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ON BOUNDED PSEUDO AND WEAK SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION IN BANACH SPACES

Abstract. In this paper, we prove an existence theorem for bounded pseudo and weak solution of the differential equation

$$x'(t) = A(t)x(t) + f(t, x(t))$$

where $f(\cdot, x(\cdot))$ is Pettis-integrable for each strongly absolutely continuous function x and $f(t, \cdot)$ is weakly-weakly sequentially continuous. We also assume some condition expressed in terms of De Blasi's measure of weak noncompactness.

1. Introduction

In this paper we shall consider the nonlinear differential equation

$$x' = A(t)x + f(t, x). \tag{1}$$

Existence theorem for the strong solutions in Banach space of (1) are well-know (see for example Szufla [14], Dawidowski-Rzepecki [9]). An existence theorem for the weak solutions (where x' denotes the weak derivative of x) have been obtained by Cichoń [4].

We will use some of the notation, definitions and results from the book of Massera-Schäffer [11].

Let $R_+ = [0, \infty)$ and let $(E, \| \, \|)$ denote a Banach space such that the space $E_w = (E, \sigma(E, E^*))$ is sequentially complete. Moreover, we introduce the following notations: $L(E)$ is the algebra of continuous linear operators from E into itself with the standard norm $| \, |$; $L = L(R_+, E)$ is the space of all measurable functions $u : R_+ \to E$ integrable in the Bochner sense on every bounded interval $I \subset R_1$, with the topology of convergence in the mean

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on every such I, $C_w(R_+, E)$ is the space of all weakly continuous functions from R_+ to E with the topology of almost uniform weak convergence.

Let $B = B(R_+, E)$ be a Banach space of measurable functions such that $\|x\| \in B(R_+, R)$ with the norm $\|x\|_B = \|\|x\|\|_{B(R)}$, where $B(R) = B(R_+, R)$ is a function space such that:

(i) $B(R_+, R) \subset L(R_+, R)$ and $B(R_+, R)$ is stronger than $L(R_+, R)$,

(ii) $B(R_+, R)$ contains all essentially bounded function with compact support,

(iii) if $u \in B(R_+, R)$, $v : R_+ \to R$ is measurable and $|v| \leq |u|$, then $v \in B(R_+, R)$ and $\|v\|_{B(R)} \leq \|u\|_{B(R)}$,

(iv) if $u \in B(R_+, R)$, $v_n \in L(R_+, R)$, $(n \in N)$, $|v_n| \leq |u|$ and $\lim_{n\to\infty} v_n(t) = 0$ a.e. on R_+, then $\lim_{n\to\infty} \|v_n\|_{B(R)} = 0$.

Let B' denote the associate space to B (cf. [11], p.50).

We shall consider the nonlinear differential equation (1), where $A(t) \in L(E)$ for $t \in R_+$, x' denotes the weak derivative of x, $(t, x) \to f(t, x)$ is a function from $R_+ \times B_r$ into E which is weakly-weakly sequentially continuous and $B_r = \{x \in E : \|x\| \leq r\}$.

Let E_0 be a subspace consisting of all points of E which are values at $t = 0$ of all bounded weak solutions of the linear differential equation

$$x' = A(t)x. \tag{2}$$

We assume that E_0 is closed and has a closed complement E_1, i.e., E is the direct sum of E_0 and E_1. Take the Green function for (2):

$$G(t,s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \leq s \leq t \\ -U(t)(Id - P)U^{-1}(s) & \text{for } 0 \leq t \leq s, \end{cases}$$

where $U : R_+ \to L(E)$ is the solution of the differential equation $U' = A(t)U$ with $U(0) = Id$, and P is the projection of E onto E_0 with $\ker P = E_1$ (cf. [8] and [11]). Thus there exists a positive number N such that $\|U(t)P\| \leq N$ for every $t \in R_+$, i.e., $\|G(t, 0)\| \leq N$.

Fix $x^* \in E^*$, and consider the equation

$$(x^*x)'(t) = x^*(A(t)x(t) + f(t, x(t))), \qquad t \in R_+. \tag{1'}$$

Let us introduce the following definition:

Definition 1. A function $x : R_+ \to E$ is said to be a pseudo-solution of the equation (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous,

(ii) for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e. $\text{mes}(A(x^*)) = 0$), such that for each $t \notin A(x^*)$ $x(\cdot)$ satisfies (1').

For the Cauchy problem there exist examples which have only pseudo-solutions, but no weak solutions or strong C-solutions ([1, 6]).

For any bounded subset W of E we denote by $\gamma(W)$ the measure of weak noncompactness of W, i.e., the infimum of all $\varepsilon > 0$ for which there exists a weakly compact subset P of E such that $W \subset \varepsilon B_1 + P$. For the properties of γ the reader is referred to [3] and [2].

We will use the following lemma and theorem.

LEMMA 1 [12]. *Let $H \subset C(I,E)$ be a family of strongly equicontinuous functions. Then*

$$\gamma_c(H) = \sup_{t\in I} \gamma(H(t)) = \gamma(H(I)),$$

where $\gamma_c(H)$ denotes the measure of weak noncompactness in $C(I,E)$. Moreover, the function $t \to \gamma(H(t))$ is continuous.

THEOREM 1 [10]. *Let D be a closed convex subset of E, and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

(3) $\quad \overline{V} = \overline{conv}(\{x\} \cup F(V)) \Longrightarrow V$ *is relatively weakly compact,*

holds for every subset V of D, then F has a fixed point.

We assume that:

- (A0) $f : R_+ \times B_r \to E$ is such that for each strongly absolutaly continuous function $x : R_+ \to B_r$, $f(\cdot, x(\cdot))$ is Pettis integrable on every finite subinterval of R_+, $f(t,\cdot)$ is weakly-weakly sequentially continuous,
- (A1) $A : R_+ \to L(E)$ is Bochner integrable on each bounded interval in R_+,
- (A2) $G(t,\cdot) \in B'$ and $\|G(t,\cdot)\|'_B \leq k$ for every $t \in R_+$,
- (A3) there exists $m \in B'$ such that $k\|m\|_B < r$ and $\|f(t,x)\| \leq m(t)$ for every $t \in R_+$ and $x \in B_r$,
- (A4) $p : R_+ \to R_+$ is non-decreasing continuous function such $p(0) = 0$, $p(u) < u$ for all $u > 0$,
- (A5) there exists $h \in B$ such that $k\|h\|_B < 1$ and

  $$\forall a > 0 \ \forall \varepsilon > 0 \ \forall X \subset B_r \ \exists \text{ open } Z \subset [0,a] \text{ mes } (Z) < \varepsilon \quad \text{and}$$

  $\gamma(f(T \times X)) \leq \sup\{h(t) : t \in T\} p(\gamma(X))$ for every compact subset T of $[0,a] \setminus Z$.

THEOREM 2. *If* (A0)–(A5) *hold then for every $x_0 \in E_0$ such that $\|x_0\| \leq (r - k\|m\|_B)/N$ there exists at least one bounded pseudo-solution $x(\cdot)$ of* (1) *on any interval $I =< 0, a >$, $a > 0$ with $px(0) = x_0$.*

Proof. Denote by $C(I,E)$ the space of continuous functions $I \to E$ with the topology of almost uniform convergence.

Let

$$K = \{x \in C(I,E) : \|x(t) - x(\tau)\| \leq r \int_\tau^t |A(s)|\, ds + \int_\tau^t m(s)\, ds, \qquad 0 \leq \tau \leq t\}.$$

We see that K is convex closed and strongly equicontinuous subset of $C(I,E)$. We fix some $x_0 \in E_0$ such that $\|x_0\| \leq (r - k\|m\|_B)/N$. Then the function $G(t,0)x_0$ is a solution of (2). Hence $\|G(t,0)x_0\| \leq N\|x_0\|$ for $t \in R_+$. Put for a compact interval I

$$F_a(x)(t) = G(t,0)x_0 + \int_0^a G(t,s)f(s,x(s))\, ds$$

for every $x \in K$. Here $\int$ denotes the Pettis integral.

By (A0), for $x \in K$, the function $G(t,\cdot)f(t,x(\cdot))$ is Pettis integrable on every compact subset of R_+. We have

$$\|F_a(x)(t)\| \leq N\|x_0\| + k\|m\|_B \leq r$$

and since $z = F_a(x)$ is a weak solution of the differential equation $z' = A(t)z + f(t,x(t))$, by assumption (A3) and the mean value theorem we have

$$\|F_a(x)(t) - F_a(x)(\tau)\| \leq \int_\tau^t \|A(t)F(x)(s) + f(s,x(s))\|\, ds \leq$$
$$\leq r \int_\tau^t |\, A(s)\,|\, ds + \int_\tau^t m(s)\, ds.$$

Hence we conclude that $F_a : K \to K$.

Since K is strongly equicontinuous, the sequence (x_n) of K converges weakly to $x \in C(I,E)$ if $x_n(t) \to x(t)$ in (E,w) for each $t \in R_+$ (cf. [12], Lemma 1.9).

Since $f(t,\cdot)$ is weakly-weakly sequentially continuous, then, by using the Lebesqu'e Dominated Convergence Theorem for the Pettis integral for each $x^* \in E^*$ and each $t \in I$, we have

$$x^*(F_a(x_n)(t)) \to x^*(F_a(x)(t))$$

whenever $x_n \to x$ in $(C(I,E),w)$. So F_a is weakly-weakly sequentially continuous on K.

Now we will show that F_a satisfies (3). Let V be a subset of K such that

$$\overline{V} = \overline{\text{conv}}(F_a(V) \cup \{0\}).$$

Fix $t \geq 0$ and $\varepsilon > 0$. By (A3) we can choose $\alpha \geq t$ such that $\|m\chi_{[\alpha,\infty)}\|_B < \varepsilon$. Let $\delta = \delta(\varepsilon)$ be a number such that $\|m\chi_A\|_B < \frac{\varepsilon}{2}$ for $A \subset S$, $(S; = [0,\alpha)$

such that mes $A < \delta$. By the Luzin Theorem there exists a closed subset K'_ε of S with mes $(S \setminus K'_\varepsilon) < \delta$ and such that h is continuous on K'_ε. Put $J'_\varepsilon = S \setminus K'_\varepsilon$. Since mes $J'_\varepsilon < \delta$, hence $\|m\chi_{J_\varepsilon}\|_B < \frac{\varepsilon}{2}$ and h is continuous on $S \setminus J'_\varepsilon$. By (A5) there exists an open subset J''_ε of S such that mes $J''_\varepsilon < \delta$ and

$$(*) \qquad \gamma(f(T \times X)) \leq \sup\{h(t) : t \in T\}p(\gamma(X))$$

for every compact subset T of $S \setminus J''_\varepsilon$. Also $\|m\chi_{J''_\varepsilon}\|_B < \frac{\varepsilon}{2}$.

Let $J_\varepsilon = J'_\varepsilon \cup J''_\varepsilon$, $A := S \setminus J_\varepsilon$. Then h is continuous on A, the inequality $(*)$ holds and $\|m\chi_{J_\varepsilon}\|_B < \varepsilon$.

We observe that (see also [4])

$$(3) \qquad F_a(V)(t) \subset G(t,0)x_0 + \int\limits_{A\cap I} G(t,s)f(s,V(s))\,ds + \\ + \int\limits_{J_\varepsilon \cap I} G(t,s)f(s,V(s))\,ds + \int\limits_{\langle\alpha,\infty)\cap I} G(t,s)f(s,V(s))ds,$$

where

$$\int\limits_T G(t,s)f(s,V(s))ds = \Big\{ \int\limits_T G(t,s)f(s.x(s))\,ds : x \in V \Big\}.$$

For any $x \in V$ we have

$$\Big\| \int\limits_{J_\varepsilon \cap I} G(t,s)f(s,x(s))\,ds \Big\| \leq k\|mx_{J_\varepsilon}\|_B < k\varepsilon, w$$

and

$$\Big\| \int\limits_{\langle\alpha,\infty)\cap I} G(t,s)f(s,x(s))\,ds \Big\| \leq k\|mx_{[\alpha,\infty)}\|_B < k\varepsilon.$$

Since $\gamma(W) \leq$ diam W for any bounded W, we have

$$(4) \qquad \gamma\Big(\int\limits_{J_\varepsilon} G(t,s)f(s,V(s))\,ds \Big) < 2k\varepsilon$$

and

$$(5) \qquad \gamma\Big(\int\limits_{\langle\alpha,\infty)\cap I} G(t,s)f(s,V(s))\,ds \Big) < 2k\varepsilon.$$

Let $Z = V(A) = \{x(s) : x \in V,\ s \in A\}$.

Since $G(t,\cdot)$ and $(h(\cdot)$ are uniformly continuous on the (compact) set A, for every $\varepsilon' > 0$ there exists a number $\eta > 0$ such that $\mid h(t') - h(t'') \mid < \varepsilon'$ and

$$(**) \qquad \|G(t,t') - G(t,t'')\| < \varepsilon',$$

whenever $|t' - t''| < \eta \ \ (t'.t'' \in A)$.

Let $t_0 < t_1 < \ldots < t_n$ be a partition of the interval S, $I_i :=< t_{i-1}, t_i >$, $J_i = I_i \setminus J_\varepsilon,\ i = 1, 2, \ldots, n$ and let s_i, r_i be points in J_i such that $h(s_i) = \sup_{s \in J_i} h(s)$ and $\|G(t, r_i)\| = \sup_{t \in J_i} \|G(t, s)\|$.

Hence, by $(*)$:

$$\gamma(f(J_i \times Z)) \leq \sup\{h(t) : t \in J_i\} p(\gamma(Z)).$$

By the Pettis-integral mean value theorem [13] for $x \in V$ we have

$$\int\limits_{A \cap I} G(t,s) f(s, x(s))\, ds = \sum_{i=1}^{n} \int\limits_{J_i \cap I} G(t,s) f(s, x(s))\, ds \in$$
$$\in \sum_{i=1}^{n} \operatorname{mes} J_i' \overline{\operatorname{conv}}\{G(t,s) f(s,z) : s \in J_i',\ z \in Z\}.$$

So

$$\int\limits_{A \cap I} G(t,s) f(s, V(s))\, ds \subset \sum_{i=1}^{n} \operatorname{mes} J_i' \overline{\operatorname{conv}}\{G(t,s) f(s,z) : s \in J_i',\ z \in Z\}.$$

From above, by the properties γ and Lemma 1

$$\gamma\Big(\int\limits_{A \cap I} G(t,s) f(s, V(s)\, ds\Big)$$
$$\leq \sum_{i=1}^{n} \operatorname{mes} J_i' \gamma(\{G(t,s) f(s,z) : s \in J_i',\ z \in Z\})$$
$$\leq \sum_{i+1}^{n} \operatorname{mes} J_i' \gamma\Big(\bigcup_{s \in J_i'} \{G(t,s) f(J_i' \times Z\}\Big)$$
$$\leq \sum_{n=1}^{n} \operatorname{mes} J_i' \sup_{s \in J_i'} \|G(t,s)\| \gamma(f(J_i' \times Z))$$
$$\leq \sum_{i+1}^{n} \operatorname{mes} J_i' \|G(t, r_i)\| h(s_i) p(\gamma(Z)).$$

But from $(**)$ it follows that

$$\operatorname{mes} J_i' \|G(t, r_1)\| h(s_i) p(\gamma(Z)) \leq p(\gamma(Z)) \int\limits_{J_i'} \|G(t,s)\| h(s)\, ds$$
$$+ \sup_{t \in J_i'} h(s) \varepsilon' \operatorname{mes} J_i' \gamma(Z) + \sup_{t \in J_i'} \|G(t,s)\| \varepsilon' \operatorname{mes} J_i' \gamma(Z).$$

Moreover

$$\sum_{i+1}^{n} \text{mes } J_i' \|G(t,r_i)\| h(s_i) p(\gamma(Z))$$

$$\leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds$$

$$+ \varepsilon' \gamma(Z) \text{mes } (S) (\sup_{s\in A\cap I} h(s) + \sup_{s\in A\cap I} \|G(t,s)\|)$$

$$\leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s) ds + \varepsilon' Q,$$

From this and (A5) we deduce that

$$(6) \qquad \gamma\Big(\int_{A\cap I} G(t,s) f(s, V(s))\, ds\Big) \leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds + \varepsilon' Q.$$

By (3), (4), (5), and (6)

$$\gamma(F_a(V)(t)) \leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds + \varepsilon' Q + 4k\varepsilon.$$

This inequality holds for every $\varepsilon' > 0$ and $\varepsilon > 0$. Hence

$$\gamma(F_a(V)(t)) \leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds.$$

From Lemma 1 we have

$$\gamma(F_a(V)(t)) \leq p(\gamma(Z)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds \leq p(\gamma(V)) \int_{A\cap I} \|G(t,s)\| h(s)\, ds$$

$$\leq p(\gamma(V)) k \|h\|_B < \sup_t \gamma(V(t)).$$

If $\gamma(V) > 0$, because $\overline{V} = \overline{\text{conv}}(F_a(V) \cup \{o\})$, by the properties of measure of weak noncompactness we have

$$\gamma(V) = \gamma(\overline{V}) \leq \gamma(F_a(V)) = \sup_t \gamma(F_a(V(t))) < \sup_t \gamma(V(t)) = \gamma(V),$$

a contradiction. So $\gamma(V) = 0$ and V is relatively weakly compact in $C(I, E)$.

Applying now Theorem 1 we conclude that F_a has a fixed point, which completes the proof.

REMARKS. Theorem 2 is true if in the assumption (A5) we use the axiomatic measure of weak-noncompactnes (cf. [2], [5]) or if (A5) is replaced by other assumptions (as for strong solution).

If f is weakly continuous function then a pseudo-solution is a weak solution.

Moreover, in some classes of spaces our pseudo solutions are also strong C-solutions (in separable Banach spaces for instance).

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