DEMONSTRATIO MATHEMATICA
Vol. XXXII No 2 1999

Ali A. Mehemmed

ALGEBRAIC PROOFS OF KWAPIEŃ'S AND LAMPERTI'S REPRESENTATION THEOREMS

Abstract. The main purpose of this paper is to give proofs of modification of Kwapień's representation theorem for continuous linear operators on L_0, and modification of the Lamperti representation theorem for linear operators between L_p spaces, where $1 \le p < \infty$, $p \neq 2$.

The previous proofs of these theorems were given, using the methods of functional analysis. In our paper we would like to show that they can be proved also by algebraic methods, with an aid of the theory of Boolean algebras and measure theory. In the proofs we use a theorem from [7].

1. Introduction

For definitions and standard results concerning Boolean algebras and measure theory the reader is referred to books of H. L. Royden [8], R. Sikorski [9] and P. R. Halmos [1]. We recall here the most important definitions and notation.

A measure space $(X, \boldsymbol{F}, \mu)$ is said to be *finite*, when $\mu(X) < \infty$.

DEFINITION 1.1. A Boolean σ-algebra $\boldsymbol{A}$ together with a measure μ such that

(i) $\mu(A) = 0$ iff $A = 0$;

(ii) $\mu(\bigvee_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_n \wedge A_m = 0$ for $n \neq m$

is called a *measure algebra.*

An element $A \neq 0$ in a measure algebra $\boldsymbol{A}$ is called an *atom* if $B \le A$ implies $B = A$ or $B = 0$.

A measure μ, as well as its corresponding measure algebra $\boldsymbol{A}$, is called *non-atomic* if whenever $F \in \boldsymbol{A}$ with $\mu(F) > 0$, there exists $E \in \boldsymbol{A}$ with $E \le F$ such that $\mu(F) > \mu(E) > 0$.

If $\boldsymbol{A}$ is a measure algebra and $A \in \boldsymbol{A}$, by $\boldsymbol{A}(A)$ we mean the restriction of the measure algebra $\boldsymbol{A}$ to A, i.e. $\boldsymbol{A}(A) = \{B \in \boldsymbol{A} : B \le A\}$.

DEFINITION 1.2. Let X and Y be two spaces and $\varphi : X \to Y$ be a point mapping. The mapping h from the family of all subsets of Y into the family of all subsets of X, defined by $h(E) = \varphi^{-1}(E), \forall E \subset Y$, is called a *set mapping induced by* φ.

If $(X_1, \boldsymbol{F}_1, \mu_1)$ and $(X_2, \boldsymbol{F}_2, \mu_2)$ are measure spaces, the point mapping $\varphi : X_1 \to X_2$ is called *measurable* if $\varphi^{-1}(E) \in \boldsymbol{F}_1$ for each $E \in \boldsymbol{F}_2$. And φ is called *non-singular*, if whenever $E \in \boldsymbol{F}_2$ and $\mu_2(E) = 0$, then $\mu_1(\varphi^{-1}(E)) = 0$.

DEFINITION 1.3. Let $(X_1, \boldsymbol{F}_1, \mu_1)$ and $(X_2, \boldsymbol{F}_2, \mu_2)$ be measure spaces and let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be the corresponding measure algebras. A set mapping $h : \boldsymbol{A}_2 \to \boldsymbol{A}_1$ is called a *homomorphism*, if

(i) $h(X_2) = X_1$;
(ii) $h(-E) = -h(E), \ \forall E \in \boldsymbol{A}_2$;
(iii) $h(E \vee F) = h(E) \vee h(F)$, for any $E, F \in \boldsymbol{A}_2$.

If, in addition, h has the property that for any sequence $\langle E_n \rangle$ in $\boldsymbol{A}_2$, $h(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} h(E_n)$, then h is called a *σ-homomorphism* from $\boldsymbol{A}_2$ into $\boldsymbol{A}_1$.

By $L_0(\mu)$ we mean, as usual, the set of all measurable functions with respect to the measure μ. If $f \in L_0(\mu)$, we put $\operatorname{supp} f = \{x : f(x) \neq 0\}$ (*the support of* f). If $E \subset X$ is measurable, then $L_0(E) = \{f : f$ is measurable and $\operatorname{supp} f \subset E\}$.

For $0 < p < \infty$ we define $L_p(\mu)$ to be the space of all functions f in $L_0(\mu)$ such that $\int_X |f(x)|^p d\mu(x) < \infty$ (i.e., $L_p(\mu) \subset L_0(\mu)$). For $p \geq 1$, $L_p(\mu)$ is a Banach space.

By L_0 and L_p we are referring to L_0 and L_p of the unit interval with Lebesgue measure.

A sequence $\langle f_n \rangle$ in $L_0(\mu)$ is said to *converge to f in measure* if, given $\varepsilon > 0$, there is an integer N such that for all $n \geq N$ we have $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$.

A norm $\|\cdot\|$ is called an *F-norm* if it satisfies $\|x + y\| \leq \|x\| + \|y\|$. If $\|\cdot\|$ is an F-norm, then $d(x, y) = \|x - y\|$ is an *invariant metric* on X. (So that $L_0(\mu)$ is a topological algebra, and a complete lattice).

A metrizable topological vector space (or metric linear space) is called an *F-space* if it is complete for an invariant metric (and hence for every invariant metric).

Thus, by V. L. Klee [10], a metric linear space which is complete in any metric will also be complete in any invariant metric. By the inequality

$$\frac{\alpha + \beta}{1 + \alpha + \beta} \leq \frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta}, \quad \forall \alpha, \beta \geq 0$$

the real-valued function

$$\|f\|_0 = \int \frac{|f|}{1+|f|} d\mu, \quad \forall f \in L_0(\mu)$$

becomes an F-norm, from which $L_0(\mu)$ is an F-space with norm $\|\cdot\|_0$, and $\|f_n - f\|_0 \to 0$ iff $f_n \to f$ in measure.

DEFINITION 1.4. A function $T : X \to Y$, where X and Y are vector spaces (over the same field K of scalars) is called a *linear operator* if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, where $x, y \in X$ and $\alpha, \beta \in K$. If X and Y are two topological vector spaces and T is continuous, then T is called a continuous linear operator.

The set of all continuous linear operators $T : X \to Y$ is denoted by $\mathcal{L}(X, Y)$, and $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$.

2. Some properties of continuous linear operators

One of the most important properties of the continuous linear operators on the F-space $L_0(\mu)$ will be given in Lemma 2.3. And now we will show that the space $L_0(\mu)$ is a complete lattice.

Let us mention that the proofs of Lemmas 2.1 and 2.2 have been published in [3], but they were also included in the author's Ms.Sc. thesis (1983) (not published).

LEMMA 2.1. *Let $(X, \boldsymbol{A}, \mu)$ be a finite measure space. If $\boldsymbol{B} = \{g_\alpha : \alpha \in I\} \subset L_0(\mu)$, then there exists an extended real-valued function h such that*

(i) *$g_\alpha \le h$ μ-almost everywhere (a.e.), $\forall \alpha \in I$*

(ii) *if $g_\alpha \le f$ μ-a.e. for some f in $L_0(\mu)$, then $h \le f$ μ-a.e. and there is a sequence $\langle g_{\alpha_n} \rangle$ in $L_0(\mu)$ with $h = \bigvee_{n=1}^{\infty} g_{\alpha_n} = \sup\{g_{\alpha_n} : n \in N\}$.*

Proof. Case 1. Let $g_\alpha : X \to [0, 1]$.

Let $\beta = \sup\{\int (g_{\alpha_1} \vee g_{\alpha_2} \vee \ldots \vee g_{\alpha_n}) d\mu : g_{\alpha_i} \in \boldsymbol{B}, 1 \le i \le n \in \boldsymbol{N}\}$. For every $n \in \boldsymbol{N}$ there is $h_n = (g_{\alpha_1} \vee g_{\alpha_2} \vee \ldots \vee g_{\alpha_n})$ with $\int h_n \nearrow \beta$.

Put $h = \bigvee\limits_{n=1}^{\infty} h_n$ (note that (*) $\int (g_\alpha \vee h_n) \le \beta$, $\forall n$). We claim that $g_\alpha \le h$ μ–a.e., $\forall \alpha \in I$. Indeed, suppose $g_\alpha > h$ for some $\alpha \in I$ on a set of positive measure, i.e. $g_\alpha - h > \varepsilon$ on a set E with $\mu(E) > 0$, for some $\varepsilon > 0$. Then

$$\begin{aligned}\int (g_\alpha \vee h_n) &= \int_E (g_\alpha \vee h_n) + \int_{-E} (g_\alpha \vee h_n) \ge \int_E g_\alpha + \int_{-E} h_n \\ &> \int_E h_n + \int_E \varepsilon + \int_{-E} h_n = \int h_n + \varepsilon\mu(E) \to \beta + \varepsilon\mu(E) > \beta\end{aligned}$$

which contradicts (*). Thus $g_\alpha \le h$ $\mu - a.e.$ $\forall \alpha \in I$.

Case 2. If $\{g_\alpha : \alpha \in I\}$ is an arbitrary subset of $L_0(\mu)$, let us define a mapping $\varphi : (-\infty, \infty] \to (0,1]$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } x = \infty \\ \frac{1}{2}\left(\frac{x}{1+|x|}\right) + \frac{1}{2} & \text{otherwise.} \end{cases}$$

Let $h = \sup\{\varphi \circ g_\alpha : \alpha \in I\}$, so that, by the Case 1:

(a) if $g_\alpha \le f$ μa.e. for some f, then $\varphi^{-1} \circ h = \sup g_\alpha \le f$ μ-a.e.

(b) $\varphi^{-1} \circ h = \sup\{g_\alpha : \alpha \in I\} \ge g_\alpha$ μ-a.e., and also there is a sequence $\langle g_{\alpha_n} \rangle$ such that $\bigvee_{n=1}^{\infty} \varphi \circ g_{\alpha_n} = h$. Thus $\bigvee_{n=1}^{\infty} g_{\alpha_n} = \varphi^{-1} \circ h$ and hence $\boldsymbol{B}$ has the required properties (i) and (ii). ∎

Every σ-finite measure algebra is complete (see [9], p.73).

LEMMA 2.2. *Suppose* $(\boldsymbol{A},\mu)$ *is a measure algebra of* σ*-finite measure, and* $S = \{E_\alpha : \alpha \in I\}$ *is a subcollection in it. Then there exists a least upper bound of* S *in* $\boldsymbol{A}$.

Proof. Let $\boldsymbol{B} = \{\chi_{E_\alpha} : \alpha \in I\} \subseteq L_0(\mu)$. By Lemma 2.1. there exists a least upper bound for $\boldsymbol{B}$ in $L_0(\mu)$. Let $h = \bigvee \boldsymbol{B} = \bigvee\{\chi_{E_\alpha} : \alpha \in I\}$. Then there is a sequence $\langle \chi_{E_{\alpha_n}} \rangle$ in $\boldsymbol{B}$ such that $h = \bigvee_{n=1}^{\infty} \chi_{E_{\alpha_n}} = \chi_{\bigcup_{n=1}^{\infty} E_{\alpha_n}} = \chi_E$, where $E = \bigcup_{n=1}^{\infty} E_{\alpha_n}$, from which $\chi_{E_\alpha} \le \chi_E$ for each $\alpha \in I$, and hence $E_\alpha \le E$. Also, if $E_\alpha \le F$ for all $\alpha \in I$, then $\chi_{E_\alpha} \le \chi_F$ for all $\alpha \in I$. Thus $\chi_E \le \chi_F$ which implies $E \le F$. Therefore, E is the least upper bound of $S = \{E_\alpha : \alpha \in I\} \subset \boldsymbol{A}$. We denote E by $\bigvee E_\alpha$ or $\bigvee S$. ∎

LEMMA 2.3. *Suppose* $T \in \mathcal{L}(L_0)$ *and* $\langle f_n \rangle$ *is a sequence of functions in* L_0 *such that* $m(\operatorname{supp} f_n) \to 0$. *Then* $m(\operatorname{supp} T(f_n)) \to 0$.

Proof. Let $\boldsymbol{A}$ be a measure algebra induced by the measure space $([0,1], \boldsymbol{F}, m)$. Let $E \in \boldsymbol{A}$, and $\alpha \in \boldsymbol{R}$. Then

$$\|\alpha\chi_E\|_0 = \int_0^1 \frac{|\alpha|\chi_E}{1+|\alpha|\chi_E}dm = \int_E \frac{|\alpha|}{1+|\alpha|}dm = \frac{|\alpha|}{1+|\alpha|}m(E) \to$$
$$\to m(E) \text{ when } \alpha \to \pm\infty,$$

and therefore $\sup_{\alpha\in\boldsymbol{R}} \|\alpha\chi_E\| = m(E)$.

Now, let $f \in L_0$. Then

$$\|\alpha f\|_0 = \int_0^1 \frac{|\alpha f|}{1+|\alpha f|}dm = \int_0^1 \frac{|\alpha f|}{1+|\alpha f|}\chi_{\operatorname{supp} f}dm \le \int_0^1 \chi_{\operatorname{supp} f}dm = m(\operatorname{supp} f)$$

and so

$$\sup_{\alpha\in\boldsymbol{R}} \|\alpha f\|_0 \le m(\operatorname{supp} f). \tag{1}$$

CLAIM: $\sup_{\alpha\in\boldsymbol{R}} \|\alpha f\|_0 = m(\operatorname{supp} f)$.

For each $n \in \boldsymbol{N}$, let $E_n = \{x \in [0,1] : |f(x)| \geq \frac{1}{n}\}$ so that the sequence $\langle E_n \rangle$ is increasing and $E_n \nearrow \operatorname{supp} f$, and we have $m(E_n) \nearrow m(\operatorname{supp} f)$. But $|f| \geq \frac{1}{n}\chi_{E_n}$. Thus $\|\alpha f\|_0 \geq \|\frac{\alpha}{n}\chi_{E_n}\|_0$ and therefore $\sup_{\alpha\in\boldsymbol{R}} \|\alpha f\|_0 \geq \sup \|\frac{\alpha}{n}\chi_{E_n}\|_0 = m(E_n)$.

Hence

$$m(\operatorname{supp} f) = \lim_{n\to\infty} m(E_n) \leq \sup_{\alpha\in\boldsymbol{R}} \|\alpha f\|_0. \tag{2}$$

From (1) and (2) we have $\|\alpha f\|_0 = m(\operatorname{supp} f)$, $\forall f \in L_0$.

Now suppose $m(\operatorname{supp} f_n) \to 0$, but $m(\operatorname{supp} T(f_n)) \not\to 0$. By passing to a subsequence, if necessary, we may assume that $m(\operatorname{supp} f_n) \to 0$, but $m(\operatorname{supp} T(f_n)) > \varepsilon$ for some $\varepsilon > 0$, which means that

$$\forall_{n\in N} \quad \sup_{\alpha\in\boldsymbol{R}} \|T(\alpha f_n)\|_0 > \varepsilon.$$

For each $n \in \boldsymbol{N}$, there exists $\alpha_n \in \boldsymbol{R}$ such that $\|T(\alpha_n f_n)\|_0 = \|\alpha T(f_n)\|_0 > \varepsilon$. So, we have $\|\alpha_n f_n\|_0 = m(\operatorname{supp} f_n) \to 0$, but $\|T(\alpha_n f_n)\|_0 \not\to 0$ which contradicts the continuity of T. ■

LEMMA 2.4. *Let $(f_i)_{i=1}^n$ be a finite sequence in L_0, and let $A_i = \operatorname{supp} f_i$ $(1 \leq i \leq n)$. Then there exists a sequence $\langle c_i \rangle_{i=1}^n$ of scalars such that if $f = \sum_{i=1}^n c_i f_i$ then $m(\bigcup_{i=1}^n A_i \setminus \operatorname{supp} f_i) = 0$.*

Proof. It suffices to prove the result for the case $n = 2$, and apply induction. For any $\alpha \in \boldsymbol{R}$, let

$$\begin{aligned} B_\alpha &= A_2 \setminus \operatorname{supp}(f_1 + \alpha f_2) = A_2 \cap \{x \in [0,1] : (f_1 + \alpha f_2)(x) = 0\} \\ &= \{x \in [0,1] : f_2(x) \neq 0, \text{ and } \alpha f_2(x) = -f_1(x)\}. \end{aligned}$$

If $\alpha \neq \beta$ in $\boldsymbol{R}$, then $\alpha f_2(x) \neq \beta f_2(x)$, or otherwise $f_2(x) = 0$. So that $B_\alpha \cap B_\beta = \emptyset$, since if $x \in B_\alpha$, then $x \notin B_\beta$. Hence the collection $\mathfrak{U} = \{B_\alpha : \alpha \in \boldsymbol{R}\}$ is an uncountable collection of pairwise disjoint subsets of A_2, and since the Lebesgue measure m is σ-finite there is a set $B_{\alpha_0} \in \mathfrak{U}$ such that $m(B_{\alpha_0}) = 0$. Thus if $x \in ((A_1 \cup A_2) \setminus \operatorname{supp}(f_1 + \alpha f_2))$ then $x \in B_{\alpha_0}$, because $A_2 \setminus \operatorname{supp}(f_1 + \alpha_0 f_2) = \{x \in A_2 : (f_1 + \alpha_0 f_2)(x) = 0\} = B_{\alpha_0}$. Therefore $m((A_1 \cup A_2) \setminus \operatorname{supp}(f_1 + \alpha_0 f_2)) = 0 = m(B_{\alpha_0})$. ■

LEMMA 2.5 ([6]). *Suppose $T \in \mathcal{L}(L_0)$, and let $h : \boldsymbol{A} \to \boldsymbol{A}$ be a set function defined by $h(E) = \sup\{\operatorname{supp} T(f) : \operatorname{supp} f \subset E\}$, for all E in the measure algebra $\boldsymbol{A}$ induced by the Lebesgue measure space $([0,1], \boldsymbol{F}, m)$. Then the set mapping h satisfies the following conditions*:

i) *$h(A \cup B) = h(A) \cup h(B)$ for A, B in $\boldsymbol{A}$;*
ii) *if $\langle E_n \rangle$ is a sequence in $\boldsymbol{A}$ and $m(E_n) \to 0$, then $m(h(E_n)) \to 0$;*
iii) *$h(\bigvee_{\alpha\in I} E_\alpha) = \bigvee_{\alpha\in I} h(E_\alpha)$.*

Proof. i) Suppose $f \in L_0$ such that $\operatorname{supp} f \subset A \cup B$ in $\boldsymbol{A}$. Then there are two functions $f_1, f_2 \in L_0$ such that $f_1 + f_2 = f$ with $\operatorname{supp} f_1 \subset A$ and $\operatorname{supp} f_2 \subset B$. Therefore $h(A \cup B) \subset h(A) \cup h(B)$. Now if $E_1 \subset E_2$, then $h(E_1) \subset h(E_2)$ by definition of h. So that for any sets A and B, $h(A) \subset h(A \cup B)$ and $h(B) \subset h(A \cup B)$. Hence $h(A) \cup h(B) \subset h(A \cup B)$. Thus $h(A \cup B) = h(A) \cup h(B)$.

ii) Suppose $\langle E_n \rangle$ is a sequence in $\boldsymbol{A}$ such that $m(E_n) \to 0$, but $m(h(E_n)) \nrightarrow 0$; by passing, if necessary, to a subsequence, we may assume that there is a positive real number α such that $m(h(E_n)) > \alpha$ for all $n \in \boldsymbol{N}$. So for each $n \in \boldsymbol{N}$ there exist $f_1, f_2, \dots, f_{k_n}$ in L_0 with $\operatorname{supp} f_i \subset E_n$ (all $1 \le i \le k_n$) and such that $m(\bigcup_{i=1}^{k_n} \operatorname{supp} T(f_i)) > \alpha$. By Lemma 2.4. above, there is a function $g_n = \sum\limits_{i=1}^{k_n} c_i f_i$ with $m(\operatorname{supp} T(g_n)) > \alpha$, and $g_n \in L_0$.

So we have $\operatorname{supp} g_n = \operatorname{supp}(\sum_{i=1}^{k_n} c_i f_i) \subset \bigcup_{i=1}^{k_n} \operatorname{supp} f_i \subset E_n \to \emptyset$, i.e. $m(\operatorname{supp} g_n) \to 0$, but $m(\operatorname{supp} T(g_n)) \nrightarrow 0$, which contradicts Lemma 2.3.

iii) Since $\bigvee_{\alpha \in I} E_\alpha = \sup_{\alpha \in I} E_\alpha$, then $E_\alpha \subset \bigvee_{\alpha \in I} E_\alpha$, for each $\alpha \in I$, from which $h(E_a) \subset h(\bigvee_{\alpha \in I} E_\alpha)$. Thus $\bigvee_{\alpha \in I} h(E_\alpha) \subset h(\bigvee_{\alpha \in I} E_\alpha)$. The existence of a sequence $\langle E_{\alpha_n} \rangle$ with $\bigcup_{n=1}^{\infty} E_{\alpha_n} = \bigvee_{\alpha \in I} E_\alpha$ is guaranteed by Lemma 2.2, so showing that $h(\bigvee_{\alpha \in I} E_\alpha) = \bigvee_{\alpha \in I} h(E_\alpha)$ is equivalent to showing that $h(\bigcup_{n=1}^{\infty} E_{\alpha_n}) = \bigcup_{n=1}^{\infty} h(E_{\alpha_n})$.

Since $h(E_{\alpha_n}) \subset h(\bigcup_{n=1}^{\infty} E_{\alpha_n})$ implies $\bigcup_{n=1}^{\infty} h(E_{\alpha_n}) \subset h(\bigcup_{n=1}^{\infty} E_{\alpha_n})$, we need only to show that $h(\bigcup_{n=1}^{\infty} E_{\alpha_n}) \subset \bigcup_{n=1}^{\infty} h(E_{\alpha_n})$. To this end, let $E = \bigcup_{n=1}^{\infty} E_{\alpha_n}$. By (i) we have

$$h(E) = h\Big(E \setminus \bigcup_{i=1}^{n} E_{\alpha_i}\Big) \cup h\Big(\bigcup_{i=1}^{n} E_{\alpha_i}\Big) = h\Big(E \setminus \bigcup_{i=1}^{n} E_{\alpha_i}\Big) \cup \Big(\bigcup_{i=1}^{n} h(E_{\alpha_i})\Big).$$

But then

$$h(E) \setminus \bigcup_{i=1}^{n} h(E_{\alpha_i}) \subset h(E \setminus \bigcup_{i=1}^{n} E_{\alpha_i})$$

for all $n \in \boldsymbol{N}$. Since

$$E \setminus E_{\alpha_1} \supset E \setminus (E_{\alpha_1} \cup E_{\alpha_2}) \supset \dots \supset E \setminus \bigcup_{i=1}^{n} E_{\alpha_i},$$

we have $m(E \setminus \bigcup_{i=1}^{n} E_{\alpha_i}) \searrow 0$. Therefore, by (ii),

$$m(h(E \setminus \bigcup_{i=1}^{n} E_{\alpha_i})) \searrow 0,$$

which implies

$$m(h(E)\setminus\bigcup_{i=1}^{\infty}h(E_{\alpha_i}))=0.$$

This means that

$$h(E)\subset\bigcup_{i=1}^{\infty}h(E_{\alpha_i})\text{ a.e.},$$

i.e. $h(\bigcup_{n=1}^{\infty}E_{\alpha_n})\subset\bigcup_{n=1}^{\infty}h(E_{\alpha_n})$, which was to be proved. ∎

Now we recall and apply a theorem describing the structure of set mappings between measure algebras, proved by the author in a previous paper [7].

THEOREM 2.6 ([7]). *Let $(X_1,\boldsymbol{F}_1,\mu_1)$ and $(X_2,\boldsymbol{F}_2,\mu_2)$ be two finite non-atomic measure spaces and let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be the corresponding measure algebras of μ_1 and μ_2 respectively. Suppose $h:\boldsymbol{A}_1\to\boldsymbol{A}_2$ is a mapping satisfying the following conditions*:

(i) *If $\langle E_n\rangle$ is a decreasing sequence in $\boldsymbol{A}_1$ with $\mu_1(E_n)\to 0$, then $\mu_2(h(E_n))\to 0$;*

(ii) *$E_1,E_2\in\boldsymbol{A}_1$ implies $h(E_1\cup E_2)=h(E_1)\cup h(E_2)$.*

Then there is a sequence of pairwise disjoint $\langle A_n\rangle$ in A_1 such that $h(X_1)=\bigcup_{n=1}^{\infty}A_n$, and for each $n\in N$, there is a partition $\{E_{1n},\dots,E_{k_nn}\}$ of X such that the mapping $h_{in}(E)=A_n\cap h(E)$, $\forall E\in A_1(E_{in})$ is a σ-homomorphism of $\boldsymbol{A}_1(E_{in})$ into $\boldsymbol{A}_2(A_n\cap h(E_{in}))$.

THEOREM 2.7. *Let A and B be two measurable subsets of $[0,1]$ of a positive measure. Suppose $T:L_0(A)\to L_0(B)$ is a continuous linear operator, and T satisfies the following conditions*:

(i) $T(1)=1$, *i.e.* $T(\chi_A)=\chi_B$;

(ii) *if* $\mu(\operatorname{supp}f_1\cap\operatorname{supp}f_2)=0$ *then* $\mu(\operatorname{supp}T(f_1)\cap\operatorname{supp}T(f_2))=0$.

Then there exists a non-singular measurable function $\varphi:B\to A$ such that $T(f)=f\circ\varphi$ for each $f\in L_0(A)$.

THEOREM 2.8 ([6]). *Suppose $T\in\mathcal{L}(L_0(A),L_0(B))$ where A,B are subsets of $[0,1]$ of positive measure such that if $\mu(\operatorname{supp}f_1\cap\operatorname{supp}f_2)=0$, $f_1,f_2\in L_0(A)$, then $\mu(\operatorname{supp}T(f_1)\cap\operatorname{supp}T(f_2))=0$. Then there exists a function $g\in L_0(B)$ and a non-singular measurable mapping $\varphi:\operatorname{supp}g\to A$ such that $T(f)=gf\circ\varphi$ for each $f\in L_0(A)$.*

Proof. Let us consider $g=T(\chi_A)$. We claim that $\operatorname{supp}T(f)\subset\operatorname{supp}g$ for any $f\in L_0(A)$. Since g is the image of the whole space $L_0(A)$ under continuous linear operator T, then for each measurable subset E of A, $T(\chi_E)+T(\chi_{-E})=g$, and $\mu(\operatorname{supp}T(\chi_E)\cap\operatorname{supp}T(\chi_{-E}))=0$. Therefore

$\operatorname{supp} T(\chi_E) \subset \operatorname{supp} g$. And hence $\operatorname{supp} T(\psi) \subset \operatorname{supp} g$ for any simple function ψ in $L_0(A)$. Thus $\operatorname{supp} T(f) \subset \operatorname{supp} g$, $\forall f \in L_0(A)$. Now define a new operator $T' : L_0(A) \to L_0(\operatorname{supp} g)$ by $T'(f) = \frac{1}{g}\chi_{\operatorname{supp} g} T(f)$. Therefore T' is also a continuous linear operator, and $T'(1) = T'(\chi_A) = \frac{1}{g}\chi_{\operatorname{supp} g} T(\chi_A) = 1$ on $\operatorname{supp} g$; also if $f_1, f_2 \in L_0(A)$ with $\mu(\operatorname{supp} f_1 \cap \operatorname{supp} f_2) = 0$, then

$$\operatorname{supp} T'(f_i) = \operatorname{supp}\left(\frac{1}{g}\chi_{\operatorname{supp} g} T(f_i)\right) \subset \operatorname{supp} T(f_i), i = 1, 2,$$

so that

$$\mu(\operatorname{supp} T'(f_1) \cap \operatorname{supp} T'(f_2)) \le \mu(\operatorname{supp} T(f_1) \cap \operatorname{supp} T(f_2)) = 0,$$

from which $T'(f) = f \circ \varphi$ (by Theorem 2.7.) for some non-singular measurable function $\varphi : \operatorname{supp} g \to A$. Thus $\frac{1}{g}T(f) = f \circ \varphi$ and $T(f) = gf \circ \varphi$. ∎

3. A modification of Kwapień's theorem

In [4], S. Kwapień has presented his theorem on the form of a linear operator in the space of all measurable functions as follows:

THEOREM 3.1 ([4]). *Let $(E, \mathfrak{M}, \lambda)$ be a standard measure space and $(F, \mathfrak{N}, \mu)$ a measure space. A linear operator $T : S(E, \mathfrak{M}, \lambda) \to S(F, \mathfrak{N}, \mu)$ is continuous iff T is of the form $T(f)(t) = \sum_{i=1}^{\infty} \varphi_i(t) f(\Phi_i(t))$, where*

I^o $\langle \varphi_i \rangle$ is a sequence of elements of $S(F, \mathfrak{N}, \mu)$ such that $\mu(\{t : \varphi_i(t) \neq 0$ for infinitely many $i\}) = 0$.

II^o $\langle \Phi_i \rangle$ is a sequence of mappings from F into E such that for $\forall i$ and $\forall A \in \mathfrak{M}$, $\Phi_i^{-1}(A) \in \overline{\mathfrak{N}}$ (the completion of $\mathfrak{N}$), and $\Phi_i^{-1}(A) \cap \{t : \varphi_i(t) \neq 0\}$ is a set of μ-measure zero whenever A is a set of λ-measure zero.

THEOREM 3.2 ([6]). *If $T \in \mathcal{L}(L_0)$, then $T(f) = \sum\limits_{n=1}^{\infty} \sum\limits_{i=1}^{k_n} g_{in} f \circ \varphi_{in}$ for each $f \in L_0$, where*

1. *$\langle A_n \rangle$ is a collection of pairwise disjoint subsets of $[0,1]$ of positive measure;*
2. *$\{E_{1n}, \dots, E_{k_n n}\}$ is a partition of $[0,1]$ into the sets of positive measure;*
3. *$\operatorname{supp} g_{in} \subset A_n$, i.e. $g_{in} \in L_0(A_n)$ for all $\in N$;*
4. *$\varphi_{in} : \operatorname{supp} g_{in} \to E_{in}$.*

In particular (*) $\quad T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n, \quad$ where

(i) each $\varphi_n : \operatorname{supp} g_n \to [0,1]$ is a non-singular measurable mapping;
(ii) each $g_n \in L_0$;
(iii) for almost all x in $[0,1]$, $g_n(x) \neq 0$ for only finitely many n.

Conversely, every map defined in the above way (*) is a continuous linear operator on L_0. (See also Theorem 8.4. in [3], p.171.)

P r o o f. Let $E \in \boldsymbol{A}$, where $\boldsymbol{A}$ is the measure algebra on $[0,1]$ with Lebesgue measure m. Define $h : \boldsymbol{A} \to \boldsymbol{A}$ by $h(E) = \sup\{\operatorname{supp} T(f) : \operatorname{supp} f \subset E\}$. From Lemma 2.5 we know that h satisfies the following:

i) $h(E_1 \cup E_2) = h(E_1) \cup h(E_2)$ for $E_1, E_2 \in \boldsymbol{A}$, and

ii) if $\langle E_n \rangle$ is a sequence in $\boldsymbol{A}$ and $m(E_n) \to 0$, then $m(h(E_n)) \to 0$.

Moreover, it is known that the Lebesgue measure m is finite and non-atomic. Therefore, by Theorem 2.6, there is a sequence $\langle A_n \rangle$ of pairwise disjoint elements in $\boldsymbol{A}$, so that $\bigcup_{n=1}^{\infty} A_n = h([0,1])$, and for each $n \in \boldsymbol{N}$ there is a partition $\{E_{1n}, \ldots, E_{k_n n}\}$ of $[0,1]$ such that for all $E \in \boldsymbol{A}(E_{in})$, the mapping $h_{in}(E) = A_n \cap h(E)$ is a σ-homomorphism of $\boldsymbol{A}(E_{in})$ into $\boldsymbol{A}(A_n \cap h(E_{in}))$. Also, if $A, B \in \boldsymbol{A}(E_{in})$ with $A \cup B = E_{in}$ and $A \cap B = \emptyset$, we have $m(h_{in}(A) \cap h_{in}(B)) = m(A_n \cap h(A) \cap h(B)) = 0$. Now let $f_1, f_2 \in L_0(E_{in})$ with $\operatorname{supp} f_1 \subset A$ and $\operatorname{supp} f_2 \subset B$; then

$$m(\operatorname{supp} T(f_1) \cap \operatorname{supp} T(f_2)) \leq m(h_{in}(A) \cap h_{in}(B)) = 0.$$

At this point we apply Theorem 2.4. Hence for each $n \in N$ and each $1 \leq i \leq k_n$ there exists a function $g_{in} \in L_0(A_n)$ with $\operatorname{supp} g_{in} \subset A_n$, so that $\chi_{A_n} T(f) = g_{in}(f \circ \varphi_{in})$ for all $f \in L_0(E_{in})$, where $\varphi_{in} : \operatorname{supp} g_{in} \to E_{in}$ is a non-singular measurable function. Hence for all $f \in L_0$, $\chi_{A_n} T(f) = \sum_{i=1}^{k_n} g_{in} f \circ \varphi_{in}$. Thus

$$T(f) = \chi_{\bigcup\limits_{n=1}^{\infty} A_n} T(f) = \sum_{n=1}^{\infty} \chi_{A_n} T(f) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} g_{in} f \circ \varphi_{in}.$$

Now we are going to show the particular representation form $T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n$ where g_n and φ_n as in conditions above.

Since $\{(g_{in}, \varphi_{in}) : n \in N,\ 1 \leq i \leq k_n\}$ is countable, we may write this sequence as the range of a sequence $\langle (g_n, \varphi_n) \rangle$.

Now if $x \in A_k$, then $g_{im}(x) \neq 0$ only if $m = k$, because $\operatorname{supp} g_{in} \subset A_m$, and $\langle A_m \rangle$ is a sequence of pairwise disjoint sets.

Therefore $|\{x \in N : g_n(x) \neq 0\}| \leq k$ for some k. Thus $T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n$ for all $f \in L_0$, and for each $x \in [0,1]$, $g(x) \neq 0$ for only finitely many n.

At last, we show the converse. Suppose $T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n$, $\forall f \in L_0$, and the conditions (i)–(iii) hold. Let $B_n = \{x \in [0,1] : g_k(x) = 0$ for each $k > n\}$. So that $B_1 \subset B_2 \subset \ldots$ and $m\left(\bigcap_{n=1}^{\infty}(-B_n)\right) = 0$, i.e. $m(-B_n) \searrow 0$.

To show that T is continuous suppose $f_k \to 0$ in L_0. Let $\varepsilon > 0$, by condition (iii) there exists n such that $m(-B_n) < \varepsilon/2$. Now $\chi_{B_n} T(f) = \sum_{i=1}^{n} g_i(f_k \circ \varphi_i) \to 0$ as $k \to \infty$, so there exist N such that whenever $k \geq N$, $\|\chi_{B_n} T(f_k)\|_0 < \varepsilon/2$. Also,

$$\|\chi_{(-B_n)} T(f_k)\|_0 \leq m(\operatorname{supp} \chi_{(-B_n)}) = m(-B_n) < \varepsilon/2.$$

Therefore, for each $k \geq N$ we have

$$\|T(f_k)\|_0 = \|\chi_{B_n} T(f_k)\|_0 + \|\chi_{(-B_n)} T(f_k)\|_0 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus if $f_n \to 0$ in L_0, we have $T(f_n) \to 0$, and hence $T \in \mathcal{L}(L_0)$. ■

4. A modification of Lamperti's theorem

The original Lamperti's theorem ([5]) reads as follows:

THEOREM 4.1 ([5]). *Let U be a linear operator on a space L_p for some positive $p \neq 2$, such that*

(i) $\|U(f)\|_p = \|f\|_p$ *for all* $f \in L_p$.

Then there exists a regular set-isomorphism T and a function h such that U is given by

(ii) $(Uf)(x) = h(x)T(f(x))$.

If a measure μ^* is defined by $\mu^*(A) = \mu(T^{-1}(A))$, then

(iii) $|h(x)|^p = \frac{d\mu^*}{d\mu}$ *a.e. on* $T(x)$.

Conversely, for any regular S-isomorphism T and any h satisfying (ii) *the operator U defined by* (ii) *satisfies* (i).

We will construct a similar form of $T(f)$ as in the Kwapień representation theorem for continuous linear operator. And also, in our modification of Lamperti's theorem, we will use the same technique as in the proof of the modification of Kwapień's theorem. In this modification we choose the algebra $\boldsymbol{A}$ (where $\operatorname{supp} f \in \boldsymbol{A}$ for each $f \in L_p$) to be finite non-atomic measure space $([0,1], \boldsymbol{A}, \mu)$. Recall that $L_p(\mu)$, $1 \leq p < \infty$ was defined to be the space of all functions $f \in L_0(\mu)$ such that $\int |f(x)|^p d\mu(x) < \infty$, and that for $p \geq 1$, $L_p(\mu)$ is a Banach space.

Now we state and prove our modification of Lamperti's theorem.

THEOREM 4.2 ([6]). *Let $1 \leq p < \infty$, $p \neq 2$, and let $([0,1], \boldsymbol{A}, \mu)$ be a non-atomic finite measure space. Let $T : L_p \to L_p$ be a linear operator which preserves norm (i.e. $\|T(f)\|_p = \|f\|_p$). Then there are measurable mappings $\varphi_n : \operatorname{supp} g_n \to A_n$ (where $\langle A_n \rangle$ is a partition of* $[0,1]$) *such that*

$$T(f) = \sum_{n=1}^{\infty} g_n(f \circ \varphi_n) \text{ for all } f \in L_p(\mu) \text{ on } [0,1].$$

Proof. We know that any function $f \in L_p(\mu)$ has a norm

$$\|f\|_p = \Big[\int_X |f|^p d\mu\Big]^{1/p} < \infty \quad \text{for every } 0 < p < \infty.$$

Let $\boldsymbol{A}$ be the non-atomic finite measure algebra on $[0,1]$ induced by the non-atomic finite measure μ such that $\operatorname{supp} f \subset \boldsymbol{A}$ for each $f \in L_p(\mu)$ on $[0,1]$. Define the set mapping $h : \boldsymbol{A} \to \boldsymbol{A}$ by $h(E) = \sup\{\operatorname{supp} T(f) : \operatorname{supp} f \subset E\}$.

Since μ is finite non-atomic measure, then by Lemma 2.5, h satisfies

i) for any decreasing sequence $\langle E_n \rangle$ in $\boldsymbol{A}$ if $\mu(E_n) \to 0$, then $\mu(h(E_n)) \to 0$;

ii) $h(A \cup B) = h(A) \cup h(B)$ for any $A, B \in \boldsymbol{A}$.

But since T is a linear operator of L_p into itself, then by Theorem 2.6. there is a sequence $\langle A_n \rangle$ of pairwise disjoint sets in $\boldsymbol{A}$ as a partition of $[0,1]$ such that $h([0,1]) = \bigcup_{n=1}^{\infty} A_n$. So that the mapping $h_n(E) = A_n \cap h(E)$ for all $E \in \boldsymbol{A}(A_k)$ is a σ-homomorphism from $\boldsymbol{A}(A_k)$ into $\mathrm{A}(A_n \cap h(A_k))$.

Also, if $A, B \in \boldsymbol{A}(A_k)$ with $A \cup B = A_k$ and $A \cap B = \emptyset$, we have $\mu(h_n(A) \cap h_n(B)) = \mu(A_n \cap h(A) \cap h(B)) = 0$.

Now, let $f_1, f_2 \in \boldsymbol{A}(A_k)$ with $\operatorname{supp} f_1 \subset A$ and $\operatorname{supp} f_2 \subset B$, so that $\mu(\operatorname{supp} T(f_1) \cap \operatorname{supp} T(f_2)) \le \mu(h_n(A) \cap h_n(B) = 0$. So, by Theorem 2.8, for each $n \in \boldsymbol{N}$ there is a function $g_n \in L_p(A_n)$ with $\operatorname{supp} g_n \subset A_n$, and a non-singular measurable mapping $\varphi_n : \operatorname{supp} g_n \to A_n$, such that $\chi_{A_n} T(f) = g_n f \circ \varphi_n$, $\forall f \in L_p(A_n)$. Hence for any $f \in L_p([0,1])$ we have

$$T(f) = \chi_{\bigcup\limits_{n=1}^{\infty} A_n} T(f) = \sum_{n=1}^{\infty} \chi_{A_n} T(f) = \sum_{n=1}^{\infty} g_n f \circ \varphi_n. \ \blacksquare$$

Acknowledgments. The author expresses his gratitude to Professor M. Mączyński and Professor T. Traczyk for their help and advice during the preparation of this paper. The author also thanks dr W. Zarębski for his help in typesetting the paper in TEX and revisions in the final version of this paper.

References

[1] P. R. Halmos, *Measure Theory*, New York, Van Nostrand, 1950.

[2] N. J. Kalton, *The endomorphism of L_p $(0 \le p < 1)$*, Indiana University Math. J. 27 (1978a), 353–381.

[3] N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-space Sampler*, Cambridge University Press, 1984.

[4] S. Kwapień, *On a form of a linear operator in the space of all measurable functions*, Bull. Acad. Polon. Sci. 21 (1973), 951–954.

[5] J. Lamperti, *On the isometries of certain function space*, Pacific J. Math. 8 (1958), 459–466.

[6] A. A. Mehemmed, *On analogies between the theory of homomorphisms in Boolean algebras and the theory of linear operators*, Doctoral thesis, Warsaw University of Technology, Warsaw, Poland, 1997.
[7] A. A. Mehemmed, *A theorem on the structure of set mappings between measure algebras*, Demonstratio Math. 32 (199), 13–20.
[8] H. L. Royden, *Real Analysis*, Second Edition, Macmillan Publishing Co. inc. New York, 1963.
[9] R. Sikorski, *Boolean Algebras*, Second Edition, Springer Verlag, Berlin-Göttingen-Heidelberg-New York, 1964.
[10] V. L. Klee, *Invariant metrics in groups*, Proc. Amer. Math. Soc. 3 (1952), 484–487.

WARSAW UNIVERSITY OF TECHNOLOGY
INSTITUTE OF MATHEMATICS
Plac Politechniki 1
00-661 WARSZAWA, POLAND

Mail address of the author
THE UNIVERSITY OF EL-FATEH
P.O.Box 13611
TRIPOLI, LIBYA

Received June 26, 1998.