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A NOTE ON THE RELATIVE ERROR FOR THE EIGENVALUES OF THE STURM–LIOUVILLE PROBLEM

1. Introduction

This work is motivated as an attempt to deal with an estimation of the relative error for the eigenvalues of the Sturm–Liouville problem. We are interested in finding upper bounds for such error. The uniform spectral continuity of this operator was treated in [14] in very simple manner. It is worth pointing out that the knowledge of the bounds for these two errors allows us to consider the eigenvalue problem for Sturm–Liouville operator in normal form to be perfectly posed for each eigenvalue regardless of its index.

Similar problem in the finite dimensional space setting was considered in [8] and [22]. Yet another approach to the bounds for the absolute errors of eigenvalues by the Gerschgorin estimate under unitary similarity has been recently investigated in [24].

2. Notation and the case of the absolute error

Suppose that

$$\mathcal{D} = \{u \in H^2(a,b) : \alpha u(a) + \alpha' u'(a) = 0,\ \beta u(b) + \beta' u'(b) = 0\}.$$

The constants $\alpha, \alpha', \beta, \beta'$ are assumed to be real with $\alpha^2 + \alpha'^2 > 0$, $\beta^2 + \beta'^2 > 0$, and the interval $[a,b]$ is finite.

For $u \in \mathcal{D}$, let

$$Lu = -u'' + qu,$$
$$\widetilde{L}u = -u'' + \widetilde{q}u,$$

where the real-valued functions $q, \widetilde{q}$ are assumed to belong to $C([a,b])$. Now

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consider the eigenvalue problems for operators L and $\widetilde{L}$:

$$Lu = \lambda u, \quad u \in \mathcal{D}, \tag{1}$$

$$\widetilde{L}u = \widetilde{\lambda}u, \quad u \in \mathcal{D}. \tag{2}$$

It is known that the operators L and $\widetilde{L}$ are self-adjoint (see [23]), and eigensystems (1) and (2) with the separated boundary conditions given, are regular and have sequences of simple real distinct eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, $\{\widetilde{\lambda}_k\}_{k=1}^{\infty}$ such that

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad \text{and} \quad \widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \widetilde{\lambda}_3 < \ldots,$$

and corresponding sequences of orthonormal eigenfunctions $\{u_k\}_{k=1}^{\infty}$, $\{\widetilde{u}_k\}_{k=1}^{\infty}$ such that $\|u_k\|_2 = \|\widetilde{u}_k\|_2 = 1$ (see [5]). With this notation we have the following theorem (see [14], [20] for more details).

THEOREM 1. *If* $\|\widetilde{q} - q\|_\infty \equiv \sup_{x\in[a,b]} |\widetilde{q}(x) - q(x)| \le \varepsilon$, *then* $|\widetilde{\lambda}_k - \lambda_k| \le \varepsilon$, *for each* $k = 1, 2, 3, \ldots$.

The important point to note here is that the estimation in Theorem 1 is sharp (see e.g. [14]). The bound with the L^2-norm is not valid; that is the estimation $|\widetilde{\lambda}_k - \lambda_k| \le \|\widetilde{q} - q\|_2$, where $\|u\|_2 = (\int_a^b |u|^2\,dx)^{1/2}$ does not hold, contrary to what was claimed in [13]. Here is an example.

EXAMPLE 1. Consider two equations $-u'' + q_i(x)u = \lambda u$ $(i = 1, 2)$ on the interval $[0, 1]$, where $q_1(x) = 0.1 \cdot x$ and $q_2(x) = 0.1 \cdot x^2$ with the same boundary conditions $u(0) = u(1) = 0$. We compute an approximation of the smallest eigenvalue of these equations. To do this we use the finite element method with the AAdHP correction (see [2]); for other alternative numerical schemes we refer the reader to [1], [4], [7], [11], [12] and [19]. Then the results with the accuracy of 5 digit after the period are as follows: $\lambda_1^{(1)} = 9.91959$, $\lambda_1^{(2)} = 9.89786$. Thus the difference $|\lambda_1^{(1)} - \lambda_1^{(2)}| = 0.02173$ is greater than $\|q_1 - q_2\|_2 = 0.018257$.

3. The spectral bound of the relative error

To simplify our discussion it helps to assume that L is positive definite which is in view of the Courant-Fischer-Poincaré minmax theorem equivalent to having the lowest eigenvalue $\lambda_1 > 0$. This can be achieved by replacing $q(x)$ by $q(x) + \lambda^*$, where λ^* is such that $\lambda^* + \lambda_1 > 0$; this just shifts all the eigenvalues up by λ^* without essentially changing the problem. From now on we make the assumption stated above; so the sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ from Section 2 satisfies one more inequality

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots. \tag{3}$$

Let F denote the bounded symmetric operator $Fu = (\widetilde{q} - q)u$ defined for $u \in L^2(a,b)$. We begin with a straightforward lemma, an operator-norm estimate of the relative error.

LEMMA 1. *Under the above assumption the following inequality holds, for each* $k = 1, 2, 3, \ldots$,

$$\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \le \|L^{-1}\| \cdot \|F\|. \tag{4}$$

Proof. From Theorem 1 it follows that

$$|\widetilde{\lambda}_k - \lambda_k| \le \|\widetilde{q} - q\|_\infty = \|F\|_{[L^2]}.$$

We will write simply $\|.\|$ instead of $\|\cdot\|_{[L^2]}$ when it causes no confusion. Remembering (3) we see that

$$\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \le \|F\| \cdot \frac{1}{\lambda_k} \le \|F\| \cdot \frac{1}{\lambda_1} = \|F\| \cdot \|L^{-1}\|.$$

This completes the proof. ■

Now we are in position to state the following strengthening of Lemma 1. The bound in the next theorem is a bit sharper.

THEOREM 2. *Under the assumption of this section we have, for each* $k = 1, 2, 3, \ldots$,

$$\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \le \rho(L^{-1}F), \tag{5}$$

where $\rho(\cdot)$ *denotes the spectral radius.*

Proof. It is known that $L^{-1}\colon C([a,b]) \to \mathcal{D}$ exists and is a an integral operator with continuous kernel (see e.g. [16, Th. V.5.4]), hence its extension on $L^2(a,b)$ is compact. By ([16, Th. IX.5.1]), L^{-1} is also positive definite, hence it has the unique square root $L^{-\frac{1}{2}}$, which is a positive definite operator. Let $\hat{F} = L^{-\frac{1}{2}} F L^{-\frac{1}{2}}$. It is well known that each eigenvalue of the Sturm–Liouville problem satisfies Poincaré's minmax principle (see [6]), which asserts that

$$\lambda_k = \min_{H_k \subset \mathcal{D}} \max_{0 \ne u \in H_k} R[u], \tag{6}$$

where H_k varies over any k-dimensional subspaces of $\mathcal{D}$, and $R[u]$ is the Rayleigh quotient of (1), that is,

$$R[u] = \frac{(Lu, u)}{(u, u)}.$$

Since $(Lu, u) > 0$ for $0 \neq u \in \mathcal{D}$ we can write the Rayleigh quotient of (2) as follows

$$\widehat{R}[u] = \frac{(\widetilde{L}u, u)}{(u, u)} = \frac{(Lu, u)}{(u, u)} \cdot \left(1 + \frac{(Fu, u)}{(Lu, u)}\right). \tag{7}$$

It is easy to check that

$$\frac{(Fu, u)}{(Lu, u)} = \frac{(\widehat{F}\widehat{u}, \widehat{u})}{(\widehat{u}, \widehat{u})}, \tag{8}$$

where $0 \neq \widehat{u} \in L^2([a, b])$ is defined as $L^{-\frac{1}{2}}\widehat{u} = u$. Since $\widehat{F}$ is also self-adjoint we have

$$\left|\frac{(\widehat{F}\widehat{u}, \widehat{u})}{(\widehat{u}, \widehat{u})}\right| \leq \|\widehat{F}\| = \rho(\widehat{F}). \tag{9}$$

This yields that

$$-\rho(\widehat{F}) \leq \frac{(\widehat{F}\widehat{u}, \widehat{u})}{(\widehat{u}, \widehat{u})} \leq \rho(\widehat{F}). \tag{10}$$

Since $\rho(AB) = \rho(BA)$ provided that A, B are bounded we see that $\rho(\widehat{F}) = \rho(L^{-\frac{1}{2}}FL^{-\frac{1}{2}}) = \rho(L^{-1}F)$. From the above and (7), (8), (10) it follows that

$$R[u](1 - \rho(L^{-1}F)) \leq \widehat{R}[u] \leq R[u](1 + \rho(L^{-1}F)).$$

Applying "minmax" characterization (6) we deduce that

$$\lambda_k(1 - \rho(L^{-1}F)) \leq \widetilde{\lambda}_k \leq \lambda_k(1 + \rho(L^{-1}F))$$

and thus

$$\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \leq \rho(L^{-1}F),$$

which is the estimate desired. ∎

Let us denote by $E = C([a, b])$ and set

$$r(x) = \widetilde{q}(x) - q(x).$$

Therefore if $r(x) > 0$ on $[a, b]$, the positive compact operator $L^{-1}F : E \to E$ is irreducible (see [21, p. 269]) and hence has positive spectral radius $\rho(L^{-1}F)$. The Krein-Rutman theorem (see [21, Th. 3.2, p. 270]) guarantees that $\tau_1 \equiv \rho(L^{-1}F)$ is an eigenvalue of $L^{-1}F$; it is the only eigenvalue of $L^{-1}F$ whose associated eigenspace contains a positive function. Moreover the geometric multiplicity of τ_1 is one, and since τ_1 is a pole of order 1 of the resolvent of $L^{-1}F$, the algebraic and geometric multiplicities of τ_1 are equal. Due to the above and as the numerical examples are involved for the regular Sturm–Liouville problem, from now on we make the assumption that $r(x) > 0$ on $[a, b]$. In the general case, i.e. when r may change sign in

$[a,b]$ we are led to the linear eigenvalue problem with an indefinite weight function (see [3], [17]). However, this topic exceeds the scope of this paper and the author intends to consider this problem in another paper.

Furthermore it follows easily from the proof of the "square root theorem" that if we add an extra assumption that a positive operator is also compact then its positive square root operator is compact as well. Thus we obtain that $L^{-\frac{1}{2}}$ is compact; so as the compact operators form a closed ideal we have that $\widehat{F}$ is also compact. According to the principle of related operators for Riesz operators (see [15], [18]) we can deduce that $\lambda_k(\widehat{F}) = \lambda_k(L^{-1}F)$ for $k = 1, 2, \dots$. So one could expect that it would be possible to strengthen the result in Theorem 2, namely

$$\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| \le |\lambda_k(L^{-1}F)|, \qquad k = 1, 2, \dots. \tag{11}$$

However this may not be true as the following example shows:

EXAMPLE 2. Let $[a,b] = [0, \frac{e^\pi - 1}{10}]$. We will consider the perturbation of the operator $Lu = -u''$ to $\widetilde{L}u = -u'' + \frac{1}{(x+0.1)^2}u$. The boundary conditions are Dirichlet's ones $u(a) = u(b) = 0$. For the unperturbed eigenvalue problem we have the eigenvalues $\lambda_k = (\frac{k\pi}{b})^2$, $k = 1, 2, \dots$. The eigenvalues of $\widetilde{L}$ are computed as in Example 1. This example is so conceived as to have the exact formulas for $\lambda_k(L^{-1}F) = \frac{1}{k^2+0.25}$. Then we can present the results in the table given below:

k	λ_k	$\widetilde{\lambda}_k$	$\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}$	$\lambda_k(L^{-1}F)$	$\rho(L^{-1}F)$	$\|L^{-1}\| \cdot \|F\|$
1	2.01334	2.99276	0.4865	0.8	0.8	49.6687
2	8.05336	9.76398	0.2124	0.2353	0.8	49.6687
3	18.12007	20.37210	0.1243	0.1081	0.8	49.6687
4	32.21346	34.86920	0.0824	0.0615	0.8	49.6687
5	50.33353	53.29400	0.0588	0.0396	0.8	49.6687
10	201.3341	205.05909	0.0185	0.00997	0.8	49.6687

4. Numerical examples

To provide numerical confirmation of the preceding results of Section 3 we intend to present two more examples. It is convenient for the purpose of this section to choose the interval $I = [a,b] = [0,\pi]$. In each of these examples we consider the perturbation of the operator L, which is simply $L = -\frac{d^2}{dx^2}$. Since the relative error ε_k is of order $O(k^{-2})$, we will examine only the case when $k = 1, 2, 3$.

EXAMPLE 3. In this example we perturb the operator $L = -\frac{d^2}{dx^2}$ to the operator $\widetilde{L} = -\frac{d^2}{dx^2} + \widetilde{q}$ with the same boundary conditions $u(0) = u(\pi) = 0$, where the function $\widetilde{q} = \widetilde{q}(x)$ is as follows:

$$\widetilde{q}(x) = \begin{cases} 2x, & \text{for } x \in (0, 1/2] \\ 1, & \text{for } x \in [1/2, \pi - 1/2] \\ 2(\pi - x), & \text{for } x \in [\pi - 1/2, \pi). \end{cases}$$

For the unperturbed Sturm–Liouville problem we have the eigenvalues $\lambda_k = k^2$ and the eigenfunctions $u_k(x) = \sqrt{\frac{2}{\pi}} \sin kx \quad (k = 1, 2, \ldots)$. The method of computation remains the same and the results are presented in the table given below:

k	$\widetilde{\lambda}_k$	$\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}$	$\rho(L^{-1}F)$	$\|L^{-1}\| \cdot \|F\|$
1	1.98691	0.98691	0.98742	1
2	4.95277	0.23819	0.98742	1
3	9.91021	0.10113	0.98742	1

EXAMPLE 4. If we consider the perturbed equation $-u'' + \epsilon u = \lambda u$ with boundary conditions $u(0) = u(\pi) = 0$, $\epsilon > 0$ we obtain the eigenvalues $\widetilde{\lambda}_k = k^2 + \epsilon$. Further $\rho(L^{-1}F) = \|L^{-1}\| \cdot \|F\| = \epsilon$ and $\left|\frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k}\right| = \lambda_k(L^{-1}F) = \frac{\epsilon}{k^2}$, $k = 1, 2, \ldots$. Hence for $k = 1$ equality $\left|\frac{\widetilde{\lambda}_1 - \lambda_1}{\lambda_1}\right| = \rho(L^{-1}F) = \epsilon$ holds.

5. Conclusion

To sum up, we have shown in Section 3, using accurate methods, that the absolute and relative errors of eigenvalues λ_k can be bounded independently of k when the data (function q) are perturbed to function $\widetilde{q}$. For numerical analysis, the inequality in Theorem 1 provides bounds on the absolute differences $|\widetilde{\lambda}_k - \lambda_k|$ between approximate eigenvalues $\widetilde{\lambda}_k$ and the true eigenvalues λ_k of the eigenproblem (1). Such bounds are sometimes unsatisfactory for small eigenvalues, which thereby suffer worse relative uncertainity than large ones. However Theorem 2 yields information about such uncertainity and its estimation of the relative error for the eigenvalues λ_k $(k = 1, 2, \ldots)$ is in some sense sharp (see Section 4).

6. Possible extensions

The idea of our proof can be extended to higher dimensions, so one can consider the eigenvalue problems of the form

$$\begin{aligned} Lu(x) + \lambda u(x) &= 0, \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned}$$

Here L is a linear second order selfadjoint elliptic differential operator, D is a bounded domain in R^d and ∂D is the boundary of D. Under sufficient smoothness conditions on the coefficients in L and ∂D, an infinite sequence of real eigenvalues, λ_k, say

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots,$$

exists. Thus obtaining the bounds for the relative error of the above eigenvalue problem could be an interesting question.

Despite the negative answer to the validity of the bound with the L^2-norm in Theorem 1, Example 1 seems to be of independent interest. Namely, consider the equations

$$-u'' + s_n u = \lambda^{(n)} u,$$

with given separated boundary conditions. Let us assume that $s_n \in L^2([0,1])$, $n = 0, 1, 2, \dots$ and $\|s_n - s_0\|_2 \to 0$. It is of interest to know whether for each fixed $k \in \{1, 2, \dots\}$: $\lambda_k^{(n)} \to \lambda_k^{(0)}$ as $n \to \infty$. One may even ask if this stays still true uniformly with respect to k, i.e. whether the following estimation holds

$$|\lambda_k^{(n)} - \lambda_k^{(0)}| \le \text{const} \cdot \|s_n - s_0\|_2.$$

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