DEMONSTRATIO MATHEMATICA
Vol. XXXII No 2 1999

Stanisław Stoiński

$(VC)^{(n)}$-ALMOST PERIODIC FUNCTIONS

Abstract. In this note we give the definition and some properties of $(VC)^{(n)}$-almost periodic functions, i.e. uniformly almost periodic and almost periodic in variation functions with first n derivatives.

Let us denote for an arbitrary $t \in \mathbb{R}$ by $V(t; f)$ *the Jordan variation* of a function f on the interval $\langle t-1, t+1 \rangle$. Let us put

$$X_0^{(n)} = \{f : \mathbb{R} \to \mathbb{R} : f \in C^{(n)}(\mathbb{R}) \text{ and } V(t; f^{(k)}) < \infty \text{ for } k = 0, 1, 2, \ldots, n \text{ and for every } t \in \mathbb{R}\}$$

and for $f \in X_0^{(n)}$

$$(VD)^{(n)}(f) = \sup_{t \in \mathbb{R}} \sum_{k=0}^{n} (|f^{(k)}(t)| + V(t; f^{(k)})).$$

We say that $f \in X_0^{(n)}$ is *a* $(VC)^{(n)}$*-bounded function* if $(VD)^{(n)}(f) < \infty$. Let us write $f_h(x) \equiv f(x+h)$, where $h \in \mathbb{R}$. If for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that $(VD)^{(n)}(f - f_h) \leq \varepsilon$ for every $h \in \mathbb{R}$ with $|h| < \delta$, we say that $f \in X_0^{(n)}$ is *a* $(VC)^{(n)}$*-continuous function.* A sequence (f_m), where $f_m \in X_0^{(n)}$ for $m = 1, 2, \ldots$, will be called $(VD)^{(n)}$*-convergent to* $f \in X_0^{(n)}$ if for an arbitrary $\varepsilon > 0$ there exists a positive number M such that $(VD)^{(n)}(f - f_m) \leq \varepsilon$ for every $m > M$.

It is easily seen that if a sequence (f_m) of $(VC)^{(n)}$-continuous functions is $(VD)^{(n)}$-convergent to a function $f \in X_0^{(n)}$, then f is $(VC)^{(n)}$-continuous.

The number $\tau \in \mathbb{R}$ is called *a* $((VD)^{(n)}, \varepsilon)$*-almost period* $(((VD)^{(n)}, \varepsilon)$-a.p.) of a function $f \in X_0^{(n)}$ if $(VD)^{(n)}(f - f_\tau) \leq \varepsilon$ for $\varepsilon > 0$. Let $VE^{(n)}\{\varepsilon; f\}$ denote the set of $((VD)^{(n)}, \varepsilon)$-almost periods of f.

1991 *Mathematics Subject Classification*: 42A75.
Key words and phrases: almost periodic function, derivative of order n, variation.

A function $f \in X_0^{(n)}$ is called $(VC)^{(n)}$-*almost periodic* ($\widetilde{(VC)}^{(n)}$-a.p.) if for each $\varepsilon > 0$ the set $VE^{(n)}\{\varepsilon; f\}$ is relatively dense. By $\widetilde{(VC)}^{(n)}$ we denote the set of $(VC)^{(n)}$-a.p. functions.

The class of V-a.p. functions (see [3]) is identical with the class of $(VC)^{(0)}$-a.p. functions. Moreover, every $(VC)^{(n)}$-a.p. function is $C^{(n)}$-a.p. function (see [1]).

THEOREM 1. *If f is a $(VC)^{(n)}$-a.p. function, then f is $(VC)^{(n)}$-bounded.*

Proof. Let $f \in \widetilde{(VC)}^{(n)}$. For an arbitrary $t \in \mathbb{R}$ and any $\varepsilon > 0$ there exists a $((VD)^{(n)}, \varepsilon)$-a.p. $\tau \in (-t, -t+l)$, where $l = l(\varepsilon) > 0$ is a number which characterizes the relative density of the set $VE^{(n)}\{\varepsilon; f\}$, such that we have

$$\begin{aligned}&\sum_{k=0}^{n}(|f^{(k)}(t)| + V(t; f^{(k)}))\\ &\leq \sum_{k=0}^{n}(|f^{(k)}(t) - f_\tau^{(k)}(t)| + V(t; f^{(k)} - f_\tau^{(k)})) + \sum_{k=0}^{n}(|f_\tau^{(k)}(t)| + V(t; f_\tau^{(k)})).\end{aligned}$$

Hence, because $f \in X_0^{(n)}$, we obtain the following estimation

$$(VD)^{(n)}(f) \leq (VD)^{(n)}(f - f_\tau) + \sup_{t\in\langle 0,l\rangle}\sum_{k=0}^{n}(|f^{(k)}(t)| + V(t; f^{(k)})) \leq \varepsilon + M,$$

where $M > 0$ is a constant. ∎

THEOREM 2. *Assume that f is a $(VC)^{(n)}$-a.p. function which satisfies the $(VC)^{(n)}$-condition: for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{0<t<l}\sum_{k=0}^{n} V(t; f^{(k)} - f_h^{(k)}) \leq \varepsilon$$

for every $h \in \mathbb{R}$ with $|h| < \delta$, where $l = l(\varepsilon) > 0$ is a number which characterizes the relative density of the set $VE^{(n)}\{\varepsilon; f\}$. Then f is $(VC)^{(n)}$-continuous.

Proof. For an arbitrary $t \in \mathbb{R}$ and for $\tau \in VE^{(n)}\{\varepsilon; f\}$, where $\tau \in (-t, -t+l)$, we obtain that for $h \in \mathbb{R}$

$$\begin{aligned}&\sum_{k=0}^{n} V(t; f^{(k)} - f_h^{(k)})\\ &\leq \sum_{k=0}^{n} V(t; f^{(k)} - f_\tau^{(k)}) + \sup_{0<t'<l}\sum_{k=0}^{n} V(t'; f^{(k)} - f_h^{(k)}) + \sum_{k=0}^{n} V(t; f_{\tau h}^{(k)} - f_h^{(k)}),\end{aligned}$$

where $f_{\tau h}^{(k)}(x) \equiv f^{(k)}(x+\tau+h)$. Hence

$$(VD)^{(n)}(f-f_h) \leq 2(VD)^{(n)}(f-f_\tau) + \sup_{t\in\mathbb{R}} \sum_{k=0}^{n} |f^{(k)}(t) - f_h^{(k)}(t)| + \sup_{0<t<l} \sum_{k=0}^{n} V(t; f^{(k)} - f_h^{(k)}).$$

Since f is $(VC)^{(n)}$-a.p., so f is $C^{(n)}$-a.p. Therefore for an arbitrary $\varepsilon > 0$ there exists a $\delta' > 0$ such that for $|h| < \delta'$ we have

$$\sup_{t\in\mathbb{R}} \sum_{k=0}^{n} |f^{(k)}(t) - f_h^{(k)}(t)| \leq \varepsilon.$$

For $|h| < \min(\delta, \delta')$ we obtain $(VD)^{(n)}(f-f_h) \leq 4\varepsilon$, i.e. f is $(VC)^{(n)}$-continuous. ∎

Analogously as in [2] and [3] we prove the following:

THEOREM 3. *A linear combination of two $(VC)^{(n)}$-a.p. functions f, g, which satisfy the $(VC)^{(n)}$-condition, is a $(VC)^{(n)}$-a.p. function.*

THEOREM 4. *If a sequence (f_m) of $(VC)^{(n)}$-a.p. functions is $(VD)^{(n)}$-convergent to a function $f \in X_0^{(n)}$, then f is a $(VC)^{(n)}$-a.p. function.*

Proof. For an arbitrary $\varepsilon > 0$ there exists m_0 such that $(VD)^{(n)}(f - f_{m_0}) \leq \varepsilon/3$. Therefore for $\tau \in VE^{(n)}\{\varepsilon/3; f_{m_0}\}$ we obtain the following estimation

$$(VD)^{(n)}(f-f_\tau) \leq (VD)^{(n)}(f-f_{m_0}) + (VD)^{(n)}(f_{m_0} - f_{m_0\tau}) + (VD)^{(n)}(f_{m_0\tau} - f_\tau) \leq \varepsilon,$$

where $f_{m_0\tau}(x) \equiv f_{m_0}(x+\tau)$, i.e. $VE^{(n)}\{\varepsilon/3; f_{m_0}\} \subset VE^{(n)}\{\varepsilon; f\}$. This proves that $f \in \widetilde{(VC)}^{(n)}$. ∎

In the following we shall investigate the derivative and the indefinite integral of a $(VC)^{(n)}$-a.p. function.

THEOREM 5. *If f is a $(VC)^{(n)}$-a.p. function and f is $(VC)^{(n+1)}$-continuous, then the derivative f' is a $(VC)^{(n)}$-a.p. function.*

Proof. Let us write

$$\frac{f_h(t) - f(t)}{h} - f'(t) = \frac{1}{h}\int_0^h \Big(\frac{\partial}{\partial t} f(v+t) - f'(t)\Big) dv$$

for $t \in \mathbb{R}$, $h \neq 0$ and

$$(VD)^{(n)}\Big(\frac{f_h - f}{h} - f'\Big) \leq \sup_{t\in\mathbb{R}} \sum_{k=0}^{n} \left| \frac{d^k}{dt^k}\Big(\frac{1}{h}\int_0^h \Big(\frac{\partial}{\partial t} f(v+t) - f'(t)\Big) dv\Big) \right|$$

$$+\sup_{t\in\mathbb{R}}\sum_{k=0}^{n} V(t; \frac{d^k}{du^k}\Big(\frac{1}{h}\int_0^h\Big(\frac{\partial}{\partial u}f(v+u)-f'(u)\Big)dv\Big))$$
$$=\sup_{t\in\mathbb{R}} W_1(h,t)+\sup_{t\in\mathbb{R}} W_2(h,t).$$

Because f is $(VC)^{(n+1)}$-continuous, so for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\sup_{t\in\mathbb{R}}\sum_{k=0}^{n+1}(|(f_h-f)^{(k)}(t)|+V(t;(f_h-f)^{(k)}))\le\varepsilon$$

for every $h \in \mathbb{R}$, $|h| < \delta$. Hence it follows that

$$\sup_{t\in\mathbb{R}} W_1(h,t)\le\varepsilon$$

and

$$\sup_{t\in\mathbb{R}} W_2(h,t)\le\frac{1}{h}\int_0^h\sum_{k=1}^{n+1}V\Big(t;\frac{\partial^k}{\partial u^k}f(v+u)-f^{(k)}(u)\Big)dv\le\varepsilon$$

for $0 < |h| < \delta$. Therefore we obtain that for every sequence (h_m), where $h_m \neq 0$, $h_m \to 0$,

$$(\widetilde{VC})^{(n)}\ni\frac{f_{h_m}-f}{h_m}\longrightarrow f'\in X_0^{(n)}$$

in the sense of $(VD)^{(n)}$-convergence, and so, by Theorem 4, f' is $(VC)^{(n)}$-a.p. ∎

REMARK 1. Let f be a B-a.p. function. Then for an arbitrary $\varepsilon > 0$ there exists $\varepsilon' = \varepsilon'(\varepsilon) > 0$ such that $\varepsilon' < \varepsilon/3$ and every ε'-a.p. of f is an $\varepsilon/3$-a.p. of the bounded indefinite integral F of the function f (see [2], p.29). It is known (see [3]) that $E\{\varepsilon'; f\} \subset E_V\{\varepsilon, F\}$.

THEOREM 6. *If f is a $(VC)^{(n)}$-a.p. function and the indefinite integral F of f is bounded, then F is a $(VC)^{(n+1)}$-a.p. function.*

PROOF. Since f is $(VC)^{(n)}$-a.p. and $F^{(k)}(x) = f^{(k-1)}(x)$ for every $x \in \mathbb{R}$ and $k = 1, 2, \dots, n+1$, so $F \in X_0^{(n+1)}$. By Remark 1, for $\tau \in VE^{(n)}\{\varepsilon'; f\}$ we have

$$(VD)^{(n+1)}(F-F_\tau)\le(VD)^{(0)}(F-F_\tau)+(VD)^{(n)}(f-f_\tau)<\frac{4}{3}\varepsilon,$$

and so F is $(VC)^{(n+1)}$-a.p. ∎

By Theorem 6 it follows the following:

COROLLARY. *If f is a V-a.p. function and the indefinite integral F of f is bounded, then F is a $(VC)^{(1)}$-a.p. function.*

THEOREM 7. *Let us assume that f is a bounded function on $\mathbb{R}$. If the derivative f' is V-a.p., then f is $(VC)^{(1)}$-a.p.*

Proof. For every $x \in \mathbb{R}$ we have

$$g(x) = \int_0^x f'(u)du, \qquad \text{where} \quad g = f + c,\ c = -f(0).$$

By Remark 1 it follows that for an arbitrary $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $\varepsilon' < \varepsilon/3$ and $E\{\varepsilon'; f'\} \subset E_V\{\varepsilon; g\}$. Hence for $\tau \in E_V\{\varepsilon'; f'\}$ we obtain

$$(VD)^{(1)}(g - g_\tau) \le V(g - g_\tau) + V(f' - f'_\tau) < \frac{4}{3}\varepsilon,$$

and so $E_V\{\varepsilon'; f'\} \subset VE^{(1)}\{(4/3)\varepsilon; g\}$. Moreover $g \in X_0^{(1)}$. Therefore g and hence f are $(VC)^{(1)}$-a.p. ∎

Now, we shall give an example of a $(VC)^{(1)}$-a.p. function.

EXAMPLE 1. Let us put

$$f(x) = \sin x + \sin(\sqrt{2}x) \qquad \text{for } x \in \mathbb{R}.$$

By Property 4 (see [3]) it follows that the derivative f' is V-a.p. Because f is the bounded indefinite integral of f', so f, by Theorem 6, is $(VC)^{(1)}$-a.p.

REMARK 2. If a function $f : \mathbb{R} \to \mathbb{R}$ is locally integrable and is periodic with the period $T > 0$, then the integral

$$F(x) = \int_0^x [f(t) - M(f)]dt \quad \text{for } x \in \mathbb{R}, \text{ where } M(f) = \frac{1}{T}\int_0^T f(u)du,$$

is continuous on $\mathbb{R}$ and is periodic with the period T.

We shall give an example of a $(VC)^{(n-1)}$-a.p. function which is not a $(VC)^{(n)}$-a.p. function, where $n \in \{2, 3, \ldots\}$.

EXAMPLE 2. Let

$$f(x) = \begin{cases} (x-k)\cos\dfrac{\pi}{2(x-k)} & \text{for } x \in (k, k+1), \\ 0 & \text{for } x = k, \end{cases}$$

$$g(x) = \begin{cases} (\sqrt{2}x - k)\cos\dfrac{\pi}{2(\sqrt{2}x - k)} & \text{for } x \in \left(\frac{\sqrt{2}k}{2}, \frac{\sqrt{2}(k+1)}{2}\right), \\ 0 & \text{for } x = \dfrac{\sqrt{2}k}{2}, \end{cases}$$

where $k = 0, \pm 1, \pm 2, \ldots$

Functions f and g are continuous on $\mathbb{R}$ and periodic with periods $T_f = 1$, $T_g = \sqrt{2}/2$, respectively. Let us denote by $V(f; a, b)$ the Jordan variation of a function f on the interval $\langle a,b\rangle$. Then for every $k=0,\pm1,\pm2,\dots$ we have

$$V(f;k,k+1)=V\Big(g;\frac{\sqrt{2}k}{2},\frac{\sqrt{2}(k+1)}{2}\Big)=\infty.$$

Moreover $f,g\notin C^{(1)}(\mathbb{R})$. We denote $F_0=f$, $G_0=g$ and

$$F_1(x)=\int_0^x[f(t)-M(f)]dt,\qquad G_1(x)=\int_0^x[g(t)-M(g)]dt,$$
$$F_2(x)=\int_0^x[F_1(t)-M(F_1)]dt,\qquad G_2(x)=\int_0^x[G_1(t)-M(G_1)]dt,$$
$$\vdots\qquad\qquad\vdots$$
$$F_n(x)=\int_0^x[F_{n-1}(t)-M(F_{n-1})]dt,\quad G_n(x)=\int_0^x[G_{n-1}(t)-M(G_{n-1})]dt,$$

where

$$M(F_i)=\frac{1}{T_f}\int_0^{T_f}F_i(t)dt,\qquad M(G_i)=\frac{1}{T_g}\int_0^{T_g}G_i(t)dt$$

for $i=0,1,2,\dots,n-1$. Functions F_i and G_i, $i=0,1,2,\dots,n$, are continuous on $\mathbb{R}$. In the following we obtain for $n\in\{2,3,\dots\}$

$$F_n'(x)=F_{n-1}(x)-M(F_{n-1}),\qquad G_n'(x)=G_{n-1}(x)-M(G_{n-1}),$$
$$F_n''(x)=F_{n-2}(x)-M(F_{n-2}),\qquad G_n''(x)=G_{n-2}(x)-M(G_{n-2}),$$
$$\vdots\qquad\qquad\vdots$$
$$F_n^{(n-1)}(x)=F_1(x)-M(F_1),\qquad G_n^{(n-1)}(x)=G_1(x)-M(G_1),$$
$$F_n^{(n)}(x)=f(x)-M(f)\in C(\mathbb{R}),\quad G_n^{(n)}(x)=g(x)-M(g)\in C(\mathbb{R}),$$
$$F_n^{(n+1)}(x)=f'(x)\notin C(\mathbb{R}),\qquad G_n^{(n+1)}(x)=g'(x)\notin C(\mathbb{R}).$$

Hence $F_n,G_n\in C^{(n)}(\mathbb{R})$, but $F_n,G_n\notin C^{(n+1)}(\mathbb{R})$. By Remark 2 it follows that F_n,G_n are periodic, and so F_n,G_n are $C^{(n)}$-a.p. functions. For every $t\in\mathbb{R}$ and for $i=0,1,2,\dots,n-1$ we have

$$V(t;F_n^{(i)})\le\int_{t-1}^{t+1}|F_{n-i-1}(u)-M(F_{n-i-1})|du<\infty,\qquad V(t;G_n^{(i)})<\infty$$

and

$$V(t;F_n^{(n)})=V(t;f)=\infty,\qquad V(t;G_n^{(n)})=\infty.$$

Hence $F_n,G_n\in X_0^{(n-1)}$ and $F_n,G_n\notin X_0^{(n)}$ for $n\ge2$, i.e. F_n,G_n are $(VC)^{(n-1)}$-a.p. functions. Since derivatives $F_n^{(i)}$ and $G_n^{(i)}$, $i=0,1,2,\dots$ $\dots,n-1$, are absolutely continuous, so $F_n^{(i)},G_n^{(i)}$ are V-continuous and hence F_n,G_n are $(VC)^{(n-1)}$-continuous. Periods T_f and T_g are incommensurate.

Thus $H = F_n + G_n$ is not a periodic function. By Theorem 3 it follows that H is the $(VC)^{(n-1)}$-a.p. function and H is not $(VC)^{(n)}$-a.p.

References

[1] M. Adamczak, $C^{(n)}$-*almost periodic functions*, Comment. Math. 37 (1997), 1–12.
[2] B. M. Levitan, *Almost periodic functions*, Moscow 1953 (in Russian).
[3] S. Stoiński, *Real-valued functions almost periodic in variation*, Func. Approx. 22 (1993), 141–148.

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
ADAM MICKIEWICZ UNIVERSITY
ul. Jana Matejki 48/49
60-769 POZNAŃ, POLAND

Received August 24, 1998; revised version November 30, 1998.