DEMONSTRATIO MATHEMATICA
Vol. XXXII No 2 1999

Guang Zhang, Sui Sun Cheng

ON CONNECTED HALF-LINEAR DIFFERENTIAL EQUATIONS

Abstract. In this paper connections between several classes of "half-linear" differential equations with or without delays are established. By means of these connections, existence of eventually positive solutions can be inferred from the properties of either one of these families of equations.

1. Introduction

To motivate our concerns in this paper, let us first look at the following four differential equations in a formal manner:

$$((x(t) - x(t-\tau))^\alpha)' + q(t)x^\alpha(t-\sigma) = 0, \tag{1}$$

$$((x'(t))^\alpha)' + Q(t)x^\alpha(t-\sigma) = 0, \tag{2}$$

$$(((x(t) + p(t)x(t-\mu))')^\alpha)' + Q(t)x^\alpha(t-\sigma) = 0, \tag{3}$$

and

$$((x'(t))^\alpha)' + Q(t)x^\alpha(t) = 0, \tag{4}$$

where α is a quotient of positive odd integers and τ a positive number. First of all it is clear that an obvious relation exists between (2) and (4) by taking $\sigma = 0$ in (2). Next, if we divide equation (1) by τ^α, then

$$\left(\frac{(x(t) - x(t-\tau))^\alpha}{\tau^\alpha}\right)' + \frac{q(t)}{\tau^\alpha}x^\alpha(t-\sigma) = 0,$$

so that one of its "limiting equation" as $\tau \to 0$ occurs to be in the form (2) for some appropriate function $Q(t)$. Next, if we take $p(t) \equiv 0$ in (3), we also obtain equation (2). The above observations suggest that qualitative properties of the above equations are related when appropriate conditions are imposed.

1991 *Mathematics Subject Classification*: 34A35.
Key words and phrases: half-linear differential equations, eventually positive solutions, delay.

Several results related to equation (4) have already been reported by Kusano and Norio [1], and Li and Yeh [2, 3, 4] and others [5, 6, 7]. Therefore, once we have made clear the relationships between these equations, the known results will immediately yield qualitative information for the others. In this paper, we will be mainly concerned with the relations between the eventually positive solutions of these equations. In order to obtain these relations, the relations between these equations and their companion some functional inequalities will be needed first.

We remark that if $x = x(t)$ is a "solution" of one of the above equations, then a constant multiple of it is also a solution of the same equation. For this reason, some authors have called these equations half-linear.

2. First connection

We first consider the following functional differential equation

$$((x(t) - p(t)x(t-\tau))^\alpha)' + q(t)f(x(t-\sigma)) = 0, \qquad t \geq 0, \tag{5}$$

and one of its companion functional differential inequality

$$((x(t) - p(t)x(t-\tau))^\alpha)' + q(t)f(x(t-\sigma)) \leq 0, \qquad t \geq 0, \tag{6}$$

where α is a quotient of positive odd integers, τ is a positive number, σ is a nonnegative number, p and q are continuous functions defined on $[0, \infty)$ such that $q(t) \geq 0$ for $t \geq 0$ and $q(t_k) > 0$ for some increasing sequence $\{t_k\}$ which diverges to ∞. The function $f : R \to R$ is assumed to be continuous and nondecreasing with $f(x) > 0$ for $x > 0$.

A function $x = x(t)$ is said to be a solution of (5) if it is continuous for all large t and $((x(t) - p(t)x(t-\tau))^\alpha)'$ is continuous and equals $-q(t)f(x(t-\sigma))$ for all large t. A solution of (6) is similarly defined.

LEMMA 1. *Suppose $p = p(t)$ is nonnegative for $t \geq 0$ and suppose there is some positive number N such that $p(N + j\tau) \leq 1$ for $j = 0, 1, 2, \ldots$ Then for any eventually positive solution $x = x(t)$ of (6), the function*

$$y(t) = x(t) - p(t)x(t-\tau), \qquad t \geq 0,$$

will satisfy $y'(t) \leq 0$ and $y(t) > 0$ for all large t.

Proof. In view of (6) and the properties of the functions q and f, $[y^\alpha(t)]'$ is eventually nonpositive and does not vanish identically for large t. Thus $y(t)$ is of constant sign and $y^{\alpha-1}(t)$ is positive for all large t. Therefore, $y'(t), \alpha y^{\alpha-1}(t)y'(t) \leq 0$ eventually. Suppose to the contrary that $y(t)$ is eventually negative. Then there exists t_1 such that $x(t) > 0, y'(t) \leq 0$ and $y(t) \leq y(t_1) < 0$ for $t \geq t_1$. That is,

$$x(t) \leq y(t_1) + p(t)x(t-\tau), \qquad t \geq t_1.$$

Choosing k^* so large that $N + k^*\tau \geq t_1$, we have

$$\begin{aligned} x(N + k^*\tau + j\tau) &= y(N + k^*\tau + j\tau) \\ &\quad + p(N + k^*\tau + j\tau)x(N + k^*\tau + (j-1)\tau) \\ &\leq y(t_1) + x(N + k^*\tau + (j-1)\tau) \\ &\leq \ldots \leq jy(t_1) + x(N + k^*\tau), \quad j \geq 0. \end{aligned}$$

Letting j tend to infinity, we see that the right-hand side diverges to $-\infty$, which is contrary to our assumption that $x(t) > 0$ for $t \geq t_1$. The proof is complete.

THEOREM 1. *Suppose $p = p(t)$ is nonnegative for $t \geq 0$ and suppose there is some positive number N such that $p(N + j\tau) \leq 1$ for $j = 0, 1, 2, \ldots$ Suppose further that either $p(t) > 0$ for $t \geq 0$, or, $\sigma > 0$ and $q(t) > 0$ for $t \geq 0$. Then, equation* (5) *has an eventually positive solution if and only if the functional differential inequality* (6) *has an eventually positive solution.*

Proof. It suffices to show that if (6) has an eventually positive solution $x(t)$, then so does the equation (5). To see this, note that by means of Lemma 1, we may assume that there is M such that for $t \geq M - \max(\tau, \sigma)$, we have $x(t) > 0, x(t-\tau) > 0, x(t-\sigma) > 0$ and $y(t) = x(t) - p(t)x(t-\tau) > 0$. In view of (6),

$$(y^\alpha(t))' \leq -q(t)f(x(t-\sigma)), \quad t \geq M,$$

so that by integrating from t to infinity, we obtain

$$x(t) \geq p(t)x(t-\tau) + \Big(\int_t^\infty q(s)f(x(s-\sigma))\,ds\Big)^{\frac{1}{\alpha}}, \; t \geq M. \tag{7}$$

Let Ω be the set of all real functions defined on $[M - \max(\tau, \sigma), \infty)$. Define an operator $F : \Omega \to \Omega$ by

$$(Fw)(t) = 1$$

for $M - \max(\tau, \sigma) \leq t < M$, and

$$(Fw)(t) = \frac{1}{x(t)}\Big\{p(t)x(t-\tau)w(t-\tau) + \Big(\int_t^\infty q(s)f(x(s-\sigma)w(s-\sigma))\,ds\Big)^{\frac{1}{\alpha}}\Big\}$$

for $t \geq M$. Consider the sequence of successive approximations: $w_{(0)}(t) \equiv 1$ and $w_{(k+1)}(t) = (Fw_{(k)})(t)$ for $k = 0, 1, 2, \ldots$ Clearly, in view of (7),

$$0 \leq w_{(k+1)}(t) \leq w_{(k)}(t) \leq 1, \quad t \geq M, k = 0, 1, 2, \ldots$$

Thus as $k \to \infty$, $w_{(k)}$ converges pointwise to some nonnegative function $w(t)$, and

$$x(t)w(t) = p(t)x(t-\tau)w(t-\tau) + \Big(\int_t^\infty q(s)f(x(s-\sigma)w(s-\sigma))\,ds\Big)^{\frac{1}{\alpha}}, \quad t \geq M.$$

Let $v(t) = x(t)w(t)$ for $t \geq M - \max(\tau, \sigma)$. We assert that $v(t) > 0$ for $t \geq M - \max(\tau, \sigma)$. Indeed, since $v(t) = x(t) > 0$ for $M - \max(\tau, \sigma) \leq t < M$, we may assume that $v(t) > 0$ for $M - \max(\tau, \sigma) \leq t < t^*$, where $t^* \geq M$, and $v(t^*) = 0$. But then

$$0 = x(t^*)w(t^*) = p(t^*)x(t^* - \tau)w(t^* - \tau) + \Big(\int_{t^*}^{\infty} q(s) f(x(s - \sigma)w(s - \sigma))\, ds \Big)^{\frac{1}{\alpha}}.$$

Thus $p(t^*) = 0$ and $q(s)f(x(s - \sigma)w(s - \sigma)) = 0$ for $s \geq t^*$, which is a contradiction. Finally, it is easily verified that $v(t)$ is a solution of (5). The proof is complete.

3. Second connection

Next, we consider the following differential equation

$$((x'(t))^{\alpha})' + Q(t)f(x(t)) = 0, \qquad t \geq 0, \tag{8}$$

and one of its companion differential inequality

$$((x'(t))^{\alpha})' + Q(t)f(x(t)) \leq 0, \qquad t \geq 0, \tag{9}$$

where α is a quotient of positive odd integers, $Q = Q(t)$ is a continuous function defined on $[0, \infty)$ such that $Q(t) \geq 0$ for $t \geq 0$ and $Q(t_k) > 0$ for some increasing sequence $\{t_k\}$ which diverges to ∞. The function $f : R \to R$ is assumed to be continuous and nondecreasing and satisfies $f(x) > 0$ for $x > 0$.

A function $x = x(t)$ is said to be a solution of (8) if it is continuous for all large t and $((x'(t))^{\alpha})'$ is continuous and equals $-Q(t)f(x(t))$ for all large t. A solution of (9) is similarly defined.

THEOREM 2. *The equation* (8) *has an eventually positive solution if and only if the differential inequality* (9) *has an eventually positive solution.*

Proof. It suffices to show that if (9) has an eventually positive solution $x(t)$, then so does equation (8). To see this, note that in view of (9), $((x'(t))^{\alpha})'$ is eventually nonpositive and does not vanish identically for large t. Thus $x'(t)$ is of constant sign for all large t. If $x'(t) < 0$ for all large t, then there is some number M such that $x(t) > 0, x'(t) < 0$ and $(x'(t))^{\alpha} \leq (x'(M))^{\alpha}$ for $t \geq M$. Hence

$$x(t) = x(M) + \int_{M}^{t} x'(s)\, ds \leq x(M) + x'(M)(t - M), \qquad t \geq M.$$

As $t \to \infty$, the right hand side tends to $-\infty$, which is a contrary to the fact that $x(t) > 0$ for $t \geq M$. In other words, $x'(t) > 0$ for all large t.

We may now suppose there is some number T such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq T$. Integrating (9) twice, we obtain

$$x(t) \geq x(T) + \int_T^t \left\{ \int_s^\infty Q(u) f(x(u))\, du \right\}^{\frac{1}{\alpha}} ds$$

for $t \geq T$. Let Ω be the set of all real functions defined on $[T, \infty)$. Define an operator $F : \Omega \to \Omega$ by

$$(Fw)(t) = x(T) + \int_T^t \left\{ \int_s^\infty Q(u) f(w(u))\, du \right\}^{\frac{1}{\alpha}} ds, \qquad t \geq T.$$

Consider the sequence of successive approximations: $w_{(0)}(t) = x(t)$ for $t \geq T$, and $w_{(k+1)}(t) = (Fw)_{(k)}(t)$ for $k = 0, 1, \ldots$ and $t \geq T$. Clearly, we have

$$0 < x(T) \leq w_{(k+1)}(t) \leq w_{(k)}(t) \leq x(t), \qquad t \geq T, \qquad k = 0, 1, 2, \ldots$$

Thus as $k \to \infty$, $w_{(k)}(t)$ converges pointwise to some positive function $w^*(t)$, and

$$w^*(t) = x(T) + \int_T^t \left\{ \int_s^\infty Q(u) f(w^*(u))\, du \right\}^{\frac{1}{\alpha}} ds, \qquad t \geq T.$$

In view of this, it is easily verified that $w^*(t)$ is a solution of (8). The proof is complete.

4. Third connection

We now set $p(t) \equiv 1$ in the equation (5) and $Q(t) = q(t)/\tau^\alpha$ in (8) to obtain

(10) $$((x(t) - x(t-\tau))^\alpha)' + q(t) f(x(t-\sigma)) = 0, \qquad t \geq 0,$$

and

(11) $$((x'(t))^\alpha)' + \frac{q(t)}{\tau^\alpha} f(x(t)) = 0, \qquad t \geq 0,$$

respectively.

THEOREM 3. *The equation* (10) *has an eventually positive solution if and only if the equation* (11) *has an eventually positive solution.*

Proof. Let $x(t)$ be an eventually positive solution of (10) and let $y(t) = x(t) - x(t-\tau)$ for $t \geq 0$. By Lemma 1, we may assume that $x(t-\tau) > 0$, $y(t) > 0$ and $y'(t) \leq 0$, for $t \geq M - \max(\tau, \sigma)$. For any $t \geq M + \tau + \sigma$, there is a positive integer k such that

$$t - \sigma - k\tau \in [M - \tau, M].$$

Thus we have

$$x(t-\sigma) = x(t - \sigma - k\tau) + \sum_{j=0}^{k-1} y(t - \sigma - j\tau)$$

$$\geq \min_{M-\tau\leq t\leq M} x(t) + \frac{1}{\tau}\sum_{j=0}^{k-1} \tau y(t-\sigma-j\tau)$$

$$\geq \min_{M-\tau\leq t\leq M} x(t) + \frac{1}{\tau}\int_{M+2\tau}^{t-\sigma} y(s)\,ds,$$

where the last inequality holds since $y'(t) \leq 0$ for $t \geq M - \max(\tau, \sigma)$. Note that

$$\int_{M+2\tau}^{t-\sigma} y(s)\,ds \geq \int_{M+2\tau+\sigma}^{t-\sigma} y(s)\,ds,$$

thus

$$x(t-\sigma) \geq \min_{M-\tau\leq t\leq M} x(t) + \frac{1}{\tau}\int_{M+2\tau+\sigma}^{t-\sigma} y(s)\,ds.$$

Let

$$Z(t) = \min_{M-\tau\leq t\leq M} x(t) + \frac{1}{\tau}\int_{M+2\tau+\sigma}^{t-\sigma} y(s)\,ds$$

for $t \geq M + 2\tau + \sigma$. Then

$$\tau^{\alpha}[(Z'(t))^{\alpha}]' = -q(t)f(x(t-\sigma)) \leq -q(t)f(Z(t))$$

for all large t, as desired.

Conversely, let $w(t)$ be an eventually positive solution of (11). It is clear that $w(t)$ is eventually increasing and concave. We may thus assume that $0 < \tau w'(t) \leq w(t-\tau)$ for $t \geq T$, where T is a positive integer. Let

$$H(t) = \begin{cases} \tau w'(t) & t \geq T, \\ (t-T+\tau)w'(T) & T-\tau \leq t < T, \\ 0 & t < T-\tau, \end{cases}$$

and set

$$u(t) = \sum_{i=0}^{\infty} H(t-i\tau).$$

In view of the definition of $H(t)$, it is clear that $0 < u(t) < \infty$ for $t > T-\tau$,

$$u(t) - u(t-\tau) = H(t) = \tau w'(t), \qquad t \geq T$$

and

$$\max_{T-\tau\leq t\leq T} u(t) = \tau w'(T) > 0.$$

For any $t \geq T+\sigma$, there is a positive integer k such that $t-k\tau-\sigma \in [T-\tau, T]$.

Thus

$$u(t-\sigma) = u(t-\sigma-k\tau) + \tau\sum_{i=0}^{k-1} w'(t-\sigma-i\tau)$$
$$\leq \max_{T-\tau\leq t\leq T} u(t) + \int_{T-\tau}^{t-\sigma} w'(s)\,ds$$
$$\leq \tau w'(T) + \int_{T-\tau}^{t} w'(s)\,ds$$
$$= \tau w'(T) + w(t) - w(T-\tau) \leq w(t).$$

Finally,

$$[(u(t)-u(t-\tau))^\alpha]' = \tau^\alpha[(w'(t))^\alpha]' \leq -q(t)f(w(t))$$
$$\leq -q(t)f(u(t-\sigma))$$

for all large t. The proof is complete.

Since σ is an arbitrary nonnegative number in (10), we see that if we replace σ by $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$ in (10) then we obtain

$$((x(t)-x(t-\tau))^\alpha)' + q(t)f(x(t-\sigma_1)) = 0, \quad t \geq 0,$$

and

$$((x(t)-x(t-\tau))^\alpha)' + q(t)f(x(t-\sigma_1)) = 0, \quad t \geq 0.$$

We see that one of these two equations has an eventually positive solution if and only if the other one has an eventually positive solution.

As another immediate corollary, if we replace the function $f(x)$ with $f(u) = u^\alpha$ in (10) and (11) we obtain

$$((x(t)-x(t-\tau))^\alpha)' + q(t)x^\alpha(t-\sigma) = 0, \quad t \geq 0,$$

and

$$((x'(t))^\alpha)' + \frac{q(t)}{\tau^\alpha}x^\alpha(t) = 0, \quad t \geq 0.$$

Then we see that one of these two equations has an eventually positive solution if and only if the other one has an eventually positive solution. In particular, note that the equation

$$((x'(t))^\alpha)' + \left(\frac{\alpha}{\alpha+1}\right)^\alpha t^{-\alpha-1}x^\alpha(t) = 0$$

has an eventually positive solution $x(t) = t^{\alpha/(\alpha+1)}$, thus the following equation

$$((x(t)-x(t-\tau))^\alpha)' + \tau^\alpha\left(\frac{\alpha}{\alpha+1}\right)^\alpha t^{-\alpha-1}x^\alpha(t-\sigma) = 0$$

has an eventually positive solution.

5. Fourth connection

We now consider the following functional differential equation

$$(((x(t)+p(t)x(t-\tau))')^{\alpha})' + q(t)f(x(t-\sigma)) = 0, \qquad t \geq 0, \tag{12}$$

where α is a quotient of positive odd integers, τ is a positive number, σ is a nonnegative number, p and q are continuous functions defined on $[0,\infty)$ such that $q(t) \geq 0$ for $t \geq 0$ and $q(t_k) > 0$ for some increasing sequence $\{t_k\}$ which diverges to ∞. The function $f : R \to R$ is assumed to be continuous and nondecreasing and satisfies $f(x) > 0$ for $x > 0$.

A function $x = x(t)$ is said to be a solution of (12) if it is continuous for all large t and $(((x(t)+p(t)x(t-\tau))')^{\alpha})'$ is continuous and equals $-q(t)f(x(t-\sigma))$ for all large t.

THEOREM 4. *Suppose that* $0 \leq p(t) < 1$ *for* $t \geq 0$ *and* $f(xy) \geq \Gamma f(x)f(y)$ *for* $x, y > 0$ *and some* $\Gamma > 0$. *If* (12) *has an eventually positive solution, then the equation*

$$((y'(t))^{\alpha})' + \Gamma q(t)f(1-p(t-\sigma))f(y(t)) = 0 \tag{13}$$

has an eventually positive solution as well.

Proof. Let $x(t)$ be an eventually positive solution of the equation (12) such that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for $t \geq t_1 \geq \max(\tau,\sigma)$. Set

$$Z(t) = x(t) + p(t)x(t-\tau). \tag{14}$$

Since $p(t) \geq 0$ for $t \geq 0$, we see that $Z(t) > 0$ for $t \geq t_1$. In view of $((Z'(t))^{\sigma})' = -q(t)f(x(t-\sigma)) \leq 0$ and the assumption on q, we see that $(Z'(t))^{\alpha}$ is non-increasing for $t \geq t_1$ and of constant sign for all large t. We assert that $Z'(t) > 0$ for all large t. Otherwise, $Z'(t) < 0$ for $t \geq t_2 \geq t_1$. In view of (12),

$$Z(t) \leq Z(t_2) + Z'(t_2)(t-t_2)$$

and hence we see that $\lim_{t\to\infty} Z(t) = -\infty$. But this is a contradiction.

Now since we have shown that $Z(t) > 0$ and $Z'(t)$ for all large t, then by our assumptions on f,

$$\begin{aligned} f(x(t-\sigma)) &\geq f(Z(t-\sigma) - p(t-\sigma)x(t-\sigma-\tau)) \\ &\geq f(Z(t-\sigma) - p(t-\sigma)Z(t-\sigma-\tau)) \\ &\geq f(Z(t-\sigma-\tau) - p(t-\sigma)Z(t-\sigma-\tau)) \\ &\geq \Gamma f(1-p(t-\sigma))f(Z(t-\sigma-\tau)). \end{aligned}$$

Thus, in view of (12), we have

$$((Z'(t))^{\alpha})' + \Gamma q(t)f(1-p(t-\sigma))f(Z(t-\sigma-\tau)) \leq 0$$

for all large t. But then by Theorem 2, we see that the equation (13) has an eventually positive solution. The proof is complete.

THEOREM 5. *Suppose that* $-1 < p^* \le p(t) \le 0$ *for* $t \ge 0$ *and* $f(xy) \ge \Gamma f(x) f(y)$ *for* $x, y > 0$ *and some* $\Gamma > 0$. *If* (12) *has an eventually positive solution which satisfies* $\liminf_{t\to\infty} x(t) = \omega > 0$, *then the following equation*

$$((y'(t))^\alpha)' + \Gamma q(t) f(1 - p(t-\sigma)) f(y(t-\tau-\sigma)) = 0$$

has an eventually positive solution.

Proof. Let $x(t)$ be an eventually positive solution of equation (12). Set $Z(t) = x(t) + p(t)x(t-\tau)$ as in (14). In view of $((Z'(t))^\sigma)' = -q(t)f(x(t-\sigma)) \le 0$ and the assumption on q, we see that $Z'(t)$ and $Z(t)$ are eventually of constant sign.

We first assert that $Z'(t) > 0$ eventually. Suppose to the contrary that $Z'(t) < 0$ for all large t. We assert that $Z(t)$ is eventually positive. Otherwise, there would be positive numbers N and β such that $Z(t) \le -\beta$ for $t \ge N$. Hence

$$x(t) = -p(t)x(t-\tau) + Z(t) \le x(t-\tau) + Z(t) \le x(t-\tau) - \beta$$

for $t \ge N$. By induction, it is easy to see that

$$\begin{aligned} x(j\tau + M) &\le Z(j\tau + M) + x((j-1) + M) \\ &\le Z(j\tau + M) + Z((j-1) + M) + x((j-2) + M) \\ &\le \dots \le -j\beta + Z(M) \end{aligned}$$

for $j \ge 1$. By taking $j \to \infty$, the right-hand side will be negative, which is a contradiction.

Thus there is some $T \ge 0$ such that $Z(t) > 0$, $Z'(t) < 0$ and $[(Z'(t))^\alpha]' \le 0$ for $t \ge T$. In view of (12),

$$Z(t) \le Z'(T)(t-T) + Z(T).$$

But then by taking $t \to \infty$, the right hand side is negative, which is a contradiction.

Now since we have shown that $Z'(t) > 0$ eventually, we assert that $Z(t)$ is again eventually positive. Suppose to the contrary that $Z(t) \le 0$ eventually. Then $\lim_{t\to\infty} Z(t) = \gamma \le 0$. However, since

$$x(t+\tau) - Z(t+\tau) = -p(t+\tau))x(t) \le -p^* x(t),$$

thus by taking limit inferior on both sides, we obtain

$$\omega - \gamma \le -p^*\omega.$$

Thus $\gamma \ge (1+p^*)\omega > 0$, which is a contradiction.

Since we have shown that $Z(t) > 0$ and $Z'(t) > 0$ for all large t, we are in the same situation as in last stage of the proof of Theorem 4. The rest of the proof is the same and it can be omitted.

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Guang Zhang
DEPARTMENT OF MATHEMATICS
DATONG ADVANCED COLLEGE
DATONG, SHANXI 037008
P. R. CHINA

Sui Sun Cheng
DEPARTMENT OF MATHEMATICS
TSING HUA UNIVERSITY
HSINCHU, TAIWAN 30043
R. O. CHINA
E-mail: sscheng@math.nthu.edu.tw

Received August 19, 1998.