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NOTE ON JENSEN AND PEXIDER FUNCTIONAL EQUATIONS

Abstract. We determine the general solutions of the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in M$$

and the Pexider functional equation

$$f(x+y) = g(x) + h(y), \quad x, y \in M$$

for $f, g, h : M \to S$, where M is an Abelian semigroup with the division by 2 and S is an abstract convex cone satisfying the cancellation law. Some applications to set-valued versions of these equations are given.

1. Introduction

Let $(X, |\cdot|)$ be a real normed space. Throughout this note $ccl(X)$ stands for the set of all non-empty, bounded, closed and convex subsets of X. Introduce a binary operation $\overset{*}{+}$ in $ccl(X)$ by the formula

$$A \overset{*}{+} B = cl(A+B),$$

where $A+B$ denotes the usual algebraic sum of A and B while clA denotes the closedness of the set A.

It is easy to see that

$$(1) \qquad A \overset{*}{+} B = cl(clA + clB) \quad \text{for all} \quad A, B \subset X.$$

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Moreover

$$\lambda(A \overset{*}{+} B) = \lambda A \overset{*}{+} \lambda B, \quad (\lambda + \mu)A = \lambda A \overset{*}{+} \mu A \tag{2}$$

for all $A, B \in ccl(X)$ and $\lambda, \mu \in \mathbb{R}_+ := [0, \infty)$.

The proof of the following generalization of the Rådström lemma (cf. [6]) can be found in [9].

LEMMA 1. *If a set $B \subset X$ is a non-empty and bounded and $C \subset X$ is closed and convex, then for every $A \subset X$*

$$A + B \subset C \overset{*}{+} B \Longrightarrow A \subset C.$$

For two non-empty subsets A and B of X and a point x of X we define

$$\rho(x, B) = \inf\{\|x - y\| : y \in B\}, \quad e(A, B) = \sup\{\rho(x, B) : x \in A\}.$$

The *Hausdorff distance* of A and B is defined by

$$\delta(A, B) = \max\{e(A, B), e(B, A)\}.$$

Write $S := \{x \in X : \|x\| \leq 1\}$. It is not difficult to prove that

$$\delta(A, B) = \inf\{t > 0 : A \subset B + tS,\ B \subset A + tS\}$$

for non-empty and bounded sets $A, B \subset X$. δ is a metric on $ccl(X)$ which is henceforth endowed with the corresponding metric space structure. Completness of $(ccl(X), \delta)$ is contained in the following (cf., e.g., [1]).

LEMMA 2. *If X is a Banach space, then $(ccl(X), \delta)$ is a complete metric space.*

The proof of the second equality of the lemma below can be found in [2]. The proof of the first one is easy to verify.

LEMMA 3. *If $A, B, C \in ccl(X)$ then*

$$\delta(A \overset{*}{+} B, C \overset{*}{+} B) = \delta(A + B, C + B) = \delta(A, C).$$

The equality

$$\delta(\lambda A, \lambda B) = |\lambda| \delta(A, B) \tag{3}$$

for $\lambda \in \mathbb{R}$ and bounded A and B, easy follows from the definition of δ.

A set-valued function F defined on an Abelian semigroup M such that the division by 2 is performable with values in a normed space is said to be **Jensen* if

$$F\left(\frac{x + y}{2}\right) = \frac{1}{2}[F(x) \overset{*}{+} F(y)],$$

for all $x, y \in M$.

Let $(X, |\cdot|)$, $(Y, |\cdot|)$ and $(Z, |\cdot|)$ be normed spaces and let $U \subset X$ be a convex set containing zero. Suppose that C is a convex cone in Y such that $0 \in C$. Define

$$lip(U, C) := \left\{ \phi : U \to C : \sup_{x \neq \bar{x}, x, \bar{x} \in U} \frac{|\phi(x) - \phi(\bar{x})|}{|x - \bar{x}|} < \infty \right\}.$$

On this set we introduce the metric

$$d(\phi_1, \phi_2) := |\phi_1(0) - \phi_2(0)| + \sup_{x \neq \bar{x}, x, \bar{x} \in U} \frac{|\phi_1(x) - \phi_2(x) - \phi_1(\bar{x}) + \phi_2(\bar{x})|}{|x - \bar{x}|}.$$

Next, we put

$$Lip(U, Z) := \{\varphi : U \to ccl(Z) : \sup_{x \neq \bar{x}, x, \bar{x} \in U} \frac{\delta(\varphi(x), \varphi(\bar{x}))}{|x - \bar{x}|} < \infty\},$$

where δ denotes the Hausdorff distance in $ccl(Z)$. On this set a metric can be defined by

$$\rho(\varphi_1, \varphi_2) := \delta(\varphi_1(0), \varphi_2(0)) + \sup_{x \neq \bar{x}, x, \bar{x} \in U} \frac{\delta(\varphi_1(x) + \varphi_2(\bar{x}), \varphi_1(\bar{x}) + \varphi_2(x))}{|x - \bar{x}|}.$$

Similarly we introduce the set $Lip(C, Z)$ with the metric ρ_1 defined analogously.

Every set-valued function $h : U \times C \to ccl(Z)$ generates the *Nemytskii operator* N defined by the formula

$$N(\phi)(x) := h(x, \phi(x)), \quad x \in U, \phi \in lip(U, C). \tag{4}$$

The operator N takes its values in the space of all set-valued functions $\varphi : U \to ccl(Z)$.

J. Matkowski proved that every Lipschitzian Nemytskii operator mapping $lip(U, Y)$ into $lip(U, Z)$ is generated by a function $h : U \times Y \to Z$ of the form $h(x, y) = A(x)y + b(x)$, $x \in U, y \in Y$, where $b \in lip(U, Z)$ and $A(x)$, $x \in U$, is linear and continuous map from Y to Z (cf. [4], Theorem 1). We are going to prove similar theorem for Nemytskii operator generated by a set-valued function h. The idea of the proof of the following proposition is due to J. Matkowski (cf. [4], the proof of Theorem 1).

PROPOSITION. *Let $(X, |\cdot|)$, $Y.|\cdot|$, $(Z, |\cdot|)$ be normed spaces and let C be a convex cone with zero in Y. Assume that $U \subset X$ is a convex set, $0 \in U$ and $h : U \times C \to ccl(Z)$. If the Nemytskii operator N defined by (4) satisfies the following conditions:*

1. $N(lip(U, C)) \subset Lip(U, Z)$;
2. *there exists $c \geq 0$ such that*

$$\rho(N\phi_1, N\phi_2) \leq c\, d(\phi_1, \phi_2), \qquad \phi_1, \phi_2 \in lip(U, C),$$

then

(a) $h(\cdot, y) \in Lip(U, Z)$ *for all* $y \in C$;

(b) *for every* $x \in U$ *the set-valued function* $h(x, \cdot)$ *is a Lipschitz function with the Lipschitz constant* c;

(c) *the function* $x \longmapsto h(x, \cdot)$ *defined on* U *is Lipschitz*;

(d) $h(x, \cdot)$ *is a *Jensen function.*

Proof. Fix $y \in C$. The constant function $\phi(x) = y$, $x \in U$ belongs to $lip(U, C)$. Consequently $h(\cdot, y) \in Lip(U, Z)$ for all $y \in C$. In particular h is continuous with respect to the first variable for every $y \in C$. On account of 2. we have

$$\frac{\delta(h(t, \phi_1(t)) + h(\bar{t}, \phi_2(\bar{t})), h(\bar{t}, \phi_1(\bar{t})) + h(t, \phi_2(t)))}{|t - \bar{t}|} \leq cd(\phi_1, \phi_2) \tag{5}$$

for $t, \bar{t} \in U$, $t \neq \bar{t}$, $\phi_1, \phi_2 \in lip(U, C)$. Let us fix $x, \bar{x} \in U$, $x \neq 0$, $|\bar{x}| < |x|$, $y_1, \bar{y_1}, y_2, \bar{y_2} \in C$. Write

$$\phi_i(t) := \begin{cases} \bar{y_i}, & |t| \leq |\bar{x}| \\ \frac{y_i - \bar{y_i}}{|x| - |\bar{x}|}(|t| - |\bar{x}|) + \bar{y_i}, & |\bar{x}| \leq |t| \leq |x| \\ y_i, & |t| \geq |x| \end{cases} \tag{6}$$

for $t \in U$ and $i = 1, 2$. It can be easy verified that $\phi_i \in lip(U, C)$ and

$$d(\phi_1, \phi_2) = |\bar{y_1} - \bar{y_2}| + \frac{|y_1 - y_2 - \bar{y_1} + \bar{y_2}|}{|x| - |\bar{x}|}. \tag{7}$$

Putting in (5) ϕ_1 and ϕ_2 given by (6) and $t = x$, $\bar{t} = \bar{x}$ we have

$$\frac{\delta(h(x, y_1) + h(\bar{x}, \bar{y_2}), h(\bar{x}, \bar{y_1}) + h(x, y_2))}{|x - \bar{x}|} \leq c\left(|\bar{y_1} - \bar{y_2}| + \frac{|y_1 - y_2 - \bar{y_1} + \bar{y_2}|}{|x| - |\bar{x}|}\right).$$

Hence we get

$$\delta(h(x, y_1) + h(\bar{x}, \bar{y_2}), h(\bar{x}, \bar{y_1}) + h(x, y_2)) \leq c\left(|\bar{y_1} - \bar{y_2}||x - \bar{x}| + |y_1 - y_2 - \bar{y_1} + \bar{y_2}|\frac{|x - \bar{x}|}{|x| - |\bar{x}|}\right). \tag{8}$$

Obviously $|x - \bar{x}|/(|x| - |\bar{x}|) \geq 1$. Moreover, for $\bar{x} = \lambda x$, where $0 < \lambda < 1$, we have

$$\frac{|x - \bar{x}|}{|x| - |\bar{x}|} = \frac{|x - \lambda x|}{|x| - \lambda|x|} = 1.$$

Thus $\liminf_{\bar{x} \to x} |x - \bar{x}|/(|x| - |\bar{x}|) = 1$. Taking the lim inf as $\bar{x} \to x$ in (8) we obtain

$$\delta(h(x, y_1) + h(x, \bar{y_2}), h(x, \bar{y_1}) + h(x, y_2)) \leq c|y_1 - y_2 - \bar{y_1} + \bar{y_2}| \tag{9}$$

for all $x \neq 0$, $x \in U$ and $y_1, y_2, \bar{y_1}, \bar{y_2} \in C$. Inequality (9) holds also for $x = 0$ in virtue of the continuity of $h(\cdot, y)$. Putting $y_1 = \bar{y_2} = \frac{y+w}{2}$, $y_2 = y$, $\bar{y_1} = w$, $y, w \in C$, in (9) we obtain

$$\delta\left(2h\left(x, \frac{y+w}{2}\right), h(x,w) + h(x,y)\right) = 0,$$

and hence

$$h\left(x, \frac{y+w}{2}\right) = \frac{1}{2}[h(x,w) \overset{*}{+} h(x,y)]$$

for all $x \in U$, $y, w \in C$. This means that for every $x \in U$ the set-valued function $y \longmapsto h(x,y)$ is *Jensen.

Putting $\bar{y_2} = y_1$ and $\bar{y_1} = y_2$ in (9) we have

$$\delta(h(x,y_1), h(x,y_2)) \leq c|y_1 - y_2|,$$

for $y_1, y_2 \in C$ and $x \in U$. Thus $h(x, \cdot)$ is a Lipschitz function for all $x \in U$.

To prove (c) take in (8) $y_1 = y$, $\bar{y_2} = \bar{y}$, $\bar{y_1} = y$, $y_2 = \bar{y}$, where $y, \bar{y} \in C$. Then we obtain

$$\delta(h(x,y) + h(\bar{x},\bar{y}), h(x,\bar{y}) + h(\bar{x},y)) \leq c|y - \bar{y}||x - \bar{x}|. \tag{10}$$

Conditions (10) and (a) imply

$$\rho_1(h(x,\cdot), h(\bar{x},\cdot)) \leq c_1|x - \bar{x}|,$$

where c_1 is a constant. ■

2. Jensen equation on a semigroup

Let $(S, +)$ be an Abelian semigroup with zero satisfying the *cancelation law*, i.e., $t + s = t' + s$ implies $t = t'$.

An Abelian semigroup S with zero is said to be an *abstract convex cone* if a map $(\lambda, s) \to \lambda s$ defined on $\mathbb{R}_+ \times S$ into S is given such that

$$1 \cdot s = s, \quad \lambda(\mu s) = (\lambda\mu)s, \quad \lambda(s+t) = \lambda s + \lambda t, \quad (\lambda + \mu)s = \lambda s + \mu s$$

for all $s, t \in S$ and $\lambda, \mu \in \mathbb{R}_+$. We will assume that an abstract convex cone is endowed with a complete metric ρ such that

$$\rho(s+t, s+t') = \rho(t,t') \quad \text{for all} \quad s,t,t' \in S \tag{11}$$

and

$$\rho(\lambda s, \lambda t) = \lambda\rho(s,t) \quad \text{for all} \quad \lambda \in \mathbb{R}_+, \; s,t \in S. \tag{12}$$

The following lemma follows easily from (11) and (12).

LEMMA 4. *The functions*

$$\mathbb{R}_+ \times S \ni (\lambda, s) \longmapsto \lambda s \in S \quad \text{and} \quad S \times S \ni (s,t) \longmapsto s + t \in S$$

are continuous.

Let $(M, +)$ be an Abelian semigroup. We say that a function $a : M \to S$ is *additive* if it satisfies the Cauchy functional equation

$$a(x+y) = a(x) + a(y) \quad \text{for all} \quad x, y \in M. \tag{13}$$

Let us add that the division by 2 is performable in M. We say that a function $f : M \to S$ is *Jensen* if

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \quad \text{for all} \quad x, y \in M. \tag{14}$$

We are concerned with the general solution of the Jensen equation (14).

THEOREM 1. *Let $(M, +)$ be an Abelian semigroup such that the division by 2 is performable and let $(S, +, \cdot)$ be an abstract convex cone satisfying the cancellation law. Assume that a complete metric ρ is given in S such that (11) and (12) hold. Then, $f : M \to S$ is a Jensen function if and only if there exists an additive function $a : M \to S$ such that*

$$f(x+y) = a(x) + f(y) \quad \textit{for all} \quad x, y \in M. \tag{15}$$

Proof. Take arbitrary $x, y \in M$. By (14)

$$f(x+y) = f\left(\frac{2x+y+y}{2}\right) = \frac{1}{2}[f(2x+y) + f(y)]$$

and

$$f(2x+y) = f\left(\frac{4x+y+y}{2}\right) = \frac{1}{2}[f(4x+y) + f(y)].$$

The above two equalities yield

$$\begin{aligned} f(x+y) &= \frac{1}{2}\left[\frac{1}{2}\Big(f(4x+y) + f(y)\Big) + f(y)\right] \\ &= \frac{1}{2^2} f(2^2x + y) + \frac{2^2-1}{2^2} f(y). \end{aligned}$$

An easy induction shows that

$$f(x+y) = \frac{1}{2^n} f(2^n x + y) + \frac{2^n - 1}{2^n} f(y) \quad \text{for all} \quad x, y \in M, \quad n \in \mathbb{N}. \tag{16}$$

Let us fix $y \in M$ and define the functions $f_{yn} : M \to S$ as follows

$$f_{yn}(x) = \frac{1}{2^n} f(2^n x + y), \qquad x \in M, \quad n \in \mathbb{N}.$$

Let $x \in M$. We will verify that $\{f_{yn}(x)\}$ is a Cauchy sequence. Take $m, n \in \mathbb{N}$ such that $n > m$. We have by (11), (12) and (16)

$$\begin{aligned}
&\rho(f_{yn}(x), f_{ym}(x)) \\
&= \rho\left(f_{yn}(x) + \frac{2^n - 1}{2^n} f(y), f_{ym}(x) + \frac{2^n - 1}{2^n} f(y) \right) \\
&= \rho\left(f(x+y), f_{ym}(x) + \left(\frac{2^m - 1}{2^m} + \frac{2^n - 1}{2^n} - \frac{2^m - 1}{2^m} \right) f(y) \right) \\
&= \rho\left(f(x+y), f(x+y) + \left(\frac{1}{2^m} - \frac{1}{2^n} \right) f(y) \right) \\
&= \rho\left(0, \left(\frac{1}{2^m} - \frac{1}{2^n} \right) f(y) \right) \\
&= \left(\frac{1}{2^m} - \frac{1}{2^n} \right) \rho\Big(0, f(y) \Big).
\end{aligned}$$

Consequently, there exists a limit of the sequence $\{f_{yn}(x)\}$. Define the function $a_y : M \to S$ by

$$a_y(x) = \lim_{n\to\infty} f_{yn}(x), \qquad x \in M.$$

We have by (16) and Lemma 4 for arbitrary $x, y \in M$

$$f(x+y) = a_y(x) + f(y). \tag{17}$$

We shall show that a_y satisfies equation (13) for every $y \in M$. Take arbitrary $x, y, z \in M$. By (14)

$$\begin{aligned}
a_y(x+z) &= \lim_{n\to\infty} f_{yn}(x+z) \\
&= \lim_{n\to\infty} \frac{1}{2^n} f(2^n(x+z)+y) \\
&= \lim_{n\to\infty} \frac{1}{2^n} f\left(\frac{2^{n+1}x + y + 2^{n+1}z + y}{2} \right) \\
&= \lim_{n\to\infty} \left[\frac{1}{2^{n+1}} f\left(2^{n+1}x + y \right) + \frac{1}{2^{n+1}} f\left(2^{n+1}z + y \right) \right] \\
&= a_y(x) + a_y(z),
\end{aligned}$$

i.e., a_y is an additive function. Now we are going to show that the function a_y does not depend on y. From (14) and (17) we obtain

$$f(x) + f(y) = 2f\left(\frac{x+y}{2} \right) = 2a_{\frac{y}{2}}\left(\frac{x}{2} \right) + 2f\left(\frac{y}{2} \right) \tag{18}$$

for all $x, y \in M$. Setting $x = y$ we get hence

$$f(y) = a_{\frac{y}{2}}\left(\frac{y}{2}\right) + f\left(\frac{y}{2}\right).$$

The last relation and (18) yield

$$f(x) + a_{\frac{y}{2}}\left(\frac{y}{2}\right) + f\left(\frac{y}{2}\right) = 2a_{\frac{y}{2}}\left(\frac{x}{2}\right) + 2f\left(\frac{y}{2}\right),$$

whence, since $a_{\frac{y}{2}}$ is additive,

$$f(x) + a_{\frac{y}{2}}\left(\frac{y}{2}\right) = a_{\frac{y}{2}}(x) + f\left(\frac{y}{2}\right)$$

for every $x, y \in M$. Inserting $2^n x$ in the place of x yields

$$\frac{1}{2^n} f\left(2^n x\right) + \frac{1}{2^n} a_{\frac{y}{2}}\left(\frac{y}{2}\right) = a_{\frac{y}{2}}(x) + \frac{1}{2^n} f\left(\frac{y}{2}\right).$$

for every $x, y \in M$. Letting $n \to \infty$ we obtain hence in view of Lemma 4

$$a_{\frac{y}{2}}(x) = \lim_{n \to \infty} \frac{1}{2^n} f\left(2^n x\right)$$

for all $x, y \in M$. We put $a(x) = a_y(x)$, $x \in M$ for some $y \in M$. The definition of a is unambiguous. Now (17) yields

$$f(x + y) = a(x) + f(y)$$

for all $x, y \in M$. The first part of theorem was proved. Conversely, we will show that every function of form (15), where $a : M \to S$ is additive actually satisfies equation (14). By (15)

$$\begin{aligned} 2f(x+y) &= 2f\left(\frac{1}{2}(x+y) + \frac{1}{2}(x+y)\right) = 2a\left(\frac{x+y}{2}\right) + 2f\left(\frac{x+y}{2}\right) \\ &= a(x) + a(y) + 2f\left(\frac{x+y}{2}\right), \end{aligned}$$

$$a(x) + f(y) = f(x+y),$$

$$a(y) + f(x) = f(x+y), \qquad x, y \in M.$$

Hence

$$2f(x+y) + a(x) + a(y) + f(y) + f(x) = a(x) + a(y) + 2f\left(\frac{x+y}{2}\right) + 2f(x+y), \quad x, y \in M.$$

Cancelling $a(x) + a(y) + 2f(x+y)$ we get

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \quad x, y \in M,$$

i.e., f is a Jensen function. ∎

REMARK 1. An additive function in formula (15) is uniquely determined.

From Theorem 1 we deduce two corollaries.

COROLLARY 1. *Let M and S be as in Theorem 1. If $f : M \to S$ is a Jensen function, then there exists an additive function $a : M \to S$ such that*

$$f(x) + a(y) = f(y) + a(x), \qquad x, y \in M.$$

COROLLARY 2. *Let $(M, +)$ be an Abelian semigroup with zero such that the division by 2 is performable and let $(S, +, \cdot)$ be as in Theorem 1. Then, $f : M \to S$ is a Jensen function if and only if there exist an additive function $a : M \to S$ and a constant $b \in S$ such that*

$$f(x) = a(x) + b \quad \textit{for all} \quad x \in M.$$

3. Pexider equation on a semigroup

In this part of the paper we will deal with the Pexider functional equation

$$f(x+y) = g(x) + h(y). \tag{19}$$

At first we will prove that f, g, h satisfying equation (19) have to be Jensen functions.

LEMMA 5. *Let $(M, +)$ be an Abelian semigroup such that the division by 2 is performable and let $(S, +)$ be an Abelian semigroup with zero satisfying the cancellation law. If $f : M \to S, g : M \to S, h : M \to S$ are solutions of equation* (19), *then they satisfy the Jensen functional equation.*

Proof. Take arbitrary $x, y \in M$. Inserting in (19) $\frac{1}{2}x$ and $\frac{1}{2}y$ instead of x and y, respectively, we get

$$f\left(\frac{x+y}{2}\right) = g\left(\frac{x}{2}\right) + h\left(\frac{y}{2}\right), \quad f\left(\frac{x+y}{2}\right) = g\left(\frac{y}{2}\right) + h\left(\frac{x}{2}\right),$$

whence

$$2f\left(\frac{x+y}{2}\right) = g\left(\frac{x}{2}\right) + h\left(\frac{x}{2}\right) + g\left(\frac{y}{2}\right) + h\left(\frac{y}{2}\right) = f(x) + f(y),$$

i.e., f is a Jensen function.

Now take arbitrary $x, y, z \in M$. Setting in (19) successively $x, y, \frac{x+y}{2}$ in the place of x and z in the place of y, we obtain

$$f(x+z) = g(x) + h(z), \quad f(y+z) = g(y) + h(z),$$

$$f\left(\frac{x+y}{2}+z\right)=g\left(\frac{x+y}{2}\right)+h(z).$$

Hence

$$2g\left(\frac{x+y}{2}\right)+2h(z)=2f\left(\frac{x+y}{2}+z\right)=2f\left(\frac{1}{2}(x+z+y+z)\right)$$
$$=f(x+z)+f(y+z)=g(x)+g(y)+2h(z).$$

Cancelling $2h(z)$ we see that g is also a Jensen function. Similar calculations leads to the equality

$$2h\left(\frac{x+y}{2}\right)=h(x)+h(y). \blacksquare$$

Lemma 5 generalizes Lemma 4 in [7].

THEOREM 2. *Assume that* $(M,+)$ *and* $(S,+,\cdot)$ *are as in Theorem* 1. *Then functions* $f:M\to S, g:M\to S, h:M\to S$ *satisfy equation* (19) *if and only if there exists an additive function* $a:M\to S$ *such that*

$$\begin{aligned} f(x+y)&=a(x)+f(y), \quad g(x+y)=a(x)+g(y),\\ h(x+y)&=a(x)+h(y)\end{aligned} \tag{20}$$

for all $x,y\in M$ *and there is an element* $u\in M$ *such that*

$$f(2u)=h(u)+g(u). \tag{21}$$

Proof. Let f,g,h satisfy equation (19). In virtue of Lemma 5 and Theorem 1 there exist additive functions $a_i:M\to S$, $i=1,2,3$, such that

$$\begin{aligned} f(x+y)&=a_1(x)+f(y),\\ g(x+y)&=a_2(x)+g(y),\\ h(x+y)&=a_3(x)+h(y)\end{aligned} \tag{22}$$

for all $x,y\in M$. Hence we have

$$f(x+y+z)=a_1(x)+f(y+z) \tag{23}$$

for all $x,y,z\in M$. On the other hand

$$f(x+y+z)=g(x+y)+h(z)=a_2(x)+g(y)+h(z)=a_2(x)+f(y+z) \tag{24}$$

for $x,y,z\in M$, whence by (23) $a_1(x)=a_2(x)$, $x\in M$. Similarly we can derive that $a_1(x)=a_3(x)$, $x\in M$. With $a:=a_1$ we obtain (20) according to (22). Relation (21) with arbitrary $u\in M$ is obvious.

Conversely, assume that functions $f:M\to S, g:M\to S$ and $h:M\to S$ satisfy (20) for all $x,y\in M$ and (21) for some $u\in M$, where $a:M\to S$ is additive. We have

$$f(x+y+2u) = a(x+y) + f(2u) = a(x) + a(y) + h(u) + g(u)$$
$$= g(x+u) + h(y+u) = g(x) + h(y) + 2a(u)$$

for all $x, y \in M$. On the other hand,

$$f(x+y+2u) = f(x+y) + a(2u) = f(x+y) + 2a(u)$$

for all $x, y \in M$. Comparing the above equalities we get

$$f(x+y) = g(x) + h(y), \qquad x, y \in M,$$

i.e., f, g, h satisfy the Pexider equation. ■

As an immediate consequence of Theorem 2 we obtain the following

COROLLARY 3. *Assume that $(S, +, \cdot)$ is as in Theorem 1 and $(M, +)$ is an Abelian semigroup with zero such that the division by 2 is performable. Then functions $f : M \to S$, $g : M \to S$, $h : M \to S$ satisfy the Pexider equation if and only if there exist an additive function $a : M \to S$ and constants $b, c \in S$ such that*

$$f(x) = a(x) + b + c, \qquad g(x) = a(x) + b, \qquad h(x) = a(x) + c$$

for all $x \in M$.

4. Applications

Let X be a real normed space. From Lemma 1, formulas (1) and (2) we derive the following result.

LEMMA 6. *The set $ccl(X)$ with the operation $\overset{*}{+}$ and the multiplication by non-negative numbers is an abstract convex cone with the cancellation law.*

The abstract convex cone $ccl(X)$ satisfies the assumptions of Theorem 1 in virtue of (3) and Lemmas 6, 2 and 3. The following result follows from Theorem 1.

THEOREM 3. *Let $(M, +)$ be an Abelian semigroup such that the division by 2 is performable and let X be a Banach space. Then a set-valued function $F : M \to ccl(X)$ is *Jensen if and only if there exists a set-valued function $A : M \to ccl(X)$ such that*

$$A(x+y) = A(x) \overset{*}{+} A(y) \tag{25}$$

and

$$F(x+y) = A(x) \overset{*}{+} F(y) \tag{26}$$

for all $x, y \in M$.

Assuming that a semigroup M contains zero we have

COROLLARY 4. *Let $(M, +)$ be an Abelian semigroup with zero such that the division by 2 is performable and let X be a Banach space. Then a set-valued function $F : M \to ccl(X)$ is *Jensen if and only if there exist a set-valued function $A : M \to ccl(X)$ and a set $B \in ccl(X)$ such that* (25) *holds for $x, y \in M$ and*

$$F(x) = A(x) \overset{*}{+} B, \qquad x \in M.$$

Similar results under the assumption that F has compact values have been obtained by Fifer (see Theorem 2 in [3]) and Nikodem (see Theorem 5.6 in [5]).

In the above two theorems the assumption that the values of the function F are convex and closed is superfluous. In fact, if a set-valued function F is *Jensen, then setting $y = x$ in the equality $F((x+y)/2) = (1/2)(F(x) \overset{*}{+} F(y))$ we obtain $2F(x) = F(x) \overset{*}{+} F(x)$ for $x \in M$. Thus $F(x)$ has to be closed and since $F(x) + F(x) \subset 2F(x)$, the set $F(x)$ is also convex for every $x \in M$.

From Theorem 2 we can derive the following

THEOREM 4. *Let $(M, +)$ be an Abelian semigroup such that the division by 2 is performable and let X be a Banach space. Then set-valued functions $F : M \to ccl(X), G : M \to ccl(X), H : M \to ccl(X)$ satisfy the functional equation*

$$F(x+y) = G(x) \overset{*}{+} H(y), \qquad x, y \in M \tag{27}$$

if and only if there exists a set-valued function $A : M \to ccl(X)$ such that (25),

$$F(x+y) = A(x) \overset{*}{+} F(y), \quad G(x+y) = A(x) \overset{*}{+} G(y), \quad H(x) = A(x) \overset{*}{+} H(y)$$

hold, for all $x, y \in M$ and there is an element $u \in M$ such that $F(2u) = G(u) \overset{*}{+} H(u)$.

If a semigroup contains zero, Theorem 4 can be improved.

COROLLARY 5. *Let $(M, +)$ be an Abelian semigroup with zero such that the division by 2 is performable and let X be a Banach space. Then set-valued functions $F : M \to ccl(X), G : M \to ccl(X), H : M \to ccl(X)$ satisfy functional equation* (27) *if and only if there exist a set-valued function $A : M \to ccl(X)$ and sets $B, C \in ccl(X)$ such that* (25) *and*

$$F(x) = A(x) \overset{*}{+} B \overset{*}{+} C, \quad G(x) = A(x) \overset{*}{+} B, \quad H(x) = A(x) \overset{*}{+} C$$

hold, for all $x \in M$.

Corollary 5 is known even in the case when X is a Hausdorff topological vector space but set-valued functions appearing there have compact values (cf. [5], Theorem 5.7).

From Proposition and Corollary 4 we can derive the following theorem concerning to Nemytskii operators

THEOREM 5. *Let $(X, |\cdot|)$, $(Y, |\cdot|)$ be normed spaces and let $(Z, |\cdot|)$ be a Banach space. Assume that C is a convex cone with zero in Y and that $U \subset X$ is a convex set, $0 \in U$ and $h : U \times C \to ccl(Z)$. If the Nemytskii operator N defined by (4) satisfies the following conditions:*

1. $N(lip(U, C)) \subset Lip(U, Z)$;
2. *there exists $c \geq 0$ such that*

$$\rho(N\phi_1, N\phi_2) \leq cd(\phi_1, \phi_2), \quad \phi_1, \phi_2 \in lip(U, C),$$

then there exist set-valued functions $A : U \times C \to ccl(Z)$, $B \in Lip(U, Z)$ such that

$$A(x, y + w) = A(x, y) \stackrel{*}{+} A(x, w) \quad \text{for } x \in U, \quad y, w \in C$$

and

(a) $A(\cdot, y) \in Lip(U, Z)$ *for all* $y \in C$;

(b) *for every $x \in U$ the set-valued function $A(x, \cdot)$ is a Lipschitz function with the Lipschitz constant c;*

(c) *the function $x \longmapsto A(x, \cdot)$ defined on U is Lipschitz with the Lipschitz constant c ;*

(d) $h(x, y) = A(x, y) \stackrel{*}{+} B(x)$ *for all $x \in U$ and $y \in C$.*

Analogous theorems to Theorem 5 for set-valued function h with compact, convex values can be found in [8] (see Theorem 1).

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