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ON REFLEXIVITY OF SCALAR-TYPE SPECTRAL OPERATORS

Introduction

The aim of the present paper is to give a dual version of the reflexivity result of scalar-type spectral operators in the quasi-complete locally convex spaces proved in [4].The method in our result is based on the barrelled locally convex $C(K)$-modules. Let $C(K)$ be the set of all continuous real or complex valued functions defined on a compact Hausdorff space K with the supremum norm.Suppose that X is a locally convex Hausdorff space. By $L^*(X)$ we denote the set of all linear mappings from X into X and by $L(X)$ the set of all continuous linear mappings from X into X. We denote by I the identity operator. The topological dual of X will be denoted by X'. We denote by X'' the topological dual of $X'[\beta(X',X)]$. We say that X is a locally convex $C(K)$-module if the bilinear mapping $\cdot : C(K) \times X \to X$, $(a,x) \to a.x$ satisfies the following conditions:

(i) $1.x = x$ for all $x \in X$, $1 \in C(K)$,
(ii) $(a.b).x = a.(b.x)$, for each $a, b \in C(K)$, $x \in X$,
(iii) The bilinear mapping $\cdot$ is separately continuous.

We say that X is a barrelled locally convex $C(K)$-module if X is a locally convex $C(K)$-module and barrelled space. Let X be a barrelled locally convex $C(K)$-module. Hence, it is accomplished by the following bilinear mappings:

$$(A) \quad X \times X' \to C(K)', \quad (x,x') \to \mu_{x,x'}(a) = x'(a.x), \quad a \in C(K),$$
$$(B) \quad C(K)'' \times X' \to X', \quad (a,x') \to (a.x')(x) = a(\mu_{x,x'}), \quad x \in X.$$

Throughout this paper, X will denote a barrelled locally convex Hausdorff space. Therefore, X' is $\sigma(X',X)$ quasi-complete, [8, 6.1]. The bilinear mapping $\cdot : C(K) \times X \to X$ defines a mapping $m : C(K) \to L(X)$, $m(a)x = a.x$, which is norm to strong operator topology continuous unital algebra homomorphism. Since $C(K)$ is an AM-space with unit, it is isomorphic to $C(S)$ with S a hyperstonian [1, Theorem 15.7], [6], [7]. It is well-known that X' is a locally convex $C(S)$-module [5], [6]. From [5],[6],[7], it is known that the linear mapping $m^* : C(K)'' \to L(X')$ defined by $m^*(a)x' = a.x'$, for each $a \in C(K)''$, $x' \in X'$ satisfies the following properties:

(i) For each $a \in C(K)''$, $m^*(a)$ is continuous from $X'[\sigma(X', X)]$ into $X'[\sigma(X', X)]$.

(ii) m^* is an algebra homomorphism.

(iii) For each $a \in C(K)$, $m^*(a) = (m(a))^*$, where $(m(a))^*$ is the adjoint of $m(a)$.

Let B be the Boolean algebra of projections in $C(S)$. For each $x' \in X'$, let $\mathrm{Ker}(x') = \{a \in C(S) : a.x' = 0\}$. We say that a projection $e_{x'} \in B$ is a carrier projection for $x' \in X'$ if $(1 - e_{x'})C(S) = \mathrm{Ker}(x')$. For unexplained notion and terminology we refer to [1], [9], [10].

LEMMA 1 [7, Lemma 1]. *For each $x' \in X'$ the carrier projection $e_{x'}$ exists in $C(S)$.*

Proof. Since S is hyperstonian, B is complete as an abstract Boolean algebra . Let D be the idempotents in $Ker(x')$ and let $e_1 = \vee D$ in B. Since D is directed upwards, $D \uparrow e_1$ in the order of $C(S)$. On the other hand $C(K)'$ consists of order continuous linear functionals on $C(S)$. So $D \to e_1$ in $\sigma(C(K)'', C(K)')$. Therefore, $e_1 x' = 0$. So $e_1 \in D$ and $e_{x'} = 1 - e_1$ is the carrier projection of x'. Also $e_1 C(S) \subset \mathrm{Ker}(x')$. Suppose $ax' = 0$ and $a(t) \neq 0$ for some $t \in S$. Then there is an open-closed neighbourhood of t on which $|a|$ is bounded away from zero. Let $e \in B$ be the characteristic function of that open-closed set. Then $aex' = 0$ and a is invertible on the support of e. From here we deduce that $ex' = 0$. Hence, the support of a is contained in the support of e_1 and $\mathrm{Ker}(x') \subset e_1 C(S)$.

We introduce the following proposition which we shall need later.

PROPOSITION 2 [2, Proposition 2.3]. *Let M be a Riesz space which is a linear subspace of the Hausdorff quasi-complete topological vector space X, such that the induced topology in M is locally solid. Then the closure $\overline{M}$ of M in X is equal to the completion of M.*

In $L^*(X')$ and $L(X')$ we consider the w^* operator topology. Once again, let B be the Boolean algebra of projections in $C(S)$ and then note that by the bilinear mapping (B), $m^*(B) = P$ is an equicontinuous Boolean algebra of projections in X', i.e., P is an equicontinuous collection of mutually commuting idempotents in $L(X')$, partially ordered by range inclusion, which is a Boolean algebra with respect to the lattice operations defined by $E \wedge F = EF$ and $E \vee F = E + F - EF$ for $E, F \in P$. We denote by A the linear span of P in $L(X')$. It should be observed that P is strongly equicontinuous Boolean algebra in $L(X')$, i.e., $\{E_n\}$ converges to zero in $L(X')$ for any disjoint sequence $\{E_n\}$ in P, [5]. An element $T \in A$ has a unique standard representation $T = \sum_{i=1}^{n} \lambda_i E_i$ where $E_i \neq 0$ in P are such that $E_i E_j = 0$ $(i \neq j)$, $\sum_{i=1}^{n} E_i = I$ and $\lambda_1, \ldots, \lambda_n$ are mutually different scalars. It is clear that A has the complex Riesz space structure, where the absolute value of $T = \sum_{i=1}^{n} \lambda_i E_i$ is given by $|T| = \sum_{i=1}^{n} |\lambda| E_i$. Furthermore, with respect to the topology induced by $L(X')$, A is a complex locally solid Riesz space [2, Lemma 3.2], [5]. By $\overline{A}$ we denote the w^*-closed subalgebra of $L(X')$ generated by P. Moreover, $\overline{A}^*$ denotes the w^* closure of A in $L^*(X')$ equipped with the w^* operator topology. Therefore, by Proposition 2 the lattice operations extend to $\overline{A}^*$. It is seen that the w^*-closure $\overline{A}^*$ of A in $L^*(X')$ is a Dedekind complete f-algebra with a complete, locally convex solid Lebesgue topology, [2, Proposition 3.6], [5]. With respect to composition as multiplication, $\overline{A}^*$ is a complex commutative f-algebra, [2, Proposition 4.2]. It is clear that $\overline{A}$ is a subalgebra of $\overline{A}^*$. We claim that $\overline{A}$ is an order ideal in $\overline{A}^*$, from which follow the expected properties of $\overline{A}$, i.e., $\overline{A}$ has the structure of a Dedekind complete, complex f-algebra such that the relative w^* operator topology in $\overline{A}$ is complete, locally solid and Lebesgue. For this, suppose that $|S| \leq |T|$ in $\overline{A}^*$ with $T \in \overline{A}$. Then there exists $R \in \overline{A}^*$ with $|R| \leq I$ such that $S = RT$, [3, Proposition 2.3]. It follows from [2, Proposition 3.11] that $R \in \overline{A}$ and therefore $S \in \overline{A}$.

Fix $x' \in X'$ and put $A(x') = \{Tx' : T \in A\}$. The mapping $x' \to Tx'$ from A onto $A(x')$ induces the structure of a complex Riesz space in $A(x')$, and the topology induced by X' in $A(x')$ is locally solid, [2, Lemma 3.3], [5]. By $\overline{A(x')}$ we denote the w^* closure of $A(x')$ in X'. Therefore, $\overline{A(x')}$ has the structure of a complex Dedekind complete Riesz space with a complete locally convex solid Lebesgue topology, [2, Proposition 3.9], [5].

Let R be a collection of linear operators on X.We denote by $AlglatR$ the algebra of all operators on X which leave left invariant all R-invariant closed subspaces of X.

In this paper,we will be concerned with densely defined linear operators $T : D(T) \to X$, where the domain $D(T)$ is a linear subspace of X. If T and S are such operators, then we write $T \subseteq S$ if $D(T) \subseteq D(S)$ and $Tx = Sx$ for all $x \in D(T)$.We recall again that $m^*(B)$ is a strongly equicontinuous Boolean algebra of projections in X' and $\overline{A}$ is the w^*-closed subalgebra of $L(X')$ generated by $m^*(B) = P$. We introduce the following similar definitions which can be found in [4].

DEFINITION 3. Let X be a barrelled locally convex $C(K)$-module. The densely defined linear operator T is said to have an adjoint spectral element T' with respect to B if there exists a sequence $\{e_n\} \subseteq B$ such that

(i) $e_n \uparrow 1$ in B,

(ii) $m^*(e_n)(X') \subseteq D(T')$ and $T'm^*(e_n) \in \overline{A}$ for all $n \in N$,

(iii) $T'x' = \lim_{n\to\infty} T'm^*(e_n)x'$ for all $x' \in D(T')$.

Such a sequence (e_n) will be called determining for T'. Note that each adjoint spectral element T' is densely defined. It is easy to check that if $(e_n) \subseteq B$ is a determining sequence for T', and if $(f_n) \subseteq B$ is a sequence such that $f_n \leq e_n$ for all $n \in N$ and $f_n \uparrow 1$ holds in B, then (f_n) is a determining sequence for T'. We see that if S and T are adjoint spectral elements with respect to B then there exists a sequence $(e_n) \subseteq B$ which is determining for S and T simultaneously.

Adjoint spectral elements T, S are called equivalent ($T \sim S$) if there exists a sequence $(e_n) \subseteq B$ with $e_n \uparrow 1$ such that $m^*(e_n)(X') \subseteq D(T) \bigcap D(S)$ and $Tm^*(e_n) = Sm^*(e_n) \in \overline{A}$ for all $n \in N$. It is easy to see that the given relation is reflexive, symmetric and transitive. So $\sim$ is an equivalence relation. By $[T]$ we denote the equivalence class of adjoint spectral elements determined by the adjoint spectral element T. It is easy to see that adjoint spectral elements T and S are equivalent if and only if there exists an adjoint spectral element R such that $R \subseteq T$ and $R \subseteq S$.

PROPOSITION 4 [4, Proposition 2.2]. *If T has an adjoint spectral element T' with respect to B, then there exists a unique maximal representative $H \in [T']$ satisfying the following properties*:

(i) *H is densely defined and w^* closed.*

(ii) *$D(H)$ is P-invariant and $H(e.x') = e.Hx'$ for all $e \in B$ and $x' \in D(H)$. In addition, if R is any representative of $[T']$ and if $(f_n) \subseteq B$ is any determining sequence for R, then*

(iii) *$m^*(f_n)(X') \subseteq D(H)$ and $Hm^*(f_n) = Rm^*(f_n)$ for all $n \in N$;*

(iv) *$x' \in D(H)$ if and only if $\lim_{n\to\infty} Rm^*(f_n)x'$ exists, in which case $Hx' = \lim_{n\to\infty} Rm^*(f_n)x'$.*

Proof. Suppose that $(e_n) \subset B$ is a determining sequence for the given adjoint spectral element T'. Define $D(H) = \{x' \in X' : \lim_n T'(e_n.x') \text{ exists}\}$ and if $x' \in D(H)$ define $Hx' = \lim_{n\to\infty} T'(e_n.x')$. After that, applying the same technique with [4, Proposition 2.2] we end the proof.

DEFINITION 5. Let T be a densely defined linear operator and suppose that T has an adjoint spectral element T' with respect to B. Then T' will be called an adjoint scalar-type spectral operator with respect to B if there exists a determining sequence (e_n) for T' such that $D(T') = \{x' \in X' : \lim_n T'(e_n.x') \text{ exists}\}$ and $T'x' = \lim_{n\to\infty} T'(e_n.x')$ for all $x' \in D(T')$.

We introduce the following dual version of [4, Corollary 2.4].

COROLLARY 6. *If T has an adjoint scalar-type spectral operator T' with respect to B, then the following assertions are true.*

(i) *T' is densely defined and w^*-closed.*

(ii) *$D(T')$ is P-invariant and $T'(e.x') = e.T'x'$ for all $x' \in D(T')$ and $e \in B$.*

(iii) *If $Y \subseteq X'$ is a $\sigma(X', X)$ closed P-invariant subspace of X', then $T'(D'(T') \bigcap Y) \subseteq Y$, i.e., $T' \in Alglatm^*(B)$.*

If S and T are linear operators in X' with domain $D(S)$ and $D(T)$ respectively, then $D(S+T) = D(S) \bigcap D(T)$ and $(S+T)x' = Sx' + Tx'$ for all $x' \in D(S+T)$, $D(ST) = \{x' \in D(T) : Tx' \in D(S)\}$ and $(ST)x' = S(Tx')$ for all $x' \in D(ST)$. The proof of the following result is straightforward, therefore it is omitted.

COROLLARY 7. *If S and T are adjoint scalar-type spectral operators with respect to B, then $S + T$ and ST are adjoint spectral elements.*

COROLLARY 8. *Let X be a barrelled locally convex $C(K)$-module. Then $\overline{A}^*$ consists precisely of the everywhere defined adjoint scalar-type spectral operators with respect to B.*

Proof. Take any $T \in \overline{A}^*$. We claim that there exists a sequence $\{E_n\}$ in P such that $E_n \uparrow I'$ and $|TE_n| \leq nI'$ for all n. Indeed, let $\{E_n\}$ be the component of the identity operator I' in the band generated by the operator $(nI' - |T|)^+$ in $\overline{A}^*$.Hence, the sequence $\{E_n\}$ has the properties that $E_n \uparrow I'$, $TE_n \in \overline{A}$ for all n and $\lim_n TE_n x' = Tx'$ for all $x' \in X'$.This implies that T is an adjoint scalar-type spectral operator with respect to B. Assume now that T is an everywhere defined scalar-type spectral operator. Then by the definition of the adjoint scalar-type spectral operator, there exists a sequence $\{e_n\} \subseteq B$ such that $Tm^*(e_n) \in \overline{A}$ for all n and $lim_n Tm^*(e_n)x' = Tx'$ for all $x' \in X'$. Therefore, $T \in \overline{A}$.

By A_∞ we denote the collection of all scalar-type spectral operators with respect to B.

DEFINITION 9. If S and T are adjoint scalar-type spectral operators with respect to B, then the adjoint scalar-type spectral operators $S \overline{\mp} T$ and $S.T$ are defined to be the maximal representatives of the corresponding classes $[S \mp T]$, $[ST]$.

With respect to defined operations A_∞ is a complex vector space and a commutative algebra with the identity operator I' as unit, which contains $\overline{A}$ as a subalgebra.

We gather some simple properties of the domains of adjoint scalar-type spectral operators.

LEMMA 10. *Let X be a barrelled locally convex $C(K)$-module and suppose that T is a densely defined linear operator with an adjoint scalar-type spectral operator $T', D(T') \subseteq X'$. Then the following are true.*

(i) *If $x' \in D(T')$ and if $S \in \overline{A}$, then $Sx' \in D(T')$ and $T'Sx' = ST'x'$.*

(ii) *If $x' \in X'$, then $D(T') \bigcap \overline{A(x')}$ is an order dense ideal in the Riesz space $\overline{A(x')}$.*

(iii) *If $x' \in X'$, then $\overline{A(x')}$ is T'-invariant, i.e., if $y \in D(T') \bigcap \overline{A(x')}$ then $T'y \in \overline{A(x')}$.*

Proof. Apply directly [4, Lemma 4.2].

We end this paper with the main theorem.

THEOREM 11. *Let X be a barrelled locally convex $C(K)$-module. Let T be a densely defined, closed linear operator in X with $m^*(B)$-invariant domain $D(T')$. Then the following statements are equivalent.*

(i) *$T' \in AlglatP$,*

(ii) *$T' \in A_\infty$, i.e., T' is an adjoint scalar-type spectral operator with respect to B.*

Proof. Apply [4, Corollary 5.8].

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