from* (M1) - (M5) (cf. [3, p.269]).

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DEFINITION 2. Let $\boldsymbol{M} = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ and $\boldsymbol{M}' = (\Pi', \Sigma'_+, \Sigma'_-, \Lambda')$ are Minkowski planes. A bijection $\varphi : \Pi \to \Pi'$ is called an isomorphism from $\boldsymbol{M}$ to $\boldsymbol{M}'$ if the following conditions hold:

$$\lambda \in \Lambda \iff \varphi(\lambda) \in \Lambda'$$

$$\sigma \in \Sigma_+ \cup \Sigma_- \iff \varphi(\sigma) \in \Sigma'_+ \cup \Sigma'_-.$$

PROPOSITION 1. *Let P be a point of a Minkowski plane $\boldsymbol{M} = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ and let*

$$\Pi_P = \Pi - ([P]_+ \cup [P]_-)$$
$$\boldsymbol{N} = \{\lambda \cap \Pi_P;\ \lambda \in \Lambda,\ P \in \lambda\} \cup \{\sigma \cap \Pi_P;\ \sigma \in \Sigma_+ \cup \Sigma_-,\ [P]_+ \neq \sigma \neq [P]_-\}.$$

Then the incidence structure $\boldsymbol{M}_P = (\Pi_P, \boldsymbol{N})$ is an affine plane, called the derrived plane with respect to P, where $\infty_1 = \{\sigma \cap \Pi_P;\ \sigma \in \Sigma_+ - \{[P]_+\}\}$ and $\infty_2 = \{\sigma \cap \Pi_P;\ \sigma \in \Sigma_- - \{[P]_-\}\}$ are two classes of parallel lines, i.e. ∞_1 and ∞_2 are ideal points in the projective extension $\overline{\boldsymbol{M}_P}$ of $\boldsymbol{M}_P$ (cf. [1, p. 116]). *Moreover, for every circle λ not passing through P the set $(\lambda \cap \Pi_P) \cup \{\infty_1, \infty_2\}$ is an oval in $\overline{\boldsymbol{M}_P}$* ([3, p. 273]).

Proof. The proof of the first part is classical (one must verify axioms of an affine plane - cf. [1]). In order to demonstrate the last part, note that three points of $(\lambda \cap \Pi_P) \cup \{\infty_1, \infty_2\}$ cannot be on a line obtained from a circle passing through P (because of (M4)) nor on a line obtained from a generator (because of (M5)). For every point $Q \in \lambda \cap \Pi_P$ a line obtained from a circle μ such that $\mu \cap \lambda = \{Q\}$, $P \in \mu$ is a unique tangent in Q. Let $[P]_+ \cap \lambda = \{R\}$, $[P]_- \cap \lambda = \{S\}$. Then $([R]_- \cap \Pi_P) \cup \{\infty_2\}$ and $([S]_+ \cap \Pi_P) \cup \{\infty_1\}$ are the unique tangents in ∞_2 and ∞_1, respectively.

DEFINITION 3. A number n $(n \geq 2)$ is called the order of a Minkowski plane $\boldsymbol{M}$ if every circle of $\boldsymbol{M}$ contains exactly $n+1$ points ([3, p. 269], see Figure 1).

PROPOSITION 2. *Let F be any commutative field and let $\overline{F} = F \cup \{\infty\}$, where $\infty \notin F$, and extend operations from F to $\overline{F}$ as follows:*

$$a + \infty = \infty + a \text{ for every } a \in \overline{F}$$

$$a \cdot \infty = \infty \cdot a \text{ for every } a \in \overline{F},\ a \neq 0.$$

Then the set of all permutations on $\overline{F}$ given by

$$x \mapsto \frac{ax+b}{cx+d},\ a, b, c, d \in F,\ ad - bc \neq 0$$

determines the set of circles in the Minkowski planes over F (cf. [3, p. 271]).

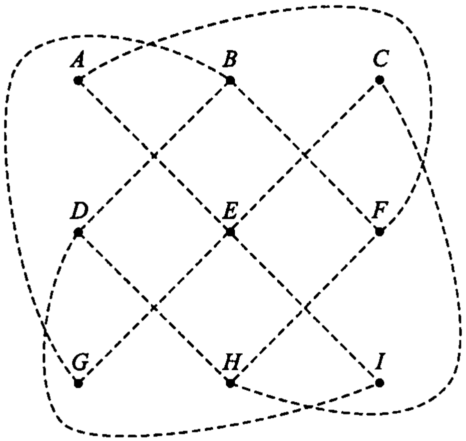

Fig. 1

Generators are given by

$$\Sigma_+ = \{\{(c, y);\ y \in \overline{F}\};\ c \in \overline{F}\},\quad \Sigma_- = \{\{(x, c);\ x \in \overline{F}\};\ c \in \overline{F}\}$$

and the point set is given by $\Pi = \overline{F} \times \overline{F}$.

COROLLARY 2. *For every pair of points Z, W on a Minkowski plane $\boldsymbol{M}$ over a field F there exists an automorphism φ of $\boldsymbol{M}$ such that $\varphi(Z) = W$.*

Proof. For $Z, W \in \overline{F} \times \overline{F}$, where $Z = (z_1, z_2)$, $W = (w_1, w_2)$ a bijection $\varphi(x, y) = (x + w_1 - z_1, y + w_2 - z_2)$ is a required automorphism.

REMARK 1. *For every $n \in \{2, \ldots, 5, 7, 8\}$ there exists exactly one (up to isomorphism) Minkowski plane of order n, namely a plane over the uniquely determined field.*

Let Ω be a set of elements called points with at least two points, Ψ a set of subsets of Ω called lines. Let $\rhd$ be an operation called *join* mapping the ordered pairs (X, Y), $X, Y \in \Omega$, $X \neq Y$ onto Ψ ($X \rhd Y$ denotes joining from X to Y), and $\equiv$ an equivalence relation called *parallelism* on Ψ. A line $X \rhd Y$ will be called *straight* if $X \rhd Y = Y \rhd X$, the remaining lines are called *proper*. The set of all straight lines will be denoted by Γ. X is called the *base point* for $X \rhd Y$.

DEFINITION 4. $(\Omega, \Psi, \rhd, \equiv)$ is called a *nearaffine plane* if the following three groups of axioms are satisfied [6, p. 53-54].

(I) Axioms of lines

(L1) $X, Y \in X \rhd Y$ for all $X, Y \in \Omega$, $X \neq Y$

(L2) $Z \in X \rhd Y - \{X\} \iff X \rhd Y = X \rhd Z$ for all $X, Y, Z \in \Omega$, $X \neq Y$

(L3) $(X \rhd Y = Y \rhd X = X \rhd Z) \Rightarrow X \rhd Z = Z \rhd X$ for all $X, Y, Z \in \Omega$, $Y \neq X \neq Z$.

(II) Axioms of parallelism

(P1) For every $k \in \Psi$, $X \in \Omega$ there exists exactly one line with base point X parallel to k. We denote this line by $(X \equiv k)$,

(P2) $X \rhd Y \equiv Y \rhd X$ for all $X, Y \in \Omega$, $X \neq Y$,

(P3) $(k \equiv l) \Rightarrow l \in \Gamma$ for all $k \in \Gamma$, $l \in \Psi$.

(III) Axioms of richness

(R1) There exist at least two non-parallel straight lines

(R2) Every line l meets every straight line k with $\neg(k \equiv l)$ in exactly one point.

COROLLARY 3. *Every proper line has exactly one base point. Each point of a straight line is its base point.*

COROLLARY 4. *A nearaffine plane is an affine plane if and only if no proper line exists in it.*

DEFINITION 5. Let $\boldsymbol{M} = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ be a Minkowski plane. Fix a point $Z \in \Pi$ and define

$$\Pi_Z = \Pi - ([Z]_+ \cup [Z]_-)$$

$$\lambda^* = \lambda \cap \Pi_Z \text{ and } \sigma^* = \sigma \cap \Pi_Z \text{ for all } \lambda \in \Lambda,\ \sigma \in \Sigma_+ \cup \Sigma_-.$$

We will use denotations $[P]_- \cap [Z]_+ = \{P^+\}$, $[P]_+ \cap [Z]_- = \{P^-\}$ (see Figure 2)

a) For $P, Q \in \Pi_Z$, $P \neq Q$ we define

$$P \rhd Q = \begin{cases} \sigma^* & \text{if } P, Q \in \sigma \text{ for some } \sigma \in \Sigma_+ \cup \Sigma_- \\ \{P\} \cup (P^+, P^-, Q)^* & \text{if } P \text{ and } Q \text{ are not on a generator,} \end{cases}$$

$$\Psi = \{P \rhd Q;\ P, Q \in \Pi_Z\}.$$

b) We define $P_1 \rhd Q_1 \equiv P_2 \rhd Q_2$ for the following cases:

i) $P_1 \rhd Q_1 = \sigma_1^*$ for some $\sigma_1 \in \Sigma_+$ ($\sigma_1 \in \Sigma_-$). Then $P_2 \rhd Q_2 = \sigma_2^*$ for some $\sigma_2 \in \Sigma_+$ ($\sigma_2 \in \Sigma_-$, respectively),

ii) $[P_1]_+ \neq [Q_1]_+$, $[P_1]_- \neq [Q_1]_-$ and $P_1, P_2 \in \sigma$ for some $\sigma \in \Sigma_+ \cup \Sigma_-$. Then $P_1 \rhd Q_1 = P_2 \rhd Q_2$ or $P_1 \rhd Q_1 \cap P_2 \rhd Q_2 = \emptyset$,

iii) $[P_1]_+ \neq [Q_1]_+$, $[P_1]_- \neq [Q_1]_-$, $[P_1]_+ \neq [P_2]_+$, $[P_1]_- \neq [P_2]_-$. Then there exists $P_3 \rhd Q_3$ such that $P_1 \rhd Q_1 \cap P_3 \rhd Q_3 = \emptyset = P_2 \rhd Q_2 \cap P_3 \rhd Q_3$, and P_3, P_i ($i = 1, 2$) are on a common generator.

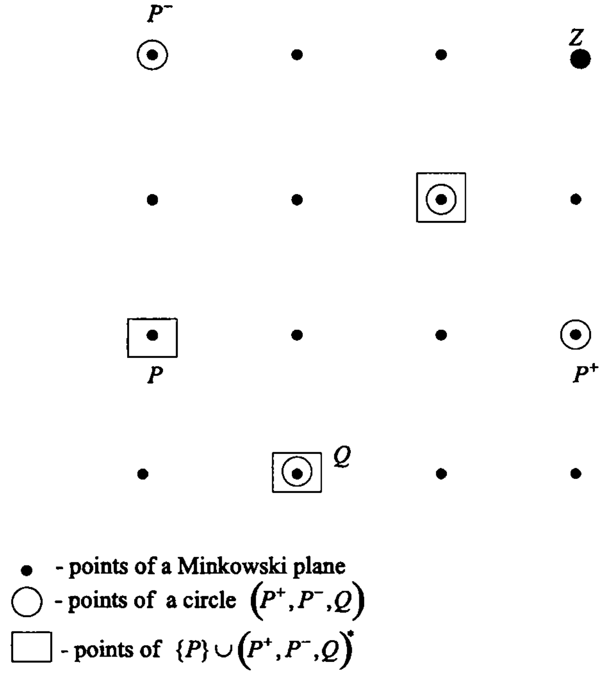

Fig. 2

THEOREM 1. *The structure* $M^Z = (\Pi_Z, \Psi, \rhd, \equiv)$ *is a nearaffine plane if and only if the following conditions hold* (cf. [5, p. 121-122])*:*

(A) *Let* $P_1, Q_1, P_2, Q_2 \in \Pi_Z$ *and suppose that* P_1 *and* Q_1, P_2 *and* Q_2, P_1 *and* P_2 *are not on a common generator. If there exists a circle* λ *touching* (P_1^+, P_1^-, Q_1) *in* P_1^- *and touching* (P_2^+, P_2^-, Q_2) *in* P_2^+ *then there exists a circle* μ *touching* (P_1^+, P_1^-, Q_1) *in* P_1^+ *and touching* (P_2^+, P_2^-, Q_2) *in* P_2^- *(see Figure 3).*

(B) *Let* P_1, Q_1, P_2, Q_2 *be points as in* (A). *If* $P_1 \in (P_2^+, P_2^-, Q_2)$ *and* $P_2 \in (P_1^+, P_1^-, Q_1)$ *then circles* λ *and* μ, *as described in* (A), *exist.*

COROLLARY 5. *If* M *is a Minkowski plane over a field then every point* Z *induces a nearaffine plane* M^Z *and for every point* W, *planes* M^Z *and* M^W *are isomorphic.*

Proof. By Corollary 2, it is enough to consider the conditions (A) and (B) with respect to $M^{(\infty,\infty)}$ (i.e. $Z = (\infty, \infty)$). Let us denote $P_i^+ = (\infty, p_i^+)$, $P_i^- = (p_i^-, \infty)$, $Q_i = (q_i, r_i)$, $i = 1, 2$. Therefore a circle (P_i^+, P_i^-, Q_i) is given by $(x - p_i^-)(y - p_i^+) = (q_i - p_i^-)(r_i - p_i^+)$. A circle λ touching (P_1^+, P_1^-, Q_1) in P_1^- and touching (P_2^+, P_2^-, Q_2) in P_2^+ must be given by

$$(x - p_1^-)(y - p_2^+) = (q_1 - p_1^-)(r_1 - p_1^+) = (q_2 - p_2^-)(r_2 - p_2^+)$$

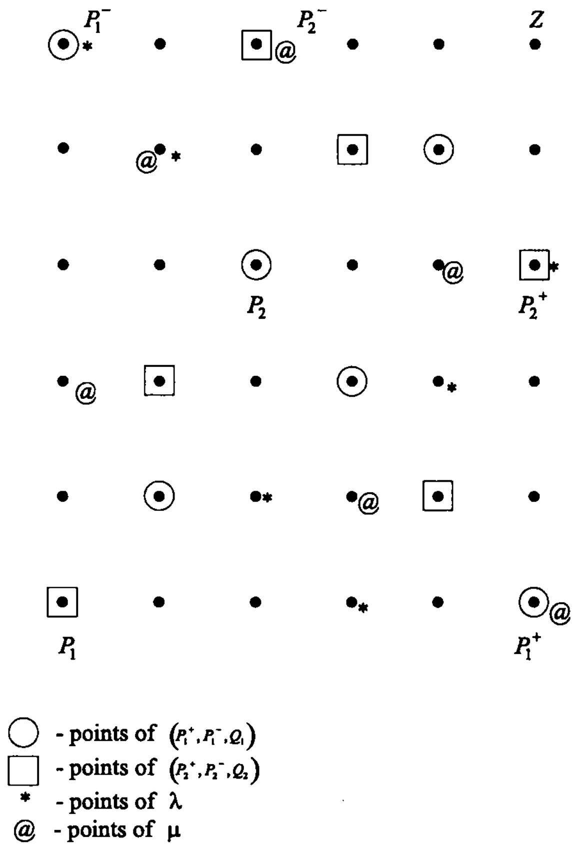

Fig. 3

The latter equality implies that a circle μ given by

$$(x - p_2^-)(y - p_1^+) = (q_1 - p_1^-)(r_1 - p_1^+) = (q_2 - p_2^-)(r_2 - p_2^+)$$

is touching (P_1^+, P_1^-, Q_1) in P_1^+ and touching (P_2^+, P_2^-, Q_2) in P_2^-. Thus (A) holds. If now $P_1 \in (P_2^+, P_2^-, Q_2)$ and $P_2 \in (P_1^+, P_1^-, Q_1)$, where $P_i = (p_i^-, p_i^+)$ then using the equation of (P_i^+, P_i^-, Q_i), $i = 1, 2$, we have

$$(p_1^- - p_2^-)(p_1^+ - p_2^+) = (q_2 - p_2^-)(r_2 - p_2^+)$$

and

$$(p_2^- - p_1^-)(p_2^+ - p_1^+) = (q_1 - p_1^-)(r_1 - p_1^+).$$

Left sides are equal and so must be right.

COROLLARY 6. *For any Minkowski plane over a field F and a point $Z = (\infty, \infty)$ we have:*

a) *every line* $P \rhd Q = \sigma^*$, *where* $\sigma \in \Sigma_+ \cup \Sigma_-$ *is a straight line* (see Corollary 3)*;*

b) *all lines of type* $\{P\} \cup (P^+, P^-, Q)^*$ *constitute a set*

$$\left\{\left\{(x,y) \in F \times F;\ y = \tfrac{ax+b}{cx+d}\right\};\ a,b,c,d \in F,\ ad-bc \neq 0,\ c \neq 0\right\} =$$
$$\left\{\left\{(x,y) \in F \times F;\ y = \tfrac{ax+b}{x+d}\right\};\ a,b,d \in F,\ ad-b \neq 0\right\} =$$
$$\{\{(x,y) \in F \times F;\ (x+d)(y-a) = b-ad\};\ a,b,d \in F,\ ad-b \neq 0\}.$$

2. General results

DEFINITION 6. Let $M = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ be a Minkowski plane such that there exists a point Z inducing a nearaffine planeM^Z and let $\lambda, \mu \in \Lambda$, $Z \notin \lambda \cup \mu$. λ is congruent to μ ($\lambda \cong \mu$) if and only if in M^Z lines obtained from λ^* and μ^* have the same set of points (more precisely: if $\lambda = (P^+, P^-, Q)$ and $\mu = (R^+, R^-, S)$ for some $P^+, R^+ \in [Z]_+$, $P^-, R^- \in [Z]_-$, and $Q, S \in \Pi_Z$ then $\{P\} \cup \lambda^* = \{R\} \cup \mu^*$).

COROLLARY 7. *The relation* $\cong$ *is an equivalence relation.*

COROLLARY 8. *A line determined by a circle* λ *(*$\lambda = (P^+, P^-, Q)$ *for some* $P, Q \in \Pi_Z$*) is proper iff* $[\lambda]_\cong = \{\lambda\}$ (see Corollary 3).

2.1. A Minkowski plane of order 3

In virtue of Remark 1, one can investigate a plane over the field $Z_3 = \{0,1,2\}$. Let

$$\Pi = \{P_1, P_2, \ldots P_{16}\},\ \Sigma_+ = \{\sigma_1, \ldots, \sigma_4\},\ \Sigma_- = \{\tau_1, \ldots, \tau_4\}$$

where

$$P_1 = (0,0),\ P_2 = (0,1),\ P_3 = (0,2),\ P_4 = (1,0),\ P_5 = (1,1),\ P_6 = (1,2),$$
$$P_7 = (2,0),\ P_8 = (2,1),\ P_9 = (2,2),\ P_{10} = (\infty,0),\ P_{11} = (\infty,1),$$
$$P_{12} = (\infty,2)\ P_{13} = (0,\infty),\ P_{14} = (1,\infty),\ P_{15} = (2,\infty),\ P_{16} = (\infty,\infty),$$
$$\sigma_1 = \{P_1,P_2,P_3,P_{13}\},\ \sigma_2 = \{P_4,P_5,P_6,P_{14}\},\ \sigma_3 = \{P_7,P_8,P_9,P_{15}\},$$
$$\sigma_4 = \{P_{10},P_{11},P_{12},P_{16}\},\ \tau_1 = \{P_1,P_4,P_7,P_{10}\},\ \tau_2 = \{P_2,P_5,P_8,P_{11}\},$$
$$\tau_3 = \{P_3,P_6,P_9,P_{12}\},\ \tau_4 = \{P_{13},P_{14},P_{15},P_{16}\}.$$

Because of Definition 5, only circles not passing through P_{16} are used to construct the nearaffine plane $M^{P_{16}}$. So Table 1 presents all such circles and the way to obtain the lines determined by them in $M^{P_{16}}$. One can easily verify that every class of congruent circles consists of three elements, e.g. $[\lambda_1]_\cong = \{\lambda_1, \lambda_{11}, \lambda_{15}\}$. Therefore the lines are straight, by Corollary 8. Let us identify every class with the set of points belonging to the line determined

Table 1

Circle λ_i	Equation of λ_i	Points on λ_i	$\{Q\} = P_{\lambda_i \cap \sigma_4}$	$\{R\} = \lambda_i \cap \tau_4$	$\{S\} = [Q]_- \cap [R]_+$	Points on $\{S\} \cup \lambda_i^*$
λ_1	$y = \frac{1}{x}$	P_5, P_9, P_{10}, P_{13}	P_{10}	P_{13}	P_1	P_1, P_5, P_9
λ_2	$y = \frac{2}{x}$	P_6, P_8, P_{10}, P_{13}	P_{10}	P_{13}	P_1	P_1, P_6, P_8
λ_3	$y = \frac{1}{x+1}$	P_2, P_6, P_{10}, P_{15}	P_{10}	P_{15}	P_7	P_2, P_6, P_7
λ_4	$y = \frac{2}{x+1}$	P_3, P_5, P_{10}, P_{15}	P_{10}	P_{15}	P_7	P_3, P_5, P_7
λ_5	$y = \frac{1}{x+2}$	P_3, P_8, P_{10}, P_{14}	P_{10}	P_{14}	P_4	P_3, P_4, P_8
λ_6	$y = \frac{2}{x+2}$	P_2, P_9, P_{10}, P_{14}	P_{10}	P_{14}	P_4	P_2, P_4, P_9
λ_7	$y = \frac{x+1}{x}$	P_6, P_7, P_{11}, P_{13}	P_{11}	P_{13}	P_2	P_2, P_6, P_7
λ_8	$y = \frac{x+2}{x}$	P_4, P_9, P_{11}, P_{13}	P_{11}	P_{13}	P_2	P_2, P_4, P_9
λ_9	$y = \frac{x}{x+1}$	P_1, P_6, P_{11}, P_{15}	P_{11}	P_{15}	P_8	P_1, P_6, P_8
λ_{10}	$y = \frac{x+2}{x+1}$	P_3, P_4, P_{11}, P_{15}	P_{11}	P_{15}	P_8	P_3, P_4, P_8
λ_{11}	$y = \frac{x}{x+2}$	P_1, P_9, P_{11}, P_{14}	P_{11}	P_{14}	P_5	P_1, P_5, P_9
λ_{12}	$y = \frac{x+1}{x+2}$	P_3, P_7, P_{11}, P_{14}	P_{11}	P_{14}	P_5	P_3, P_5, P_7
λ_{13}	$y = \frac{2x+1}{x}$	P_4, P_8, P_{12}, P_{13}	P_{12}	P_{13}	P_3	P_3, P_4, P_8
λ_{14}	$y = \frac{2x+2}{x}$	P_5, P_7, P_{12}, P_{13}	P_{12}	P_{13}	P_3	P_3, P_5, P_7
λ_{15}	$y = \frac{2x}{x+1}$	P_1, P_5, P_{12}, P_{15}	P_{12}	P_{15}	P_9	P_1, P_5, P_9
λ_{16}	$y = \frac{2x+1}{x+1}$	P_2, P_4, P_{12}, P_{15}	P_{12}	P_{15}	P_9	P_2, P_4, P_9
λ_{17}	$y = \frac{2x}{x+2}$	P_1, P_8, P_{12}, P_{14}	P_{12}	P_{14}	P_6	P_1, P_6, P_8
λ_{18}	$y = \frac{2x+2}{x+2}$	P_2, P_7, P_{12}, P_{14}	P_{12}	P_{14}	P_6	P_2, P_6, P_7

by it, and denote them by

$$\alpha_1 = \{P_1, P_5, P_9\},\ \alpha_2 = \{P_1, P_6, P_8\},\ \alpha_3 = \{P_2, P_4, P_9\},\ \alpha_4 = \{P_2, P_6, P_7\},$$
$$\alpha_5 = \{P_3, P_4, P_8\},\ \alpha_6 = \{P_3, P_5, P_7\}.$$

Further, in virtue of Definition 5, Corollary 5, and Corollary 6, a) we have straight lines

$$\sigma_1^* = \{P_1, P_2, P_3\},\ \sigma_2^* = \{P_4, P_5, P_6\},\ \sigma_3^* = \{P_7, P_8, P_9\},$$
$$\tau_1^* = \{P_1, P_4, P_7\},\ \tau_2^* = \{P_2, P_5, P_8\},\ \tau_3^* = \{P_3, P_6, P_9\}.$$

Now we put $\Pi' = \Pi_{P_{16}} = \{P_1, \dots, P_9\}$, $\Sigma'_+ = \{\sigma_1^*, \sigma_2^*, \sigma_3^*\}$, $\Sigma'_- = \{\tau_1^*, \tau_2^*, \tau_3^*\}$, $\Lambda' = \{\alpha_1, \alpha_2, \dots, \alpha_6\}$. One can easily verify that $\boldsymbol{M}' = (\Pi', \Sigma'_+, \Sigma'_-, \Lambda')$ is a Minkowski plane of order 2 (see Figure 4).

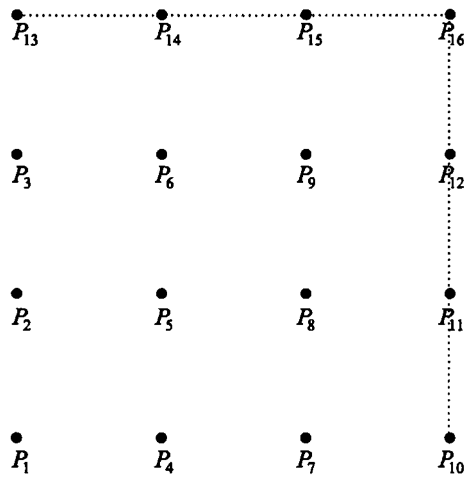

Fig. 4

From Corollary 4 and Corollary 5 we obtain

COROLLARY 9. *If $\boldsymbol{M}$ is a Minkowski plane of order 3 then for every point Z all lines of $\boldsymbol{M}^Z$ are straight and so $\boldsymbol{M}^Z$ is an affine plane.*

2.2. A Minkowski plane of order 4

In the same way as before, we investigate a plane over the field $\boldsymbol{K} = \{0, 1, a, b\}$ with $1 + 1 = a + a = b + b = 0$, $1 + a = b$, $1 + b = a$, $a + b = 1$, $a \cdot b = 1$, $a \cdot a = b$, $b \cdot b = a$ (cf. [2]), where we put:

$$P_1 = (0,0),\ P_2 = (0,1),\ P_3 = (0,a),\ P_4 = (0,b),\ P_5 = (1,0),\ P_6 = (1,1),$$
$$P_7 = (1,a),\ P_8 = (1,b),\ P_9 = (a,0),\ P_{10} = (a,1),\ P_{11} = (a,a),$$
$$P_{12} = (a,b)\ P_{13} = (b,0),\ P_{14} = (b,1),\ P_{15} = (b,a),\ P_{16} = (b,b),$$
$$P_{17} = (\infty,0)\ P_{18} = (\infty,1),\ P_{19} = (\infty,a),\ P_{20} = (\infty,b),\ P_{21} = (0,\infty),$$
$$P_{22} = (1,\infty),\ P_{23} = (a,\infty),\ P_{24} = (b,\infty),\ P_{25} = (\infty,\infty),$$
$$\sigma_1 = \{P_1, P_2, P_3, P_4, P_{21}\},\ \sigma_2 = \{P_5, P_6, P_7, P_8, P_{22}\},$$
$$\sigma_3 = \{P_9, P_{10}, P_{11}, P_{12}, P_{23}\},\ \sigma_4 = \{P_{13}, P_{14}, P_{15}, P_{16}, P_{24}\},$$
$$\sigma_5 = \{P_{17}, P_{18}, P_{19}, P_{20}, P_{25}\},\ \tau_1 = \{P_1, P_5, P_9, P_{13}, P_{17}\},$$

Table 2

Circle λ_i	Equation of λ_i	Points on λ_i	$\{Q\} = \lambda_i \cap \sigma_5$	$\{R\} = \lambda_i \cap \tau_5$	$\{S\} = [Q]_- \cap [R]_+$	Points on $\{S\} \cup \lambda_i^*$
λ_1	$y = \frac{1}{x}$	$P_6, P_{12}, P_{15}, P_{17}, P_{21}$	P_{17}	P_{21}	P_1	P_1, P_6, P_{12}, P_{15}
λ_2	$y = \frac{a}{x}$	$P_7, P_{10}, P_{16}, P_{17}, P_{21}$	P_{17}	P_{21}	P_1	P_1, P_7, P_{10}, P_{16}
λ_3	$y = \frac{b}{x}$	$P_8, P_{11}, P_{14}, P_{17}, P_{21}$	P_{17}	P_{21}	P_1	P_1, P_8, P_{11}, P_{14}
λ_4	$y = \frac{1}{x+1}$	$P_2, P_{11}, P_{16}, P_{17}, P_{22}$	P_{17}	P_{22}	P_5	P_2, P_5, P_{11}, P_{16}
λ_5	$y = \frac{a}{x+1}$	$P_3, P_{12}, P_{14}, P_{17}, P_{22}$	P_{17}	P_{22}	P_5	P_3, P_5, P_{12}, P_{14}
λ_6	$y = \frac{b}{x+1}$	$P_4, P_{10}, P_{15}, P_{17}, P_{22}$	P_{17}	P_{22}	P_5	P_4, P_5, P_{10}, P_{15}
λ_7	$y = \frac{1}{x+a}$	$P_4, P_7, P_{14}, P_{17}, P_{23}$	P_{17}	P_{23}	P_9	P_4, P_7, P_9, P_{14}
λ_8	$y = \frac{a}{x+a}$	$P_2, P_8, P_{15}, P_{17}, P_{23}$	P_{17}	P_{23}	P_9	P_2, P_8, P_9, P_{15}
λ_9	$y = \frac{b}{x+a}$	$P_3, P_6, P_{16}, P_{17}, P_{23}$	P_{17}	P_{23}	P_9	P_3, P_6, P_9, P_{16}
λ_{10}	$y = \frac{1}{x+b}$	$P_3, P_8, P_{10}, P_{17}, P_{24}$	P_{17}	P_{24}	P_{13}	P_3, P_8, P_{10}, P_{13}
λ_{11}	$y = \frac{a}{x+b}$	$P_4, P_6, P_{11}, P_{17}, P_{24}$	P_{17}	P_{24}	P_{13}	P_4, P_6, P_{11}, P_{13}
λ_{12}	$y = \frac{b}{x+b}$	$P_2, P_7, P_{12}, P_{17}, P_{24}$	P_{17}	P_{24}	P_{13}	P_2, P_7, P_{12}, P_{13}
λ_{13}	$y = \frac{x+1}{x}$	$P_5, P_{11}, P_{16}, P_{18}, P_{21}$	P_{18}	P_{21}	P_2	P_2, P_5, P_{11}, P_{16}
λ_{14}	$y = \frac{x+a}{x}$	$P_8, P_9, P_{15}, P_{18}, P_{21}$	P_{18}	P_{21}	P_2	P_2, P_8, P_9, P_{15}
λ_{15}	$y = \frac{x+b}{x}$	$P_7, P_{12}, P_{13}, P_{18}, P_{21}$	P_{18}	P_{21}	P_2	P_2, P_7, P_{12}, P_{13}
λ_{16}	$y = \frac{x}{x+1}$	$P_1, P_{12}, P_{15}, P_{18}, P_{22}$	P_{18}	P_{22}	P_6	P_1, P_6, P_{12}, P_{15}
λ_{17}	$y = \frac{x+a}{x+1}$	$P_3, P_9, P_{16}, P_{18}, P_{22}$	P_{18}	P_{22}	P_6	P_3, P_6, P_9, P_{16}
λ_{18}	$y = \frac{x+b}{x+1}$	$P_4, P_{11}, P_{13}, P_{18}, P_{22}$	P_{18}	P_{22}	P_6	P_4, P_6, P_{11}, P_{13}
λ_{19}	$y = \frac{x}{x+a}$	$P_1, P_7, P_{16}, P_{18}, P_{23}$	P_{18}	P_{23}	P_{10}	P_1, P_7, P_{10}, P_{16}
λ_{20}	$y = \frac{x+1}{x+a}$	$P_4, P_5, P_{15}, P_{18}, P_{23}$	P_{18}	P_{23}	P_{10}	P_4, P_5, P_{10}, P_{15}
λ_{21}	$y = \frac{x+b}{x+a}$	$P_3, P_8, P_{13}, P_{18}, P_{23}$	P_{18}	P_{23}	P_{10}	P_3, P_8, P_{10}, P_{13}
λ_{22}	$y = \frac{x}{x+b}$	$P_1, P_8, P_{11}, P_{18}, P_{24}$	P_{18}	P_{24}	P_{14}	P_1, P_8, P_{11}, P_{14}
λ_{23}	$y = \frac{x+1}{x+b}$	$P_3, P_5, P_{12}, P_{18}, P_{24}$	P_{18}	P_{24}	P_{14}	P_3, P_5, P_{12}, P_{14}
λ_{24}	$y = \frac{x+a}{x+b}$	$P_4, P_7, P_9, P_{18}, P_{24}$	P_{18}	P_{24}	P_{14}	P_4, P_7, P_9, P_{14}

Table 2 (cont.)

Circle λ_i	Equation of λ_i	Points on λ_i	$\{Q\} = \lambda_i \cap \sigma_4$	$\{R\} = \lambda_i \cap \tau_4$	$\{S\} = [Q]_- \cap [R]_+$	Points on $\{S\} \cup \lambda_i^*$
λ_{25}	$y = \frac{ax+1}{x}$	$P_8, P_{10}, P_{13}, P_{19}, P_{21}$	P_{19}	P_{21}	P_3	P_3, P_8, P_{10}, P_{13}
λ_{26}	$y = \frac{ax+a}{x}$	$P_5, P_{12}, P_{14}, P_{19}, P_{21}$	P_{19}	P_{21}	P_3	P_3, P_5, P_{12}, P_{14}
λ_{27}	$y = \frac{ax+b}{x}$	$P_6, P_9, P_{16}, P_{19}, P_{21}$	P_{19}	P_{21}	P_3	P_3, P_6, P_9, P_{16}
λ_{28}	$y = \frac{ax}{x+1}$	$P_1, P_{10}, P_{16}, P_{19}, P_{22}$	P_{19}	P_{22}	P_7	P_1, P_7, P_{10}, P_{16}
λ_{29}	$y = \frac{ax+1}{x+1}$	$P_2, P_{12}, P_{13}, P_{19}, P_{22}$	P_{19}	P_{22}	P_7	P_2, P_7, P_{12}, P_{13}
λ_{30}	$y = \frac{ax+b}{x+1}$	$P_4, P_9, P_{14}, P_{19}, P_{22}$	P_{19}	P_{22}	P_7	P_4, P_7, P_9, P_{14}
λ_{31}	$y = \frac{ax}{x+a}$	$P_1, P_8, P_{14}, P_{19}, P_{23}$	P_{19}	P_{23}	P_{11}	P_1, P_8, P_{11}, P_{14}
λ_{32}	$y = \frac{ax+1}{x+a}$	$P_4, P_6, P_{13}, P_{19}, P_{23}$	P_{19}	P_{23}	P_{11}	P_4, P_6, P_{11}, P_{13}
λ_{33}	$y = \frac{ax+a}{x+a}$	$P_2, P_5, P_{16}, P_{19}, P_{23}$	P_{19}	P_{23}	P_{11}	P_2, P_5, P_{11}, P_{16}
λ_{34}	$y = \frac{ax}{x+b}$	$P_1, P_6, P_{12}, P_{19}, P_{24}$	P_{19}	P_{24}	P_{15}	P_1, P_6, P_{12}, P_{15}
λ_{35}	$y = \frac{ax+a}{x+b}$	$P_4, P_5, P_{10}, P_{19}, P_{24}$	P_{19}	P_{24}	P_{15}	P_4, P_5, P_{10}, P_{15}
λ_{36}	$y = \frac{ax+b}{x+b}$	$P_2, P_8, P_9, P_{19}, P_{24}$	P_{19}	P_{24}	P_{15}	P_2, P_8, P_9, P_{15}
λ_{37}	$y = \frac{bx+1}{x}$	$P_7, P_9, P_{14}, P_{20}, P_{21}$	P_{20}	P_{21}	P_4	P_4, P_7, P_9, P_{14}
λ_{38}	$y = \frac{bx+a}{x}$	$P_6, P_{11}, P_{13}, P_{20}, P_{21}$	P_{20}	P_{21}	P_4	P_4, P_6, P_{11}, P_{13}
λ_{39}	$y = \frac{bx+b}{x}$	$P_5, P_{10}, P_{15}, P_{20}, P_{21}$	P_{20}	P_{21}	P_4	P_4, P_5, P_{10}, P_{15}
λ_{40}	$y = \frac{bx}{x+1}$	$P_1, P_{11}, P_{14}, P_{20}, P_{22}$	P_{20}	P_{22}	P_8	P_1, P_8, P_{11}, P_{14}
λ_{41}	$y = \frac{bx+1}{x+1}$	$P_2, P_9, P_{15}, P_{20}, P_{22}$	P_{20}	P_{22}	P_8	P_2, P_8, P_9, P_{15}
λ_{42}	$y = \frac{bx+a}{x+1}$	$P_3, P_{10}, P_{13}, P_{20}, P_{22}$	P_{20}	P_{22}	P_8	P_3, P_8, P_{10}, P_{13}
λ_{43}	$y = \frac{bx}{x+a}$	$P_1, P_6, P_{15}, P_{20}, P_{23}$	P_{20}	P_{23}	P_{12}	P_1, P_6, P_{12}, P_{15}
λ_{44}	$y = \frac{bx+a}{x+a}$	$P_2, P_7, P_{13}, P_{20}, P_{23}$	P_{20}	P_{23}	P_{12}	P_2, P_7, P_{12}, P_{13}
λ_{45}	$y = \frac{bx+b}{x+a}$	$P_3, P_5, P_{14}, P_{20}, P_{23}$	P_{20}	P_{23}	P_{12}	P_3, P_5, P_{12}, P_{14}
λ_{46}	$y = \frac{bx}{x+b}$	$P_1, P_7, P_{10}, P_{20}, P_{24}$	P_{20}	P_{24}	P_{16}	P_1, P_7, P_{10}, P_{16}
λ_{47}	$y = \frac{bx+1}{x+b}$	$P_3, P_6, P_9, P_{20}, P_{24}$	P_{20}	P_{24}	P_{16}	P_3, P_6, P_9, P_{16}
λ_{48}	$y = \frac{bx+b}{x+b}$	$P_2, P_5, P_{11}, P_{20}, P_{24}$	P_{20}	P_{24}	P_{16}	P_2, P_5, P_{11}, P_{16}

$$\tau_2 = \{P_2, P_6, P_{10}, P_{14}, P_{18}\},\ \tau_3 = \{P_3, P_7, P_{11}, P_{15}, P_{19}\},$$
$$\tau_4 = \{P_4, P_8, P_{12}, P_{16}, P_{20}\},\ \tau_5 = \{P_{21}, P_{22}, P_{23}, P_{24}, P_{25}\},$$
$$\Pi = \{P_1, P_2, \ldots, P_{25}\},\ \Sigma_+ = \{\sigma_1, \ldots, \sigma_5\}\ \Sigma_- = \{\tau_1, \ldots, \tau_5\}.$$

The set of all circles not passing through $P_{25} = (\infty, \infty)$ and the construction of the line determined by a circle, are presented in Table 2. Now every class of congruent circles consists of four elements, e.g. $[\lambda_1]_{\cong} = \{\lambda_1, \lambda_{16}, \lambda_{34}, \lambda_{43}\}$. Hence the lines obtained from circles are straight, by Corollary 8. Identifying each class with the set of points belonging to each of its elements, we obtain

$$\alpha_1 = \{P_1, P_6, P_{12}, P_{15}\},\ \alpha_2 = \{P_1, P_7, P_{10}, P_{16}\},\ \alpha_3 = \{P_1, P_8, P_{11}, P_{14}\},$$
$$\alpha_4 = \{P_2, P_5, P_{11}, P_{16}\},\ \alpha_5 = \{P_3, P_5, P_{12}, P_{14}\},\ \alpha_6 = \{P_4, P_5, P_{10}, P_{15}\},$$
$$\alpha_7 = \{P_4, P_7, P_9, P_{14}\},\ \alpha_8 = \{P_2, P_8, P_9, P_{15}\},\ \alpha_9 = \{P_3, P_6, P_9, P_{16}\},$$
$$\alpha_{10} = \{P_3, P_8, P_{10}, P_{13}\},\ \alpha_{11} = \{P_4, P_6, P_{11}, P_{13}\},\ \alpha_{12} = \{P_2, P_7, P_{12}, P_{13}\}.$$

Further we have straight lines

$$\sigma_1^* = \{P_1, P_2, P_3, P_4\},\ \sigma_2^* = \{P_5, P_6, P_7, P_8\},$$
$$\sigma_3^* = \{P_9, P_{10}, P_{11}, P_{12}\}, \sigma_4^* = \{P_{13}, P_{14}, P_{15}, P_{16}\},$$
$$\tau_1^* = \{P_1, P_5, P_9, P_{13}\},\ \tau_2^* = \{P_2, P_6, P_{10}, P_{14}\},$$
$$\tau_3^* = \{P_3, P_7, P_{11}, P_{15}\},\ \tau_4^* = \{P_4, P_8, P_{12}, P_{16}\}.$$

Thus for $\Pi' = \Pi_{P_{25}} = \{P_1, \ldots, P_{16}\}$, $\Sigma'_+ = \{\sigma_1^*, \sigma_2^*, \sigma_3^*, \sigma_4^*\}$, $\Sigma'_- = \{\tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*\}$ and $\Lambda' = \{\alpha_1, \alpha_2, \ldots, \alpha_{12}\}$ we have obtained a Minkowski plane $\boldsymbol{M}' = (\Pi', \Sigma'_+, \Sigma'_-, \Lambda')$ of order 3. Really, axioms (M1)-(M3), (M5) are obvious. In order to prove (M4) note that every three points of $\Pi_{P_{25}}$, no two of which on common generator, uniquely determine the 4-th point and these four points constitute a circle α_i for some $i = 1, 2, \ldots, 12$ (see Figure 5).

Analogously as Corollary 9 we obtain

Corollary 10. *If $\boldsymbol{M}$ is a Minkowski plane of order 4 then for every point Z all lines of $\boldsymbol{M}^Z$ are straight and so $\boldsymbol{M}^Z$ is an affine plane.*

2.3. A Minkowski plane of order different from 3 and 4

Lemma 1. *If $\boldsymbol{M} = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ is a Minkowski plane of order at least 5 then for every $\lambda, \mu \in \Lambda$ there is $\lambda \cong \mu \Leftrightarrow \lambda = \mu$.*

Proof. Of course, if $\lambda = \mu$ then $\lambda \cong \mu$. Fix a point Z and suppose that $\lambda \cong \mu$ for some circles λ, μ, where $Z \notin \lambda \cup \mu$. Let us denote

$$[Z]_+ \cap \lambda = \{P^+\}, [Z]_- \cap \lambda = \{P^-\}, [Z]_+ \cap \mu = \{Q^+\}$$
$$[Z]_- \cap \mu = \{Q^-\}, [P^+]_- \cap [P^-]_+ = \{P\}, [Q^+]_- \cap [Q^-]_+ = \{Q\}$$

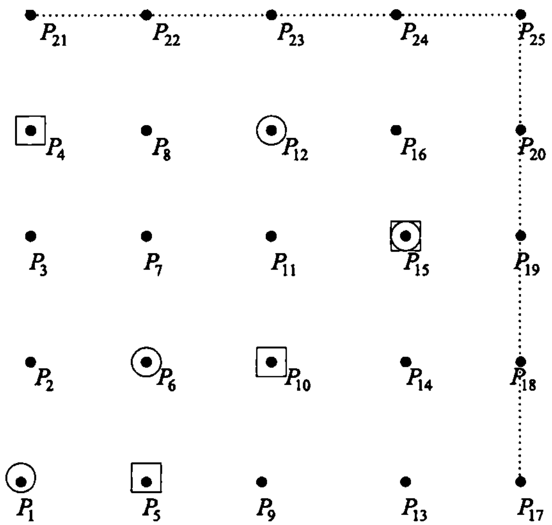

Fig. 5

Then $\{P\} \cup \lambda^* = \{Q\} \cup \mu^*$ as $\lambda \cong \mu$. Let $P^+ \neq Q^+$, $P^- \neq Q^-$, whence $P \in \mu^*$, $Q \in \lambda^*$. But, by the assumption of order at least 5, there are at least three (+)generators different from $[P]_+, [Q]_+, [Z]_+$ (see Figure 6) and points of λ lying on them must be the same as points of μ (otherwise $\{P\} \cup \lambda^* \neq \{Q\} \cup \mu^*$). In virtue of (M4) we have $\lambda = \mu$. If, for example $P^+ = Q^+$ then there are at least four (+)generators on which points of λ and μ must be the same.

LEMMA 2. *No point Z in a Minkowski plane M of order n, where $n \neq 3$ and $n \neq 4$, can induce a nearaffine plane M^Z which is also a Minkowski plane.*

Proof. It is trivial for $n = 2$ so let $M = (\Pi, \Sigma_+, \Sigma_-, \Lambda)$ be a Minkowski plane of order at least 5. We shall show that then M^Z cannot satisfy (M4). Let $\lambda \in \Lambda$, $Z \notin \lambda$ and fix a point $Q_1 \in \lambda \cap \Pi_Z$. We have two possibilities:

a) There exist distinct points $Q_2, Q_3 \in \Pi_Z$ such that $\lambda \cap (Q_1^+, Q_1^-, Q_2) = \{Q_2, Q_3\}$. Because of Lemma 1, circles λ and $\mu = (Q_1^+, Q_1^-, Q_2)$ are in distinct classes with respect to $\cong$. Thus in M^Z there would exist two distinct circles $\{P\} \cup \lambda^*$ (where $[P]_+ \cap \lambda = [Z]_- \cap \lambda$ and $[P]_- \cap \lambda = [Z]_+ \cap \lambda$) and $\{Q_1\} \cup (Q_1^+, Q_1^-, Q_2)^*$ containing Q_1, Q_2, Q_3.
b) Points $Q_2, Q_3 \in \Pi_Z$ such that $\lambda \cap (Q_1^+, Q_1^-, Q_2) = \{Q_2, Q_3\}$ do not exist. It means that for every $Q \in \lambda$, $Q \neq Q_1$, a circle (Q_1^-, Q_1^+, Q) is tangent to λ at Q (see Figure 7). But in the projective derrived

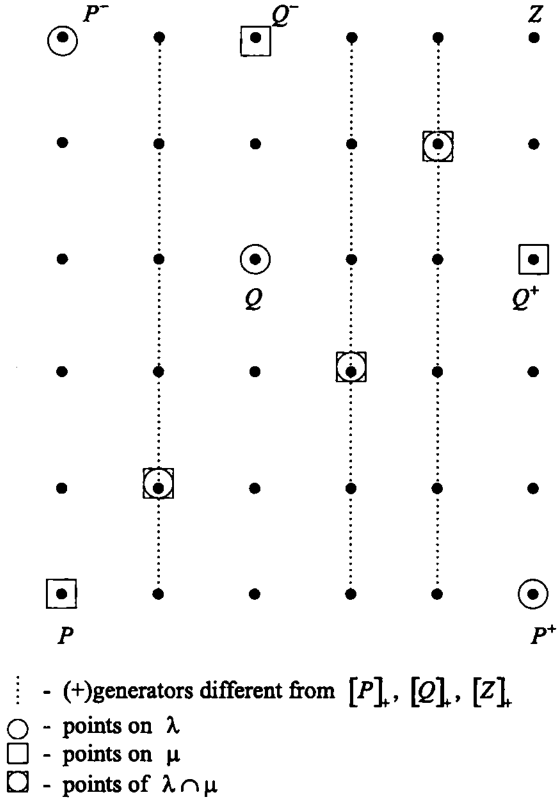

Fig. 6

plane $\overline{M_{Q_1^-}}$ such a circle becomes a tangent of an oval obtained from λ (see Proposition 1), and Q_1^+ is on this tangent. It is well known that the number of tangents of an oval passing through a point R not belonging to the oval, is 0 or 2, unless all tangents contain R ([4, p.10]). By the assumption $n \geq 5$, we have that all tangents of an oval obtained from λ should contain Q_1^+. Let $[Q_1^-]_- \cap \lambda = [Z]_- \cap \lambda = \{S\}$. In particular $p = ([S]_+ \cap \Pi_{Q_1^-}) \cup \{\infty_1\}$ is a tangent (see the proof of Proposition 1) but $[Z]_+ = [Q_1^+]_+ \neq [S]_+$ so $Q_1^+ \notin p$, a contradiction.

From Lemma 2, Corollary 5, and our consideration in 2.1 and 2.2 the following characterization is immediate.

THEOREM 2. *A Minkowski plane M is of order 3 or of order 4 if and only if every point Z induces a nearaffine plane M^Z, which is also a Minkowski plane.*

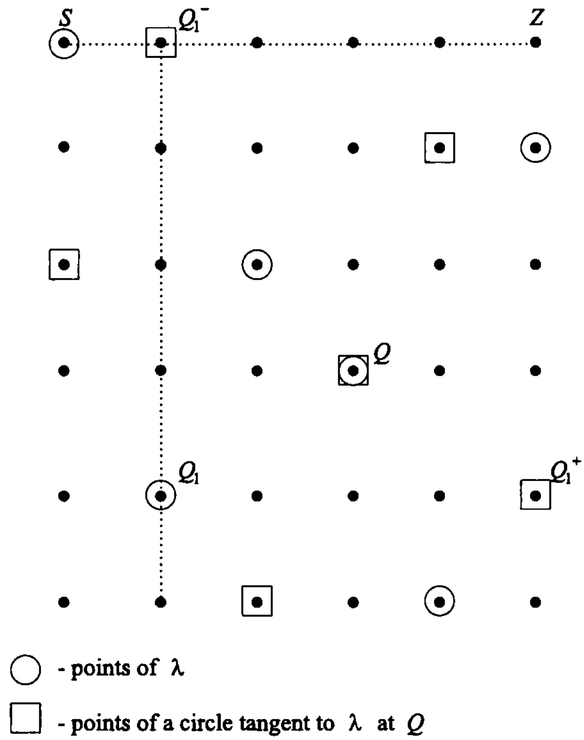

Fig. 7

REMARK 2. *The "true" reason of our result obtained in Theorem 2 is quite trivial. Simply, it is caused by very small order.*

One can trivialy show that if M is a Minkowski plane of order 2 then for every point Z, M^Z is an affine plane. Thus, using Lemma 1, Corollary 8 and Corollary 4, we obtain the following characterization of Minkowski planes of order 2, 3, and 4.

COROLLARY 11. *For any Minkowski plane M and a point Z a residual plane M^Z is an affine plane if and only if M is of order* 2, 3 *or* 4.

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