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SOME FIXED POINT THEOREMS FOR SET-VALUED MAPPINGS IN UNIFORM SPACES

1. Introduction

Let X be a complete Hausdorff space whose uniformity is generated by a family of pseudometrics $\{d_i : i \in I\}$, and let 2^X be its hyperspace equipped with the Hausdorff uniformity induced by $\{d_i : i \in I\}$. A general fixed point theorem for $F : X \to 2^X$ is proved. Then examples show that this theorem includes known fixed point theorems and also yields a new theorem.

A fixed point theorem for multi-valued contraction mappings was proved for the first time by S.B. Nadler [2]. Since then, many authors have given generalizations of this theorem in various forms, such as the one given by R. Węgrzyk. R. Węgrzyk has applied fixed point theorems to the proof of multi-valued functions and functional equations [5].

Let $(X, \mathcal{U})$ be a uniform space. A family $\{d_i : i \in I\}$ of pseudometrics on X with indexing set I, is called an *associated family for the uniformity* $\mathcal{U}$ if the family

$$\beta = \{V(i,r) : i \in I, r > 0\}$$

where

$$V(i,r) = \{(x,y) : x, y \in X,\ d_i(x,y) < r\}$$

is a subbase for the uniformity $\mathcal{U}$. We may assume that β itself is a base by adjoining finite intersection of members of β, if necessary. The corresponding family of pseudometrics is called an *augmented associated family* for $\mathcal{U}$. An associated family for $\mathcal{U}$ will be denoted by p^*. For details the reader is referred to Taraftar [3] and Thron [4].

Let A be a nonempty subset of a uniform space X. Define

$$\Delta^*(A) = \sup\{d_i(x,y) : x, y \in A,\ i \in I\},$$

where $\{d_i(x,y) : i \in I\} = p^*$. Then $\Delta^*(A)$ is called an *augmented diameter* of A. Further, A is said to be *p^*-bounded* if $\Delta^*(A) < \infty$. Let

$$2^X = \{A : A \text{ is a nonempty, closed and } p^*\text{-bounded subset of } X\}.$$

For any nonempty subsets A and B of X, define

$$\begin{aligned} d_i(x, A) &= \inf\{d_i(x, a) : a \in A,\ i \in I\}, \\ H_i(A, B) &= \max\{\sup_{a \in A} d_i(a, B),\ \sup_{b \in B} d_i(A, b)\} \\ &= \sup_{x \in X}\{|d_i(x, A) - d_i(x, B)|\}. \end{aligned}$$

It is well-known that on 2^X, H_i is a pseudometric, called the *Hausdorff pseudometric induced* by d_i, $i \in I$.

Let $(X, \mathcal{U})$ be a uniform space and let $U \in \mathcal{U}$ be an arbitrary entourage. For each subset A of X, define

$$U[A] = \{y \in X; (x, y) \in U \text{ for some } x \in A\}.$$

The *Hausdorff uniformity* $2^{\mathcal{U}}$ on 2^X is defined by the base $2^\beta = \{\widetilde{U} : U \in \mathcal{U}\}$, where

$$\widetilde{U} = \{(A, B) : A, B \in 2^X \text{ and } A \subseteq U[B],\ B \subseteq U[A]\}.$$

The augmented associated family p^* also induces a uniformity $\mathcal{U}^*$ on 2^X defined by the base $\beta^* = \{V^*(i, r) : i \in I,\ r > 0\}$, where

$$V^*(i, r) = \{(A, B) : A, B \in 2^X,\ H_i(A, B) < r\}.$$

The uniformities $2^{\mathcal{U}}$ and $\mathcal{U}^*$ on 2^X are uniformly isomorphic. The space $(2^X, \mathcal{U}^*)$ is thus a uniform space called the *hyperspace* of $(X, \mathcal{U})$.

The following theorem was proved in [5].

THEOREM 1 [5]. *If (Y, d) is complete metric space and $F : Y \to CL(Y)$ is a multi-valued function which fulfils the inequality $D(Fx, Fy) \leq \psi(d(x, y))$ for all x, y in X and some strictly increasing function ψ such that* $\lim_{k\to\infty} \psi^k(t) = 0$ *for every t, then*

(a) *for every $y_0 \in Y$ and for every fixed point $y \in Y$ of F there exists a sequence of iterates of F at y_0 which converges to y,*

(b) *if $\sum_{k=1}^{\infty} \psi^k(t) < \infty$, for $t > 0$, then the set of fixed points of F is nonempty.*

In this theorem $\psi : [0, \infty) \to [0, \infty)$, D is the Hausdorff metric and

$$CL(Y) = \{A : A \text{ is closed in } Y\}.$$

2. Main results

THEOREM 2. *Let* $(X,\mathcal{U})$ *be a complete Hausdorff uniform space defined by* $\{d_i : i \in I\} = p^*$ *and* $(2^X,\mathcal{U}^*)$ *a hyperspace, let* $F : X \to 2^X$ *be a continuous mapping and* Fx *compact for each* x *in* X. *Assume that*

$$H_i(Fx, Fy) \leq K(d_i(x,y)) \tag{1}$$

for all $i \in I$ *and* $x, y \in X$, *where* $K : [0,\infty) \to [0,\infty)$, $K(0) = 0$ *and* K *is non-decreasing. Then there exists* z *in* X *with* $z \in Fz$ *if and only if there exists* x_0 *in* X *with* $\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty$.

Note that in this theorem K is not assumed to be continuous and $K^n(t) = K(K^{n-1}(t))$.

Proof. If $z \in Fz$ then $d_i(z, Fz) = 0$, $0 = K(0) = K^2(0) = \ldots = K^n(0)\ldots$ for each $i \in I$ and $\sum_{n=1}^{\infty} K^n(d_i(z, Fz)) = 0$. Let $x_0 \in X$ and $x_1 \in Fx_0$ be arbitrary. Suppose that there exists x_0 such that $\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0)) < \infty$ for each $i \in I$. Let $U \in \mathcal{U}$ be an arbitrary entourage. Since β is a base for $\mathcal{U}$, there exists $V(i,r) \in \beta$ such that $V(i,r) \subseteq U$. Now $y \to d_i(x_0, y)$ is continuous on the compact set Fx_0 and this implies that there exists $x_1 \in Fx_0$ such that $d_i(x_0, x_1) = d_i(x_0, Fx_0)$. Similarly, Fx_1 is compact so there exists $x_2 \in Fx_1$ such that $d_i(x_n, x_{n+1}) = d_i(x_1, Fx_1)$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Fx_n$ and $d_i(x_n, x_{n+1}) = d_i(x_n, Fx_n)$. Noting that K is non-decreasing and using inequality (1), we have

$$\begin{aligned} d_i(x_n, x_{n+1}) &= d_i(x_n, Fx_n) \leq H_i(Fx_{n-1}, Fx_n) \\ &\leq K(d_i(x_{n-1}, x_n)) = K(d_i(x_{n-1}, Fx_{n-1})) \\ &\leq K(H_i(Fx_{n-2}, Fx_{n-1})) \\ &\leq K^2(d_i(x_{n-2}, x_{n-1}) \\ &\leq \ldots \\ &\leq K^n(d_i(x_0, x_1)) = K^n(d_i(x_0, Fx_0)). \end{aligned}$$

Hence we obtain

$$\begin{aligned} d_i(x_n, x_{n+m}) &\leq d_i(x_n, x_{n+1}) + d_i(x_{n+1}, x_{n+2}) + \ldots + d_i(x_{n+m-1}, x_{n+m}) \\ &\leq K^n(d_i(x_0, Fx_0)) + K^{n+1}(d_i(x_0, Fx_0)) + \ldots + \\ &\quad + K^{n+m-1}(d_i(x_0, Fx_0)) \\ &= \sum_{k=n}^{n+m-1} K^k(d_i(x_0, Fx_0)). \end{aligned}$$

Since $\sum_{n=1}^{\infty} K^n(d_i(x_0, Fx_0) < \infty$, it follows that there exists p such that $d_i(x_n, x_m) < r$ and hence $(x_n, x_m) \in U$ for all $n, m \geq p$. Therefore the sequence $\{x_n\}$ is Cauchy sequence in the d_i-topology on X.

Let $S_p = \{x_n : n \geq p\}$ for all positive integers p and let $\mathcal{B}$ be the filter basis $\{S_p : p = 1, 2, \ldots\}$. Then since $\{x_n\}$ is a d_i-Cauchy sequence for each $i \in I$, it is easy to see that the filter basis $\mathcal{B}$ is Cauchy filter in the uniform space $(X, \mathcal{U})$. To see this we first note that family $\{V(i, r) : i \in I\}$ is a base for $\mathcal{U}$ as $p^* = \{d_i : i \in I\}$. Now, since $\{x_n\}$ is a d_i-Cauchy sequence in X, there exists a positive integer p such that $d_i(x_n, x_m) < r$ for $m \geq p$, $n \geq p$. This implies that $S_p \times S_p \subset V(i, r)$. Thus given any $U \in \mathcal{U}$, we can find a $S_p \in \mathcal{B}$ such that $S_p \times S_p \subset U$. Hence $\mathcal{B}$ is a Cauchy filter in $(X, \mathcal{U})$. Since $(X, \mathcal{U})$ is complete Hausdorff space, the Cauchy filter $\mathcal{B} = \{S_p\}$ converges to a unique point $z \in X$. Since $F : X \to 2^X$ is continuous, $F(\mathcal{B}) \to Fz$. On the other hand, since

$$S_{p+1} \subseteq F(S_p) = \bigcup\{Fx_n : n \geq p\} \quad \text{for } p = 1, 2, \ldots,$$

then $F(S_p)$ in the filter induced from the filter base $\mathcal{B}$. In other words, the filter induced from $F(\mathcal{B})$ is coarser than that of $\mathcal{B}$. Therefore $F(\mathcal{B})$ converges to z. It follows that $z \in Fz$. Hence z is a fixed point of F. This completes the proof.

Remark 1. K needs only be defined on the range of d_i for all $i \in I$. If you replace the pseudometric with an equivalent pseudometric with $d_i(x, y) < 1$, then clearly Theorem 2 holds for $K : [0, 1) \to [0, \infty)$. To apply Theorem 2, one needs a non-decreasing function K and x in X with

$$\sum_{n=1}^{\infty} K^n(d_i(x, Fx) < \infty.$$

The following examples satisfy these conditions and therefore illustrate the generality of Theorem 2; let X denote complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$.

Example 1. Suppose $0 < \lambda_i < 1$. Let $K(t) = \lambda_i t$ for $t \geq 0$. Then $H_i(Fx, Fy) \leq K(d_i(x, y)) = \lambda_i d_i(x, y)$ and $K^n(d_i(x, Fx)) = \lambda_i^n d_i(x, Fx)$ for any x in X. It is known that there exists z with $z \in Fz$ without assuming that Fx is compact.

Example 2. Suppose that F satisfies $H_i(Fx, Fy) \leq \phi(d_i(x, y))d_i(x, y)$ for all x, y in X, where $\phi : [0, \infty) \to [0, 1)$ and ϕ is non-decreasing. Then $K(t) = t\phi(t)$, K is non-decreasing, and $K : [0, \infty) \to [0, \infty)$. It follows by induction that $K^n(t) \leq t[\phi(t)]^n$, since $\phi(t) < 1$ and $\sum_{n=1}^{\infty} K^n(t) < \infty$.

EXAMPLE 3. Consider $K(t) = t\phi(t)$, where $\phi : [0,\infty) \to [0,\infty)$, $\phi(t) \le t$ for $t \le 1$. If $t < 1$, it follows that $K^n(t) \le t[\phi(t)]^n$, If K is non-decreasing, then Theorem 2 can be applied.

EXAMPLE 4. $K(t) = t\phi(t)$, where $\phi : [0,\infty) \to [0,\infty)$, $\phi(\alpha_i t) \le \alpha_i\phi(t)$ for $\alpha_i \in (0,1)$. If $\phi(t) < 1$, then $K^n(t) \le K(t)(\phi(t))^n$ for all $n \ge 2$.

EXAMPLE 5. Assume that K is non-decreasing, K is convex on $[0,1)$ and $K(t) < t$ for all $0 < t < 1$. If $t < 1$, $K(t) < t$, then $K(t) = \alpha_i t$ for some $0 < \alpha_i < 1$. It can be shown that $K^n(t) \le \alpha_i^n t$ for all n and thus $\sum_{n=1}^{\infty} K^n(t) < \infty$.

THEOREM 3. *Let $(X,\mathcal{U})$ be a complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$, let $F : X \to 2^X$ be a multi-valued mapping and Fx compact for each x in X. Assume that $H_i(Fx,Fy) \le [d_i(x,y)]^q$, where $q > 1$, then F has a fixed point in X.*

Proof. Let $K(t) = t^q$ for $t \ge 0$. Then $K(0) = 0$ and K is increasing, $K(t) < t$ if $t < 1$ and K is convex. If $t = d_i(x,Fx) < 1$, then $\sum_{n=1}^{\infty} K^n(t) < \infty$ from the previous example. Also F is continuous, so Theorem 2 applies.

REMARK 2. If we replace the uniform space $(X,\mathcal{U})$ in Theorems 2 and 3 and Examples 1–5 by a metric space (i.e. a metrizable uniform space), then the results of Hicks [1] will follow as special cases of our results.

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