DEMONSTRATIO MATHEMATICA
Vol. XXXII No 2 1999

Marek Adamczak

ON COMPACTNESS OF $C^{(n)}$-ALMOST PERIODIC FUNCTIONS

Abstract. In this paper we give a compactness criterion for $C^{(n)}$-almost periodic functions. Some properties of these functions are shown.

1. Preliminaries

We first recall the basic notations related to Steklov functions and $C^{(n)}$ -almost periodic functions.

For a given positive number h and for a function $f : \mathbb{R} \to \mathbb{R}$ which is locally integrable, put

$$S_f(h)(u) = \frac{1}{2h} \int_{u-h}^{u+h} f(s)ds, \qquad u \in \mathbb{R}.$$

Then $S_f(h)$ is called *the Steklov function* for f.

Let $C^{(n)}(\mathbb{R})$ be the set of functions from $\mathbb{R}$ into itself with n-th continuous derivative.

It is easily seen that if $f \in C^{(n)}(\mathbb{R})$, then $S_f(h) \in C^{(n+1)}(\mathbb{R})$.

Let us put for $f \in C^{(n)}(\mathbb{R})$

$$D^{(n)}(f) = \sup_{t \in \mathbb{R}} \Big(|f(t)| + \sum_{k=1}^{n} |f^{(k)}(t)| \Big).$$

We say that an $f \in C^{(n)}(\mathbb{R})$ is $C^{(n)}$*-bounded* iff $D^{(n)}(f) < \infty$. Let $f_h(x) \equiv f(x+h)$. We say that f is *a* $C^{(n)}$*-continuous function* iff $\lim_{h\to 0} D^{(n)}(f - f_h) = 0$. A sequence (f_k) in $C^{(n)}(\mathbb{R})$ will be called $D^{(n)}$ *-convergent to* f iff $\lim_{k\to\infty} D^{(n)}(f - f_k) = 0$.

1991 *Mathematics Subject Classification*: 42A75.

Key words and phrases: uniformly almost periodic function, derivative of order n, indefinite integral, compactness.

A set $A \subset \mathbb{R}$ is called *relatively dense* iff there exists a positive number l such that in every open interval $(\alpha, \alpha + l)$, $\alpha \in \mathbb{R}$, there is at least one element of the set A. A number $\tau \in \mathbb{R}$ is called *a* $(D^{(n)}, \varepsilon)$*-almost period* ($(D^{(n)}, \varepsilon)$-a.p.) of a function $f \in C^{(n)}(\mathbb{R})$ iff $D^{(n)}(f - f_\tau) \leq \varepsilon$, for $\varepsilon > 0$. Let $E^{(n)}\{\varepsilon; f\}$ denote the set of $(D^{(n)}, \varepsilon)$-a.periods of f. A function $f \in C^{(n)}(\mathbb{R})$ is called $C^{(n)}$*-almost periodic* ($C^{(n)}$-a.p.) iff for each $\varepsilon > 0$ the set $E^{(n)}\{\varepsilon; f\}$ is relatively dense. By $\widetilde{C^{(n)}}$ we denote the set of $C^{(n)}$-a.p. functions.

Basic properties and examples of $C^{(n)}$-a.p. functions may be found in [1].

2. More properties of $C^{(n)}$-a.p. functions

In this section we prove some theorems on $C^{(n)}$-a.periodicity of functions and we give an example of a $C^{(1)}$-bounded and a $C^{(1)}$-continuous function which is not $C^{(1)}$-a.p.

REMARK 1. A function f is $C^{(n)}$-a.p. if and only if $f, f', \ldots, f^{(n)}$ are uniformly a.p. functions.

THEOREM 1. *The following statements hold:*

(i) *If f is a $C^{(n)}$-a.p. function, then $S_f(h)$ is a $C^{(n+1)}$-a.p. function.*
(ii) *If f is a $C^{(n)}$-continuous function, then* $\lim_{h\to 0} D^{(n)}(f - S_f(h)) = 0$.

Proof. (i) Let $f \in \widetilde{C^{(n)}}$. Then for any $t \in \mathbb{R}$ and for $\tau \in E^{(n)}\{\varepsilon h/(h+1); f\}$ with $h > 0$ we have

$$|S_f(h)(t) - S_{f_\tau}(h)(t)| \leq \varepsilon \frac{h}{h+1}.$$

Thus we obtain

$$\begin{aligned} &D^{(n+1)}(S_f(h) - S_{f_\tau}(h)) \\ &\quad \leq \sup_{t\in\mathbb{R}} |S_f(h)(t) - S_{f_\tau}(h)(t)| + \frac{1}{h} \sup_{t\in\mathbb{R}} \sum_{k=0}^{n} |f^{(k)}(t) - f_\tau^{(k)}(t)| \leq \varepsilon. \end{aligned}$$

This means that $E^{(n)}\{\varepsilon h/(h+1); f\} \subset E^{(n+1)}\{\varepsilon; S_f(h)\}$. Since $S_f(h) \in C^{(n+1)}(\mathbb{R})$, so $S_f(h) \in \widetilde{C^{(n+1)}}$.

(ii) For each $t \in \mathbb{R}$ we have

$$|f(t) - S_f(h)(t)| \leq \frac{1}{2h} \int_{-h}^{h} |f(t) - f(s+t)| ds,$$

$$|f^{(k)}(t) - S_f(h)^{(k)}(t)| \leq \frac{1}{2h} \int_{-h}^{h} \left| f^{(k)}(t) - \frac{\partial^k}{\partial t^k} f(s+t) \right| ds$$

for $k = 1, 2, \ldots, n$. Since f is $C^{(n)}$-continuous, so for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that $D^{(n)}(f - f_s) \leq \varepsilon$ for $s \in \mathbb{R}$ with $|s| < \delta$. Thus for all $t \in \mathbb{R}$ and $h > 0$ such that $|s| \leq h < \delta$ we obtain

$$D^{(n)}(f - S_f(h)) \leq \sup_{t\in\mathbb{R}} \left(\frac{1}{2h} \int_{-h}^{h} |f(t) - f(s+t)| ds + \frac{1}{2h} \sum_{k=1}^{n} \int_{-h}^{h} \left| f^{(k)}(t) - \frac{\partial^k}{\partial t^k} f(s+t) \right| ds \right) \leq \varepsilon.$$

This proves (ii). ∎

PROPOSITION 1. *If f is a $C^{(n)}$-continuous function and the indefinite integral $F(u) = \int_0^u f(s)ds$ for $u \in \mathbb{R}$ is $C^{(n)}$-a.p., then f is a $C^{(n)}$-a.p. function.*

Proof. For each $t \in \mathbb{R}$ we denote

$$G_m(t) = \frac{F(t + h_m) - F(t)}{h_m} \quad \text{for } h_m \neq 0,\ h_m \to 0 \text{ as } m \to \infty.$$

Let us observe that $G_m \in \widetilde{C^{(n)}}$, because $F \in \widetilde{C^{(n)}}$ (see [1], Th.3). We have

$$D^{(n)}(G_m - f) = \sup_{t\in\mathbb{R}}(I_1(t) + I_2(t)),$$

where

$$I_1(t) = \left| \frac{1}{h_m} \int_0^{h_m} (f(s+t) - f(t))ds \right| \leq \frac{1}{|h_m|} \int_0^{h_m} |f(s+t) - f(t)| ds = J_1(t)$$

and

$$I_2(t) = \sum_{k=1}^{n} \left| \frac{1}{h_m} \int_0^{h_m} \left(\frac{\partial^k}{\partial t^k} f(s+t) - f^{(k)}(t) \right) ds \right| \leq \sum_{k=1}^{n} \frac{1}{|h_m|} \int_0^{h_m} \left| \frac{\partial^k}{\partial t^k} f(s+t) - f^{(k)}(t) \right| ds = \sum_{k=1}^{n} J_2(k,t).$$

Moreover, since f is $C^{(n)}$-continuous, hence for an arbitrary $\varepsilon > 0$ there exist an $M > 0$ and a $\delta = \delta(\varepsilon) > 0$ such that $|h_m| < \delta$ for every $m > M$ and we have

$$\sup_{t\in\mathbb{R}} J_1(t) \leq \frac{\varepsilon}{n+1} \quad \text{and} \quad \sup_{t\in\mathbb{R}} J_2(k,t) \leq \frac{\varepsilon}{n+1}$$

for every $k = 1, 2, \ldots, n$. Therefore $D^{(n)}(G_m - f) \leq \varepsilon$ for every $m > M$. Thus the sequence (G_m) of $C^{(n)}$-a.p. functions is $D^{(n)}$-convergent to f. Finally, by Theorem 5 in [1], we get $f \in \widetilde{C^{(n)}}$. ∎

PROPOSITION 2. *If f' is a $C^{(n)}$-a.p. function and f is bounded, then f is a $C^{(n+1)}$-a.p. function.*

Proof. Since

$$f(t) = f(0) + \int_0^t f'(s)ds \quad \text{for } t \in \mathbb{R}$$

is bounded, so using Theorem 8 in [1] we obtain $f \in \widetilde{C^{(n+1)}}$. ∎

Now, we shall be occupied with $C^{(1)}$-a.periodicity of a superposition of functions. Analogously to theorem in [2], p. 429, there holds:

PROPOSITION 3. *If E is a set of values of a $C^{(1)}$-a.p. function g and f has a uniformly continuous and bounded derivative on E, then the composition $f \circ g$ is a $C^{(1)}$-a.p. function.*

Proof. We assume that $g \in \widetilde{C^{(1)}}$. Since g is $C^{(1)}$-bounded and f' is bounded on the set E, so there exists an $M > 0$ such that $\sup_{t\in\mathbb{R}} |f'(g(t))|$, $\sup_{t\in\mathbb{R}} |g'(t)| \le M$. Since f, f' are uniformly continuous on E, so for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that for $\tau \in E^{(1)}\{\delta; g\}$ we have

$$\sup_{t\in\mathbb{R}} |f^{(k)}(g(t)) - f^{(k)}(g_\tau(t))| \le \frac{\varepsilon}{1+2M} \quad \text{for } k = 0, 1.$$

Denote $\Delta = \min\{\delta, \varepsilon/(1+2M)\}$. Then for $\tau \in E^{(1)}\{\Delta; g\}$ and $G = f \circ g$, we get

$$\begin{aligned} D^{(1)}(G - G_\tau) \le \sup_{t\in\mathbb{R}} |f(g(t)) - f(g_\tau(t))| + \sup_{t\in\mathbb{R}} |f'(g(t))| \sup_{t\in\mathbb{R}} |g'(t) - g'_\tau(t)| \\ + \sup_{t\in\mathbb{R}} |g'_\tau(t)| \sup_{t\in\mathbb{R}} |f'(g(t)) - f'(g_\tau(t))| \le \varepsilon. \end{aligned}$$

Consequently, $E^{(1)}\{\Delta; g\} \subset E^{(1)}\{\varepsilon; G\}$, i.e. $f \circ g \in C^{(1)}(\mathbb{R})$ is a $C^{(1)}$-a.p. function. ∎

EXAMPLE. The function arctg on $\mathbb{R}$ is $C^{(1)}$-bounded and $C^{(1)}$-continuous. However, this function is not $C^{(1)}$-a.p.

3. Completeness

We denote

$$BC^{(n)}(\mathbb{R}) = \{f \in C^{(n)}(\mathbb{R}) : D^{(n)}(f) < \infty\}.$$

We know that $\widetilde{C^{(n)}} \subsetneq BC^{(n)}(\mathbb{R})$ and the space $\langle BC^{(n)}(\mathbb{R}), \rho^{(n)}\rangle$ is metric with respect to the metric $\rho^{(n)}$, where

$$\rho^{(n)}(f,g) = D^{(n)}(f-g) \quad \text{for } f, g \in BC^{(n)}(\mathbb{R})$$

(see [1]).

REMARK 2. Clearly, the metric space $\langle BC^{(n)}(\mathbb{R}), \rho^{(n)} \rangle$ is complete. Using Theorem 5 in [1] we obtain that the metric space $\langle \widetilde{C^{(n)}, \rho^{(n)}} \rangle$ is complete. Moreover, the space $\langle BC_C^{(n)}(\mathbb{R}), \rho^{(n)} \rangle$, where

$$BC_C^{(n)}(\mathbb{R}) = \{f \in BC^{(n)}(\mathbb{R}) : f \text{ is } C^{(n)}\text{-continuous}\},$$

is a complete metric space, as well (see [1], Th. 1).

4. Conditional $C^{(n)}$-compactness

We say that a family $A \subsetneq BC^{(n)}(\mathbb{R})$ is *conditionally $C^{(n)}$-compact* iff the set A is conditionally compact with respect to the metric $\rho^{(n)}$ in $\langle BC^{(n)}(\mathbb{R}), \rho^{(n)} \rangle$, i.e. every sequence in A includes a Cauchy subsequence.

THEOREM 2. *A nonempty set $A \subsetneq BC_C^{(n)}(\mathbb{R})$ is conditionally $C^{(n)}$-compact if and only if the following statements hold:*

(i) *for every $h > 0$ the family of Steklov functions $A_h = \{S_f(h) : f \in A\}$ is conditionally $C^{(n)}$-compact,*

(ii) *for an arbitrary $\varepsilon > 0$ there exists an $h > 0$ such that $\rho^{(n)}(f, S_f(h)) < \varepsilon$ for every $f \in A$.*

Proof. *Necessity.* We assume that A is a conditionally $C^{(n)}$-compact set. In the same way as in [3], p. 217, by the Hausdorff Theorem, there exists a finite $(\varepsilon/(n+3))$-net

$$f_1, f_2, \ldots, f_l \in A$$

for the set A. Hence for every function $f \in A$ there exists a $k \in \{1, 2, \ldots, l\}$ such that

$$\rho^{(n)}(f, f_k) < \frac{\varepsilon}{n+3}. \tag{1}$$

We shall construct a finite $(\varepsilon(n+1)/(n+3))$-net for the family A_h of Steklov functions. Namely, for each but fixed $h > 0$ we have

$$S_{f_1}(h), S_{f_2}(h), \ldots, S_{f_l}(h) \in A_h \tag{2}$$

and for an arbitrary function $f \in A$ we obtain

$$\rho^{(n)}(S_f(h), S_{f_k}(h)) \le \sup_{t \in \mathbb{R}} |S_f(h)(t) - S_{f_k}(h)(t)| + \sup_{t \in \mathbb{R}} \sum_{i=1}^{n} \left| \frac{d^i}{dt^i}(S_f(h)(t) - S_{f_k}(h)(t)) \right| < \varepsilon \frac{n+1}{n+3}$$

for k satisfying (1). Hence the set (2) is a finite $(\varepsilon(n+1)/(n+3))$-net for the family A_h for every fixed $h > 0$. This means that the family A_h is conditionally $C^{(n)}$-compact. We have to prove still (ii). Let $\varepsilon > 0$. Then, by Theorem 1, for every function f_k, $k = 1, 2, \ldots, l$, there exists an $h_k =$

$h_k(\varepsilon) > 0$ such that $\rho^{(n)}(f_k, S_{f_k}(h_k)) \leq \varepsilon/(n+3)$, because f_k is $C^{(n)}$ -continuous. Denote $0 < h_0 = \min\{h_k : k = 1, 2, \ldots, l\}$. From here

$$\rho^{(n)}(f_k, S_{f_k}(h_0)) \leq \frac{\varepsilon}{n+3} \quad \text{for } k = 1, 2, \ldots, l. \tag{3}$$

Thus for an arbitrary $\varepsilon > 0$ there exists an $h_0 > 0$ such that, according to (1) and (3), we conclude $\rho^{(n)}(f, S_f(h_0)) < \varepsilon$ for every $f \in A$.

Sufficiency. Assume that conditions (i) and (ii) hold. Let $\varepsilon > 0$. Then, by the condition (ii), there exists an $h = h(\varepsilon) > 0$ such that

$$\rho^{(n)}(f, S_f(h)) < \frac{\varepsilon}{3} \quad \text{for every } f \in A. \tag{4}$$

Analogously to the proof in [3], for this $h > 0$ we construct the set A_h which, according to (i), is conditionally $C^{(n)}$-compact. By the Hausdorff Theorem, there exist functions

$$S_{f_1}(h), S_{f_2}(h), \ldots, S_{f_l}(h) \in A_h,$$

where $f_1, f_2, \ldots, f_l \in A$ are such that for every $S_f(h) \in A_h$ there exists a $k \in \{1, 2, \ldots, l\}$ satisfying the inequality

$$\rho^{(n)}(S_f(h), S_{f_k}(h)) < \frac{\varepsilon}{3}. \tag{5}$$

Moreover, the family $\{f_1, f_2, \ldots, f_l\} \subset A$ is a finite ε-net for A, since for each $f \in A$ there exists a $k \in \{1, 2, \ldots, l\}$ such that, by (4) and (5), we have $\rho^{(n)}(f, f_k) < \varepsilon$. It follows that $A \subsetneq BC_C^{(n)}(\mathbb{R})$ is conditionally $C^{(n)}$-compact. The proof is complete. ■

5. $C^{(n)}$-normal functions

In this section we characterize $C^{(n)}$-a.p. functions in the class $BC^{(n)}(\mathbb{R})$.

A function $f \in BC^{(n)}(\mathbb{R})$ is called $C^{(n)}$-*normal* iff the family of functions $f_T = \{f_h : h \in \mathbb{R}\}$ is conditionally $C^{(n)}$-compact.

We prove some properties of $C^{(n)}$-normal functions.

THEOREM 3. *A function $f \in BC^{(n)}(\mathbb{R})$ is $C^{(n)}$-a.p. if and only if f is a $C^{(n)}$-normal function.*

Proof. *Necessity.* Let $f \in \widetilde{C^{(n)}}$. We have $f_T \subsetneq BC^{(n)}(\mathbb{R})$ and the metric space $\langle BC^{(n)}(\mathbb{R}), \rho^{(n)} \rangle$ is complete. Thus, according to the Hausdorff Theorem, we only need to construct a finite ε-net for the family f_T with respect to the metric $\rho^{(n)}$. Namely, we know that f is a $C^{(n)}$-continuous function (see [1], Th. 2). Thus for an arbitrary $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $D^{(n)}(f - f_h) \leq \varepsilon/2$ for every $h \in \mathbb{R}$ with $|h| < \delta$. Analogously as in [4], let $l = l(\varepsilon) \geq \delta$ be a number which characterizes the relative density of the set $E^{(n)}\{\varepsilon/(2+a); f\}$ with $a > 0$. We denote $h_k = k\delta$ for $k = 1, 2, \ldots, m$, where m satisfies $m\delta \le l < (m+1)\delta$. Then the set

$$f_{h_1}, f_{h_2}, \dots, f_{h_m} \in f_T$$

is a finite ε-net for the family $f_T \subsetneq BC^{(n)}(\mathbb{R})$ with respect to the metric $\rho^{(n)}$. This means that f is a $C^{(n)}$-normal function.

Sufficiency. The proof is analogous to the proof in [3], p. 220. ∎

Finally, we shall give theorems about $C^{(n)}$-normality of a linear combination of $C^{(n)}$-normal functions and next about $C^{(n)}$-normality of a product and a quotient of $C^{(n)}$-normal functions.

LEMMA. *Let f is a $C^{(n)}$-normal function and let $\inf_{t\in\mathbb{R}} |f(t)| = m > 0$. Then for an arbitrary $\varepsilon > 0$ there exists a finite $(p_n(\varepsilon))$-net for the family $(1/f)_T = \{(1/f)_h : h \in \mathbb{R}\}$, where $0 < p_n(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.*

Proof. (*Induction*) Since $f \in BC^{(1)}(\mathbb{R})$ is $C^{(1)}$-normal, it follows by the Hausdorff Theorem that for each $\varepsilon > 0$ there exists a finite ε-net

$$f_{h_1}, f_{h_2}, \dots, f_{h_m} \in f_T$$

for the family f_T. Hence for every function $f_h \in f_T$ there exists a $k \in \{1, 2, \dots, m\}$ such that

$$\rho^{(1)}(f_h, f_{h_k}) < \varepsilon.$$

We have to construct a finite $(p_1(\varepsilon))$-net for the family $(1/f)_T$, where $0 < p_1(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Let $\varepsilon > 0$. Thus for every function $(1/f)_h \in (1/f)_T$ there exists a $k \in \{1, 2, \dots, m\}$ such that

$$\rho^{(1)}\left(\frac{1}{f_h}, \frac{1}{f_{h_k}}\right) < \frac{2\varepsilon(m^2 + M_1^2)}{m^4} = p_1(\varepsilon),$$

where $\sup_{t\in\mathbb{R}} |f^{(i)}(t)| \le M_1$, for $i = 0, 1$, and $0 < p_1(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Suppose that for an arbitrary $\varepsilon > 0$ and for a $C^{(s)}$-normal function f there exists a finite $(p_s(\varepsilon))$-net for the family $(1/f)_T$, where $0 < p_s(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. We assume that $f \in BC^{(s+1)}(\mathbb{R})$ is $C^{(s+1)}$-normal. By the Hausdorff Theorem, for an arbitrary $\varepsilon > 0$,

$$f_{h_1}, f_{h_2}, \dots, f_{h_r} \in f_T$$

is a finite ε-net for the family f_T. Hence for every function $f_h \in f_T$ there exists a $q \in \{1, 2, \dots, r\}$ such that

$$\rho^{(s+1)}(f_h, f_{h_q}) < \varepsilon. \tag{6}$$

We have $(1/f)_T \subset BC^{(s+1)}(\mathbb{R})$. Moreover, f is a $C^{(s+1)}$-bounded function. Then there exists a constant $M_2 > 0$ such that $\sup_{t\in\mathbb{R}} |f^{(i)}(t)| \le M_2$, for $i = 0, 1, 2, \dots, s+1$, and $\sup_{t\in\mathbb{R}} |(1/f^2)^{(i)}(t)|, \sup_{t\in\mathbb{R}} |(1/(f_h^2 f_{h_q}^2))^{(i)}(t)| \le M_2$ for $i = 0, 1, 2, \dots, s$. We only need to construct a finite $(p_{s+1}(\varepsilon))$-net for the family $(1/f)_T$ with respect to the metric $\rho^{(s+1)}$, where $0 < p_{s+1}(\varepsilon) \longrightarrow 0$

as $\varepsilon \longrightarrow 0$. Namely, we know that f is a $C^{(s)}$-normal function, too. Then for every $(1/f)_h \in (1/f)_T$ we get

$$\rho^{(s+1)}\left(\frac{1}{f_h}, \frac{1}{f_{h_q}}\right) < p_s(\varepsilon) + \sup_{t\in\mathbb{R}}\left|\left(\frac{1}{f_h}(t) - \frac{1}{f_{h_q}}(t)\right)^{(s+1)}\right|$$

with $0 < p_s(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, and

$$\left|\left(\frac{1}{f_h}(t) - \frac{1}{f_{h_q}}(t)\right)^{(s+1)}\right| \le J(t) + K(t),$$

where

$$J(t) = \left|\left(\frac{f'_h(t) - f'_{h_q}(t)}{f^2_{h_q}(t)}\right)^{(s)}\right| < \varepsilon 2^s M_2 \tag{7}$$

and

$$K(t) = \left|\left(\frac{f'_h(t)(f^2_h(t) - f^2_{h_q}(t))}{f^2_h(t) f^2_{h_q}(t)}\right)^{(s)}\right| < \varepsilon 2^{2s+1} M_2^3. \tag{8}$$

According to (7) and (8) we obtain

$$\sup_{t\in\mathbb{R}}\left|\left(\frac{1}{f_h}(t) - \frac{1}{f_{h_q}}(t)\right)^{(s+1)}\right| < \varepsilon 2^s M_2(1 + 2^{s+1} M_2^2). \tag{9}$$

Therefore, by (9), for $q \in \{1, 2, \ldots, r\}$ satisfying (6), we have

$$\rho^{(s+1)}\left(\frac{1}{f_h}, \frac{1}{f_{h_q}}\right) < p_s(\varepsilon) + \varepsilon 2^s M_2(1 + 2^{s+1} M_2^2) = p_{s+1}(\varepsilon),$$

where $0 < p_{s+1}(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. This proves the lemma. ∎

THEOREM 4. *The following statements hold*:

(i) *A linear combination of $C^{(n)}$-normal functions is a $C^{(n)}$-normal function.*

(ii) *A product of $C^{(n)}$-normal functions is a $C^{(n)}$-normal function.*

(iii) *If $\inf_{t\in\mathbb{R}} |g(t)| = m > 0$, then a quotient f/g of $C^{(n)}$-normal functions f, g is a $C^{(n)}$-normal function.*

Proof. (i) Let $f, g \in BC^{(n)}(\mathbb{R})$ be $C^{(n)}$-normal and let $c \neq 0$ be an arbitrary constant. It is clear that the family $(cf)_T = \{cf_h : h \in \mathbb{R}\}$ is conditionally $C^{(n)}$-compact, i.e. cf is a $C^{(n)}$-normal function. Moreover, analogously as in [3], for every sequence (h_m), $h_m \in \mathbb{R}$ for $m = 1, 2, \ldots$, the sequence (f_{h_m}) includes a Cauchy subsequence $(f_{h_{m_k}})$. However, the sequence $(g_{h_{m_k}})$ includes a Cauchy subsequence $(g_{h_{m_{k_p}}})$. We conclude that for an arbitrary sequence (h_m) the sequence $((f+g)_{h_m})$ includes the Cauchy subsequence $((f+g)_{h_{m_{k_p}}})$. Therefore we obtain that $f + g$ is a $C^{(n)}$-normal function.

(ii) Let $f, g \in BC^{(n)}(\mathbb{R})$ be $C^{(n)}$-normal. Since $fg = 1/4((f+g)^2 - (f-g)^2)$ and there holds the statement (i), we only need to show that f^2 is $C^{(n)}$-normal whenever f is. Clearly, $f^2 \in BC^{(n)}(\mathbb{R})$. Moreover, by the Hausdorff Theorem, for an arbitrary $\varepsilon > 0$ there exists a finite $(\varepsilon/(2M(2^{n+1}-1)))$ -net

$$f_{h_1}, f_{h_2}, \ldots, f_{h_m} \in f_T$$

for the family f_T, where $\sup_{t\in\mathbb{R}} |f^{(i)}(t)| \le M$ for $i = 0, 1, 2, \ldots, n$. Hence for every function $f_h \in f_T$ there exists a $k \in \{1, 2, \ldots, m\}$ such that

$$\rho^{(n)}(f_h, f_{h_k}) < \frac{\varepsilon}{2M(2^{n+1}-1)}. \tag{10}$$

We need to construct a finite ε-net for the family $(f^2)_T = \{f_h^2 : h \in \mathbb{R}\}$. Let $\varepsilon > 0$. Then for every $t \in \mathbb{R}$ and for every function $f_h^2 \in (f^2)_T$ we have

$$|(f_h^2(t) - f_{h_k}^2(t))^{(l)}| < \frac{\varepsilon 2^l}{2^{n+1}-1} \quad \text{for } l = 0, 1, 2, \ldots, n,$$

for k satisfying (10). Thus for every $\varepsilon > 0$ and every $f_h^2 \in (f^2)_T$ there exists a $k \in \{1, 2, \ldots, m\}$ such that $\rho^{(n)}(f_h^2, f_{h_k}^2) < \varepsilon$. Consequently, f^2 is $C^{(n)}$-normal.

(iii) Let $f, g \in BC^{(n)}(\mathbb{R})$ be $C^{(n)}$-normal and let $\inf_{t\in\mathbb{R}} |g(t)| = m > 0$. Since $f/g = f(1/g)$, hence $C^{(n)}$-normality follows from (ii) and the Lemma. ∎

Let us remark that, by Theorems 3 and 4, we immediately obtain that a linear combination of $C^{(n)}$-a.p. functions and a product of $C^{(n)}$-a.p. functions are $C^{(n)}$-a.p. Moreover, if $\inf_{t\in\mathbb{R}} |g(t)| = m > 0$, then a quotient f/g of $C^{(n)}$-a.p. functions f, g is a $C^{(n)}$-a.p. function.

References

[1] M. Adamczak, $C^{(n)}$-*almost periodic functions*, Comment. Math. 37 (1997), 1–12.

[2] B. P. Demidovitch, *Mathematical theory of stabilization*, Warsow 1972 (in Polish).

[3] B. M. Levitan, *Almost periodic functions*, Moscow 1953 (in Russian).

[4] S. Stoiński, *On compactness in variation of almost periodic functions*, Demonstratio Math. 31 (1) (1998), 131–134.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
Jana Matejki 48/49
60-769 POZNAŃ, POLAND
E-mail: keram@amu.edu.pl

Received October 2nd, 1998.