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THE CAUCHY PROBLEM FOR CERTAIN GENERALIZED DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH SINGULARITY

The present paper is devoted to a natural generalization of differential equations for mappings from subset of a Banach space into a Banach space.The subject matter refers to studies of generalized differential equations of the first order introduced in [7].

Let X, Y be Banach spaces over the field $\mathbb{R}$ and let U and V be open subsets of X and Y, respectively. Let h be a mapping from U into X and F a mapping from $U \times V$ into Y.

We shall start with defining a derivative of a function f in a direction of the mapping h on U, denoted by $(\nabla_h f)(x)$ for $x \in U$, and generalizing the well known notion of the directional derivative [6]. From a point of view of differenitial geometry, a directional derivative $\nabla_h f$ means a derivative in the direction of a vector field (with a singularity, because $h(0) = 0$). Then we consider the Cauchy problem

$$(\nabla_h)f(x) + Af(x) = F(x, f(x)), \quad f(0) = 0$$

for mappings from a subset of a Banach space into a Banach space, which are defined in C or in $C_*^{0,1}$, with the assumption that 0 is a singular point (i.e. $h(0) = 0$). We also study the Cauchy problem

$$Df(x)h(x) + Af(x) = F(x, f(x)), \; f(0) = 0$$

for mappings from a subset of a Banach space into a Banach space which are defined in C^1 class, with the assumption that 0 is a singular point (i.e. $h(0) = 0$).

1. Introduction

In lemmas and theorems presented in this paper the real Banach space X will be considered with a semi-inner product, defined as follows [3], [4].

Let X^* be the dual space of Y and

$$T(x) = \{x^* \in X^*;\ \|x^*\| = 1,\ x^*(x) = \|x\|\} \quad \text{for} \quad x \in X$$

and X_0 a set of nonzero elements with norm equal to 1, chosen one by one from each line in X through zero. Let $\Im_0$ be any (fixed in further considerations) mapping from X_0 into X^* such that $\Im_0(y) \in T(y)$ for $y \in X_0$. Define by $\Im(\lambda y) = \lambda \Im_0(y)$ for $y \in X_0$, $\lambda \in \mathbb{R}$ the homogenous extension $\Im$ of $\Im_0$ to the whole space X and a semi-inner product by $\langle x, y\rangle = \Im(y)(x)$ for $x, y \in X$, having the following properties:

(a) it maps $X \times X$ into $\mathbb{R}$,

(b) $\langle x + y, z\rangle = < x, z > + < y, z >$, $\langle \lambda x, y\rangle = \lambda < x, y >$, $\langle x, \lambda y\rangle = \lambda\langle x, y\rangle$ for $x, y, z \in X$, $\lambda \in \mathbb{R}$,

(c) $\langle x, x\rangle = \|x\|^2$ for $x \in X$,

(d) $|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle$ for $x, y \in X$.

Denote by B_ρ the open ball in X with radius ρ and centre zero, i. e., $B_\rho = \{x \in X : \|x\| \le \rho\}$.

DEFINITION 1. A mapping $h : B_\rho \to X$ of the class C^1 will be called a regular mapping (in zero) if:

(i) is bounded with its first derivative Dh in B_ρ,

(ii) $h(0) = 0$,

(iii) there exists such a constant $C > 0$ that

$$y^*(Dh(0)y) \ge C \tag{1}$$

for $y^* \in T(y)$ and for every $y \in X$ such that $\|y\| = 1$.

It is not difficult to prove the following lemma.

LEMMA 1. *If $h : B_\rho \to X$ is a regular mapping (in zero), then for every $\alpha \in (0, C)$ there exists a constant $r \in (0, \rho)$ such that Cauchy problem*

$$\frac{\partial}{\partial t} v(t, x) = -h(v(t, x)), \quad v(0, x) = x \tag{2}$$

has in the domain $[0, \infty) \times B_r$ exactly one continuously differentiable solution $v = v(t, x)$, having the properties

$$\|v(t, x)\| \le e^{-\alpha t}\|x\| \quad \textit{for} \quad t \in [0, \infty), x \in B_r, \tag{3}$$

$$v(t, v(\tau, x)) = v(t + \tau, x) \quad \textit{for} \quad t, \tau \in [0, \infty), x \in B_r. \tag{4}$$

LEMMA 2. *If $h : B_\rho \to X$ is a regular mapping (in zero) of class C^2 then for every $\beta \in (0, C)$ there exists such $\tilde{r} \in (0, r]$ (where r denotes the constant from Lemma 1), that*

$$\|D_2 v(t, x)\| \le e^{-\beta t} \quad \textit{for} \quad t \in [0, \infty), x \in B_{\tilde{r}}. \tag{5}$$

P r o o f. Let C denote such positive constant that the inequality (1) is fulfilled. Then

$$\langle Dh(0)g, g\rangle \geq C\|g\|^2 \quad \text{for every} \quad g \in X. \tag{6}$$

Since h is a regular mapping of class C^2, we have

$$Dh(y) = Dh(0) + \check{r}(y), \quad \text{where} \quad \lim_{y\to 0} \|\check{r}(y)\| = 0.$$

Therefore for every $\beta \in (0, C)$ there exists such $\hat{r} \in (0, \rho)$, that

$$\|\check{r}(y)\| \leq C - \beta \quad \text{for} \quad y \in B_{\hat{r}}. \tag{7}$$

From (6), (7) and the Schwarz inequality for semi-inner product we obtain

$$\langle Dh(y)g, g\rangle \geq C\|g\|^2 - \|\check{r}(y)\|\|g\|^2 \geq \beta\|g\|^2 \tag{8}$$

for $y \in B_{\hat{r}}$, for every $g \in X$ and for $\beta \in (0, C)$.

The function $v = v(t, x)$ for $t \in [0, \infty)$ and $x \in B_r$ fulfills the equations

$$\frac{\partial}{\partial t} v(t, x) = -h(v(t, x)), \quad v(0, x) = x.$$

Differentiating the above equations with respect to x at the point $u \in B_r$ and considering Theorem 29 in [8] and Theorem IX.5′ in [6] we obtain

$$\frac{\partial}{\partial t}(D_2 v(t, x)u) = -Dh(v(t, x))D_2 v(t, x)u \quad \text{and} \quad D_2 v(0, x)u = u$$

for $t \in [0, \infty)$ and $x, u \in B_r$.

Hence the function $g(t, x, u) = D_2 v(t, x)u$ for $x, u \in B_r$ is the solution of the equations

$$\frac{d}{dt} g(t, x, u) = -Dh(v(t, x))g(t, x, u), \quad g(0, x, u) = u \quad \text{for} \quad t \in [0, \infty).$$

The mapping $g(\cdot, x, u)$ is continuously differentiable on $[0, \infty)$. Therefore the function $\|g(\cdot, x, u)\|$ is absolutely continuous on every interval $[0, \tau]$ ([5] p.172) and so almost every differentiable on $[0, \infty)$. By Lemma 1.3 in [2] we have

$$\frac{d}{dt}\|g(t, x, u)\|^2 = -2\langle Dh(v(t, x))g(t, x, u), g(t, x, u)\rangle$$

for almost every $t \in [0, \infty)$.

Fix now $\beta \in (0, C)$ and denote $\tilde{r} = min(r, \hat{r})$. By Lemma 1, $v(t, x) \in B_r$ if $x \in B_r$. It follows from the inequality (8) that particular for $g = g(t, x, u)$ we have

$$\langle Dh(v(t, x))g(t, x, u), g(t, x, u)\rangle \geq \beta\|g(t, x, u)\|^2$$

for almost every $t \in [0, \infty)$ and $x, u \in B_{\tilde{r}}$. Therefore we obtain

$$\frac{d}{dt}\|g(t, x, u)\|^2 \leq -2\beta\|g(t, x, u)\|^2$$

for almost every $t \in [0, \infty)$ and $x, u \in B_{\tilde{r}}$. Hence

$$\frac{d}{dt}(e^{2\beta t}\|g(t,x,u)\|^2) \leq 0$$

for almost every $t \in [0, \infty)$ and $x, u \in B_{\tilde{r}}$.

From the absolute continuity of the function $\|g(t,x,u)\|$ it follows that the function $e^{2\beta t}\|g(t,x,u)\|^2$ is decreasing on $t \in [0, \infty)$. Therefore

$$\|g(0,x,u)\|^2 \geq e^{2\beta t}\|g(t,x,u)\|^2 \quad \text{for} \quad t \in [0,\infty) \quad \text{and} \quad x, u \in B_{\tilde{r}}.$$

Since $g(0,x,u) = u$, the above inequality takes the form

$$\|g(t,x,u)\| \leq e^{-\beta t}\|u\| \quad \text{for} \quad t \in [0,\infty) \quad \text{and} \quad x, u \in B_{\tilde{r}}.$$

Hence we obtain

$$\|D_2 v(t,x)u\| \leq e^{-\beta t}\|u\| \quad \text{for} \quad t \in [0,\infty) \quad \text{and} \quad x, u \in B_{\tilde{r}}.$$

Consequently for $\beta \in (0, C)$ there exists $\tilde{r} > 0$ such that

$$\|D_2 v(t,x)\| \leq e^{-\beta t} \quad \text{for} \quad t \in [0,\infty) \quad \text{and} \quad x, u \in B_{\tilde{r}}. \ \blacksquare$$

A function v satisfying the problem (2) will be called in this paper a natural transformation generated by the mapping h.

EXAMPLE 1. Consider the mapping $h : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $h(x_1, x_2) = [8x_1 - 2x_2, -2x_1 + 5x_2]$. Then

$$v(t,x)=\left[\tfrac{2}{5}(2x_1-x_2)e^{-9t}+\tfrac{1}{5}(x_1+2x_2)e^{-4t}, -\tfrac{1}{5}(2x_1-x_2)e^{-9t}+\tfrac{2}{5}(x_1+2x_2)e^{-4t}\right]$$

is the natural transformation generalized by h. Since the symmetric matrix

$$Dh(0) = \begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}$$

has the eigenvalues $\lambda_1 = 4$, $\lambda_2 = 9$, the inequality (1) takes the form

$$[\, y_1, \quad y_2\,]\begin{bmatrix} 8 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 8y_1^2 - 4y_1y_2 + 5y_2^2 = 4\bar{y}_1^2 + 9\bar{y}_2^2 \geq 4\|y\|^2 = 4,$$

where $\bar{y}_1, \bar{y}_2$ are the coordinates of the vector $y \in \mathbb{R}^2$ in the normed ortogonal base composed of the eigenvectors. Consequently $C = 4$. This is easy to verify that

$$\|v(t,x)\|^2 - \|x\|^2 e^{-8t} = \frac{1}{5}e^{-18t}(1 - e^{10t})(2x_1 - x_2)^2 \leq 0.$$

Therefore $\|v(t,x)\| \leq \|x\|e^{-4t} \leq \|x\|e^{-\alpha t}$ for $t \in [0,\infty)$, $x \in \mathbb{R}^2$, $\alpha \in (0,4]$. Moreover, it can be shown that

$$\|D_2 v(t,x)u\|^2 - \|u\|^2 e^{-8t} = \frac{1}{5}e^{-18t}(1 - e^{10t})(2u_1 - u_2)^2 \leq 0\,.$$

Then $\|D_2 v(t,x)\| \leq e^{-4t} \leq e^{-\beta t}$ for $t \in [0,\infty)$, $x \in \mathbb{R}^2$, and $\beta \in (0,4]$.

Let $C_*^{0,1}(B_{\tau_0}, Y)$ be the space of the continuous mappings $f : B_{\tau_0} \to Y$, $B_{\tau_0} \subset X$, such that for every mapping f there exists such a constant $C_* > 0$ that $\|f(x)\| \le C_*\|x\|$ for $x \in B_{\tau_0}$. Let

$$\|f\|_* = \inf\{C_* > 0; \|f(x)\| \le C_*\|x\| \quad \text{for} \quad x \in B_{\tau_0}\}. \tag{9}$$

This is not difficult to verify that this functional is a norm.

In what follows we shall denote by $\mathcal{B}_\rho = C(B_{\tau_0}, B_\rho) \subset C(B_{\tau_0}, Y)$ the ball with centre at zero and radius $\rho > 0$ considered in the space of the continuous functions with sup norm and by $\mathcal{B}_\rho^* \subset C_*^{0,1}(B_{\tau_0}, Y)$ the ball with centre at zero and radius $\rho > 0$ considered in the space $C_*^{0,1}(B_{\tau_0}, Y)$ with the norm $\|\cdot\|_*$.

It is not difficult to prove

THEOREM 1. *The space $C_*^{0,1}(B_{\tau_0}, Y)$ with the norm defined by* (9) *is the Banach space.*

2. The Cauchy problem for the generalized differential equations of first order with singularity in a point zero, in C and $C_*^{0,1}$ classes

Let X, Y be a real Banach spaces, U an open subset of X and $h : U \to X$ be a function of class C^1, bounded together with its first derivative on U. From Theorem 10.4.5 and Theorem 10.8.2 in [1] it follows that for any $x_0 \in U$ there exists a constant $t_0 > 0$ and a neighbourhood $U_0 \subset U$ of a point x_0 such that the Cauchy problem (2) has in the domain $(-t_0, t_0) \times U_0$ exactly one continuously differentiable solution $v = v(t, x)$. Lemma 1 specifies the additional properties of the solution of the problem (2) for x in a neighbourhood of zero in the case of regular mapping h.

DEFINITION 2. We say that the mapping $f : U \to Y$ has at a point $x \in U$ a derivative in a direction of the mapping h if there exists a limit

$$(\nabla_h f)(x) := \lim_{t \to 0} \frac{f(v(0,x)) - f(v(t,x))}{t} \quad \text{for} \quad t \in \mathbb{R},$$

where $v = v(t, x)$ is the solution of the problem (2) in a neighbourhood of a point $(0, x)$.

We can use in Definition 2 the natural transformation generated by the regular mapping h (for $t \ge 0$), since

$$(\nabla_h) f(x) = -\left[\frac{\partial}{\partial t} f(v(t,x))\right]_{t=0}$$

in a neighbourhood of a point $x = 0$.

COROLLARY 1. *If the mapping $f : U \to Y$ is differentiable at a point $x \in U$, then there exists a derivative of a function f in a direction of mapping h at a point $x \in U$ and the following equality is true $(\nabla_h)f(x) = Df(x)h(x)$.*

Proof. Since f is differentiable, we have

$$\left[\frac{\partial}{\partial t} f(v(t,x))\right]_{t=0} = -\left[Df(v(t,x))h(v(t,x))\right]_{t=0} = -Df(x)h(x) \quad \text{for } x \in U.$$

Consequently, by Definition 2, we obtain the thesis. ∎

COROLLARY 2. *The directional derivative from Definition 2 is equivalent to the ordinary derivative in the case of constant h and v having the form $v(t,x) = x - th$ for $t \in (-t_0, t_0)$ and $x \in U$.*

COROLLARY 3. *Let $h : B_\rho \to X$, with $B_\rho \subset X$, be a regular mapping. Moreover, let $C > 0, r \in (0,\rho)$ be such constants that the inequalities (1), (3) hold. If there exists a derivative of a function f in a direction of the mapping h in B_ρ then*

$$(\nabla_h)f(v(t,x)) = -\frac{\partial}{\partial t} f(v(t,x)) \quad \textit{for} \quad t \in [0,\infty),\ x \in B_r.$$

Proof. Let $v = v(t,x)$, for $t \in [0,\infty)$ and $x \in B_r$, be a natural transformation generated by a mapping h. By Lemma 1, we have $v(t_0, v(t,x)) = v(t_0 + t, x)$ for $t, t_0 \in [0,\infty)$, $x \in B_r$; hence

$$\left[\frac{\partial}{\partial t} f(v(t,x))\right]_{t=t_0} = \left[\frac{\partial}{\partial t} f(v(t+t_0,x))\right]_{t=0} = \left[\frac{\partial}{\partial t} f(v(t,v(t_0,x)))\right]_{t=0}$$

From Lemma 1 it follows that the inequality $\|v(t_0,x)\| \le e^{-\alpha t_0}\|x\|$ holds for $t_0 \in [0,\infty), x \in B_r$. Consequently,

$$\left[\frac{\partial}{\partial t} f(v(t,v(t_0,x)))\right]_{t=0} = -(\nabla_h f)(v(t_0,x)) \quad \text{for} \quad t_0 \in [0,\infty), x \in B_r. \ ∎$$

DEFINITION 3. Let U and V be open subsets of Banach spaces X and Y, respectively. Let $h : U \to X$ be a mapping of class C^1, F be any function from $U \times V$ into Y, and A ($A \in L(Y,Y)$) be a linear and continuous operator in the Banach space Y. Every function $f : U \to V$ which has a derivative in a direction of mapping h in U and fulfills the equation $\nabla_h f(x) + Af(x) = F(x, f(x))$ for $x \in U$ will be called its solution.

We shall introduce the following assumption.

ASSUMPTION 1. *Let $h : B_{\rho_1} \to X$, with $B_{\rho_1} \subset X$, be a regular mapping and let $C > 0$ be a constant such that (1) holds. We assume that α is a fixed constant from $(0,C)$, $v : [0,\infty) \times B_{r_1} \to X$, where $r_1 \in (0,\rho_1)$, is a natural transformation generated by the mapping h and (3) holds for $x \in B_{r_1}$.*

For the mapping $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, and for the operator $A \in L(Y, Y)$, fulfilling one of the conditions:

1) $A = aI_Y$ for $a > -\alpha$, where I_Y is the identity operator on Y,
2) $\|A\| < \alpha$,

we consider the Cauchy problem

$$(10) \qquad (\nabla_h)f(x) + Af(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{\rho_1}, \quad f(0) = 0$$

under the Assumption 1.

LEMMA 3. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be a continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2]$ and $K > 0$ we have*

$$(11) \qquad \|F(x, y)\| \le K\|x\| \quad for \quad (x, y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}.$$

Then for $\tau_1 := min(r_1, \tilde{\rho}_1)$ (where r_1 is a constant from Assumption 1) and for any continuous function $f : B_{\tau_1} \to B_{\tilde{\rho}_2}$ two following conditions are equivalent:

I. *f is the solution of the Cauchy problem* (10) *in the class $C_*^{0,1}(B_{\tau_1}, Y)$ (in particular f is differentiable in a direction of the mapping h),*

II. *$f \in C_*^{0,1}(B_{\tau_1}, Y)$ and*

$$(12) \qquad f(x) = \int_0^\infty e^{-At} F(v(t, x), f(v(t, x)))dt$$

for $x \in B_{\tau_1}$, where the above integral is absolutely convergent. If we reduce our considerations to the mapping A of the form $A = aI_Y, a \ge 0$, two following conditions are equivalent in the class of continuous mappings:

III. *f is the solution of Cauchy problem* (10),

IV. *f fulfills the integral equation* (12).

Proof. Let $\tau_1 = \min(r_1, \tilde{\rho}_1)$. By (11) and according to the form of the mapping A, any function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_1}, Y)$ fulfills the following inequalities

$$(13) \qquad \|e^{-At} F(v(t, x), f(v(t, x)))\| \le e^{\|A\|t} K\|v(t, x)\| \le K\|x\|e^{-(\alpha - \|A\|)t},$$

or

$$(13') \qquad \|e^{-at} F(v(t, x), f(v(t, x)))\| \le K\|x\|e^{-(\alpha + a)t}$$

for $t \in [0, \infty), x \in B_{\tau_1}$. Therefore for both forms of the mapping A we have

$$(14) \qquad \|e^{-At} F(v(t, x), f(v(t, x)))\| \le K\|x\|e^{-\mathbb{A}t}$$

for $t \in [0, \infty)$, $x \in B_{\tau_1}$, where

1) $\mathbb{A} = \alpha - \|A\|$ from the inequality (13);
2) $\mathbb{A} = \alpha + a$ from the inequality (13').

Let $f \in C(B_{r_1}, Y)$ be the solution of the problem (10). By Lemma IX.7, from [6] and Corollary 3 we get the equality

$$-\frac{\partial}{\partial t}[e^{-At} f(v(t,x))] = e^{-At} F(v(t,x), f(v(t,x))) \quad \text{for} \quad t \in [0,\infty), x \in B_{r_1}$$

which, together with the initial condition $v(0,x) = x$, is equivalent to

$$e^{-At} f(v(t,x)) - f(x) = -\int_0^t e^{-A\tau} F(v(\tau,x), f(v(\tau,x)))d\tau \tag{15}$$

for $t \in [0,\infty), x \in B_{r_1}$. If in addition $f \in C^{0,1}_*(B_{r_1}, Y)$, then we have the estimate

$$\|f(v(t,x))\| \le \|f\|_* \|v(t,x)\| \le \|f\|_* e^{-\alpha t} \|x\| \quad \text{for} \quad t \in [0,\infty), \ x \in B_{r_1}$$

implying, due to the condition $f(0) = 0$ and the form of the mapping A, that $\lim_{t\to\infty} e^{-At} f(v(t,x)) = 0$. By (15), we obtain

$$f(x) = \int_0^\infty e^{-A\tau} F(v(\tau,x), f(v(\tau,x)))d\tau \quad \text{for} \quad x \in B_{r_1},$$

where the above integral is absolutely convergent, by (14).

Now we shall show that every solution $f \in C^{0,1}_*(B_{r_1}, Y)$ of the above integral equation is a solution of the problem (10). By Lemma 1, we have $v(t_1, v(t,x)) = v(t_1 + t, x)$ for $t, t_1 \in [0,\infty)$ and $x \in B_{r_1}$. Hence

$$f(v(t,x)) = e^{At} \int_t^\infty e^{-A\tau} F(v(\tau,x), f(v(\tau,x)))d\tau \quad \text{for} \quad t \in [0,\infty), \ x \in B_{r_1}.$$

Differentiating above equation with respect to t, we obtain

$$\frac{\partial}{\partial t} f(v(t,x)) = Ae^{At} \int_t^\infty e^{-A\tau} F(v(\tau,x), f(v(\tau,x)))d\tau - F(v(t,x), f(v(t,x)))$$

for $t \in [0,\infty)$, $x \in B_{r_1}$. Therefore, by (12), for $t = 0$ we have

$$\nabla_h f(x) + Af(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{r_1}.$$

Now, it is sufficient to show that $f(0) = 0$. By Lemma 1, we have $v(t,0) = 0$ for $t \in [0,\infty)$ and the assumption (11) leads to $F(0,y) = 0$ for $y \in B_{\bar{\rho}_2}$. Therefore

$$\int_0^\infty e^{-A\tau} F(v(\tau,0), f(v(\tau,0)))d\tau = 0$$

and hence $f(0) = 0$. Consequently, $f \in C^{0,1}_*(B_{r_1}, Y)$ is a solution of the problem (10).

If we only assume that $f \in C(B_{\tau_1}, Y)$ and the mapping A has the form $A = aI_Y, a \geq 0$, then

$$\|e^{-at} f(v(t,x))\| \leq \|f(v(t,x))\| \quad \text{for} \quad t \in [0,\infty), x \in B_{\tau_1}.$$

Moreover, the left-hand side of the above inequality tends to $\|f(0)\| = 0$ as $t \to \infty$. Therefore, (15) implies (12). Further, the proof runs similarly as for $f \in C_*^{0,1}(B_{\tau_1}, Y)$. ∎

THEOREM 2. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be the continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2]$ and $K, L > 0$ the following conditions take places*

(i) $\|F(x,y)\| \leq K\|x\|$ *for* $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$,
(ii) $\|F(x,y_1) - F(x,y_2)\| \leq L\|y_1 - y_2\|\|x\|$ *for* $x \in B_{\tilde{\rho}_1}$ *and* $y_1, y_2 \in B_{\tilde{\rho}_2}$.

Then for $\tau_2 := \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}\tilde{\rho}_2}{K}, \frac{\mathbb{A}}{L})$ (where $\mathbb{A} = \alpha + a$ or $\mathbb{A} = \alpha - \|A\|$ and α, r_1 denote the constants from Assumption 1) the problem (10) *has exactly one solution f in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_2}, Y)$. This solution belongs to the class $C_*^{0,1}(B_{\tau_2}, Y)$.*

Proof. Let $\tau_2 = \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}\tilde{\rho}_2}{K}, \frac{\mathbb{A}}{L})$. For every $\tau < \tau_2$ we shall prove the existence and uniqueness of the solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ of the integral equation

$$f(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_\tau$$

equivalent to the problem (10), by Lemma 3. On this purpose, using the Banach fixed point theorem ([6] Theorem VIII.1), consider a mapping S defined on the closed ball $\bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ by

$$S(f)(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_\tau, f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

Every solution of the problem (10) is the fixed point of the mapping S. From the inequality (14) we obtain

$$(16) \quad \|S(f)(x)\| \leq \int_0^\infty K e^{-\alpha t} \|e^{-At}\| \|x\| dt = \frac{K\|x\|}{\mathbb{A}} \quad \text{for} \quad x \in B_\tau, f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

Therefore, from the definition of τ_2, it follows that

$$\|S(f)\| \leq \frac{K\tau}{\mathbb{A}} < \tilde{\rho}_2 \quad \text{for} \quad f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

Hence, S is a well-defined mapping and $S : \bar{\mathcal{B}}_{\tilde{\rho}_2} \to \mathcal{B}_{\tilde{\rho}_2} \subset \bar{\mathcal{B}}_{\tilde{\rho}_2}$.

We shall prove that S is a contraction. Let $f_1, f_2 \in \bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$. Then from the assumption (ii) we have

$$\|S(f_1)(x) - S(f_2)(x)\| \leq \frac{L\|x\|}{\mathbb{A}} \sup_{t\in[0,\infty)} \|f_1(v(t,x)) - f_2(v(t,x))\| \quad \text{for } x \in B_\tau.$$

Hence

$$\|S(f_1) - S(f_2)\| \leq \frac{L\tau}{\mathbb{A}} \|f_1 - f_2\| \quad \text{for} \quad f_1, f_2 \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

The inequality $\frac{L\tau}{\mathbb{A}} < 1$ is satisfied by definition of τ_2. Therefore the mapping S is a contraction. So, the assumptions of the Banach fixed point theorem hold. Therefore, there exists only one mapping $f_\tau \in \bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ which is a fixed point of the mapping S and so the solution of the problem (10) is unique; in addition, $f_\tau = S(f_\tau)$ belongs to the open ball $\mathcal{B}_{\tilde{\rho}_2}$. From the uniqueness of the solution we obtain the equality $f_\tau = f_{\tau'}\,|_{B_\tau}$ for $\tau < \tau' < \tau_2$. The searched solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_2}, Y)$ coincide with the function f_τ on every ball B_τ, i.e., $f(x) = f_\tau(x)$ for $x \in B_\tau, \tau < \tau_2$. By (16), the following inclusion takes place $f \in C_*^{0,1}(B_{\tau_2}, Y)$. ∎

THEOREM 3. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be the continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2], L > 0$ and $K_1, K_2, K_1^2 + K_2^2 > 0$, the conditions*

(i) *$\|F(x,y)\| \leq K_1\|x\| + K_2\|y\|$ for $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$,*
(ii) *$\|F(x,y_1) - F(x,y_2)\| \leq L\|y_1 - y_2\|\|x\|$ for $x \in B_{\tilde{\rho}_1}$ and $y_1, y_2 \in B_{\tilde{\rho}_2}$.*

are fulfilled. In the case of $A = aI_Y, a \geq 0$, for $K_1 \neq 0$ let be

$$\tau_3 := \begin{cases} \min\left(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1}{L}\right) & \text{if } 0 \leq \frac{K_2}{a} \leq 1 - \frac{K_1/\tilde{\rho}_2}{L},\ a \neq 0, \\ \min\left(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1 - K_2/a}{K_1/\tilde{\rho}_2}\right) & \text{if } 1 - \frac{K_1/\tilde{\rho}_2}{L} \leq \frac{K_2}{a} \leq \frac{L}{K_1/\tilde{\rho}_2 + L},\ a \neq 0, \\ \min\left(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1}{K_1/\tilde{\rho}_2 + L}\right) & \text{if } \frac{K_2}{a} \geq \frac{L}{K_1/\tilde{\rho}_2 + L},\ a \neq 0, \\ \min\left(r_1, \tilde{\rho}_1, \frac{\alpha}{K_1/\tilde{\rho}_2 + L}\right) & \text{if } a = 0, \end{cases}$$

and for $K_1 = 0$ let be $\tau_3 := \min(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1}{L})$ (where α, r_1 denote constants from Assumption 1). The problem (10) has exactly one solution f in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_3}, Y)$. This solution is in $C_*^{0,1}(B_{\tau_3}, Y)$.

Proof. Let us denote $\tau_1 = \min(r_1, \tilde{\rho}_1)$. From (i), (ii) it follows that for every function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_1}, Y)$ the inequalities

$$(17) \qquad \|e^{-at} F(v(t,x), f(v(t,x)))\| \leq (K_1 + L\|f(v(t,x))\|) e^{-(\alpha+a)t} \|x\|$$

hold for $t \in [0, \infty), x \in B_\tau$.

The analysis similar to that in the proof of Lemma 3 (when (17) replaces (14)) shows that the problem (10) in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y), \tau \leq \tau_1$, is equivalent to the equation

$$f(x) = \int_0^\infty e^{-at} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_\tau.$$

Let τ_3 be as assumed. For every $\tau < \tau_3$ we prove existence and uniqueness of the solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ of the equation (see (12)):

$$f(x) = \int_0^\infty e^{-at} F(v(t,x), f(v(t,x)))dt, \quad x \in B_\tau,$$

equivalent to the problem (10). The condition (i) implies that for every function $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ we have

$$\|e^{-at} F(v(t,x), f(v(t,x)))\| \le K_1 \|x\| e^{-(\alpha+a)t} + K_2 \tilde{\rho}_2 e^{-at} \quad \text{for } t \in [0,\infty),\ x \in B_\tau. \tag{18}$$

Using the Banach fixed point theorem, consider a mapping S on the closed ball $\bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y)$ defined by

$$S(f)(x) = \int_0^\infty e^{-at} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_\tau,\ f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

This integral is absolutely convergent. Moreover, by (17), we have

$$\|S(f)(x)\| \le \frac{(K_1 + L\tilde{\rho}_2)\|x\|}{\alpha + a} \quad \text{for} \quad x \in B_\tau,\ f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}. \tag{19}$$

Every solution of the problem (10) is a fixed point of the mapping S. Just as in the proof of Theorem 2, we can show that

$$\|S(f_1) - S(f_2)\| \le \frac{L\tau}{\alpha + a}\|f_1 - f_2\| \quad \text{for} \quad f_1, f_2 \in \bar{\mathcal{B}}_{\tilde{\rho}_2} \subset C(B_\tau, Y). \tag{20}$$

In the case of $K_1 \neq 0$ and $0 \le \frac{K_2}{a} \le \frac{L}{K_1/\tilde{\rho}_2 + L}$, with $a \neq 0$, we have

$$\tau_3 = \min(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1}{L}, (\alpha + a)\frac{1 - K_2/a}{K_1/\tilde{\rho}_2}),$$

because the condition $\frac{K_2}{a} \ge 1 - \frac{K_1/\tilde{\rho}_2}{L}$ is equivalent to $\frac{1}{L} \ge \frac{1-K_2/a}{K_1/\tilde{\rho}_2}$ and from the condition $\frac{K_2}{a} \le \frac{L}{K_1/\tilde{\rho}_2 + L}$ it follows that $\frac{1}{K_1/\tilde{\rho}_2 + L} \le \frac{1-K_2/a}{K_1/\tilde{\rho}_2}$. By (18) and the condition $\tau_3 \le (\alpha + a)\frac{1-K_2/a}{K_1/\tilde{\rho}_2}$, we obtain

$$\|S(f)(x)\| < (1 - K_2/a)\tilde{\rho}_2 + \frac{K_2 \tilde{\rho}_2}{a} = \tilde{\rho}_2 \quad \text{for} \quad x \in B_\tau,\ f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}.$$

The inequality $\tau_3 \le (\alpha + a)\frac{1}{L}$ is true, hence we conclude from (20) that the mapping S is a contraction.

Now let $\frac{K_2}{a} \geq \frac{L}{K_1/\tilde{\rho}_2+L}$ for $a \neq 0$ or $a = 0$ or $K_1 = 0$. Then

$$\tau_3 = \min\left(r_1, \tilde{\rho}_1, (\alpha + a)\frac{1}{K_1/\tilde{\rho}_2 + L}\right).$$

From (19) and the condition $\tau_3 \leq (\alpha+a)\frac{1}{K_1/\tilde{\rho}_2+L}$ it follows that $\|S(f)\| < \tilde{\rho}_2$ for $f \in \bar{\mathcal{B}}_{\tilde{\rho}_2}$. Since $\tau_3 \leq \frac{\alpha+a}{K_1/\tilde{\rho}_2+L} \leq \frac{\alpha+a}{L}$, from the inequality (20) it follows that S is a contraction. Thus S is a well-defined mapping and $S : \bar{\mathcal{B}}_{\tilde{\rho}_2} \to \mathcal{B}_{\tilde{\rho}_2} \subset \bar{\mathcal{B}}_{\tilde{\rho}_2}$. By the Banach theorem, in the same manner as in the proof of Theorem 2, we can state that the problem (10) has exactly one solution $f \in \mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_3}, Y)$. This solution is in $C^{0,1}_*(B_{\tau_3}, Y)$, by the inequality (19). ∎

THEOREM 4. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be a continuous mapping such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2], L > 0$ and $K_1, K_2, K_1^2 + K_2^2 > 0$, the following conditions take places*

(i) $\|F(x, y)\| \leq K_1\|x\| + K_2\|y\|$ *for* $(x, y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$,

(ii) $\|F(x, y_1) - F(x, y_2)\| \leq L\|y_1 - y_2\|\|x\|$ *for* $x \in B_{\tilde{\rho}_1}$ *and* $y_1, y_2 \in B_{\tilde{\rho}_2}$.

For $K_1 \neq 0$, let τ_4 be defined by

$$\tau_4 := \begin{cases} \min(r_1, \tilde{\rho}_1, \frac{\alpha+\mathbb{A}}{L}, \frac{(\mathbb{A}-K_2)\tilde{\rho}_2}{K_1}) & \text{if } \mathbb{A} \geq K_2(1 + \frac{K_1}{\tilde{\rho}_2 L}), \\ \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}\tilde{\rho}_2}{K_1+\tilde{\rho}_2 L}) & \text{if } 0 < \mathbb{A} \leq K_2(1 + \frac{K_1}{\tilde{\rho}_2 L}), \end{cases}$$

and for $K_1 = 0$ by $\tau_4 := \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}}{L})$ (where $\mathbb{A} = \alpha + a$ or $\mathbb{A} = \alpha - \|A\|$ and α, r_1 denote the constants from Assumption 1). The problem (10) has exactly one solution f in the ball $\bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_{\tau_4}, Y)$, $\kappa = \frac{\tilde{\rho}_2}{\tau_4}$. In the case $\mathbb{A} \leq K_2(1 + \frac{K_1}{\tilde{\rho}_2 L})$ this solution is in the open ball $\mathcal{B}^*_\kappa$.

Proof. Let τ_4 be as assumed and $\kappa = \frac{\tilde{\rho}_2}{\tau_4}$. Note that, if $f \in \bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_{\tau_4}, Y)$, then for every $\tau < \tau_4$ we have

$$\|f(v(t,x))\| \leq \|f\|_*\|v(t,x)\| \leq \frac{\tilde{\rho}_2}{\tau_4}\|v(t,x)\| \leq \frac{\tilde{\rho}_2\tau}{\tau_4} < \tilde{\rho}_2, \; t \in [0,\infty), \, x \in B_\tau.$$

Hence $f(v(t,x)) \in \mathcal{B}_{\tilde{\rho}_2}$ for $t \in [0, \infty)$ and $x \in B_\tau$. In particular, the inequalities (i), (ii) hold for $y = f(v(t, x))$. Hence, for every $\tau < \tau_4$, by (i), we get

$$\|e^{-At}F(v(t,x), f(v(t,x)))\| \leq (K_1 + K_2\frac{\tilde{\rho}_2}{\tau_4})e^{-\mathbb{A}t}\|x\|, \tag{21}$$

and, by (i) and (ii), we obtain

$$\|e^{-At}F(v(t,x), f(v(t,x)))\| < (K_1 + L\tilde{\rho}_2)e^{-\mathbb{A}t}\|x\| \tag{22}$$

for $t \in [0, \infty)$, $x \in B_\tau$. The analysis similar to that in the proof of Lemma 3 (when one of the inequalities (21), (22) replaces (14)) shows that for every

$\tau < \tau_4$ the problem (10) in the ball $\bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_\tau, Y), \kappa = \frac{\tilde{\rho}_2}{\tau_4}$, is equivalent to the searching of the solution $f \in \bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_\tau, Y)$ of the integral equation (12).

In order to prove the existence and uniqueness of the solution of the equation (12), using the Banach fixed point theorem, consider the mapping S on the closed ball $\bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_\tau, Y)$, $\kappa = \frac{\tilde{\rho}_2}{\tau_4}$, defined by

$$S(f)(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_\tau,\ f \in \bar{\mathcal{B}}^*_\kappa.$$

If $K_1 \neq 0$ and

$$\mathbb{A} \geq K_2\left(1 + \frac{K_1}{\tilde{\rho}_2 L}\right) \tag{23}$$

then $\tau_4 = \min(r_1, \tilde{\rho}_1, \frac{\alpha+\mathbb{A}}{L}, \frac{(\mathbb{A}-K_2)\tilde{\rho}_2}{K_1})$, because from (23) it follows that $\frac{\mathbb{A}-K_2}{K_1} \geq \frac{\mathbb{A}}{K_1+\tilde{\rho}_2 L}$. The condition $\tau_4 \leq \frac{(\mathbb{A}-K_2)\tilde{\rho}_2}{K_1}$ is equivalent to $\frac{\tilde{\rho}_2}{\tau_4} \geq \frac{K_1}{\mathbb{A}-K_2}$. Hence $K_1\tau_4 + K_2\tilde{\rho}_2 \leq \mathbb{A}\tilde{\rho}_2$. Therefore, from (21) we have

$$\|S(f)(x)\| \leq \int_0^\infty \frac{\mathbb{A}\tilde{\rho}_2}{\tau_4} e^{-\mathbb{A}t}\|x\|dt = \kappa\|x\| \quad \text{for} \quad x \in B_\tau, f \in \bar{\mathcal{B}}^*_\kappa. \tag{24}$$

Now let

$$0 < \mathbb{A} \leq K_2\left(1 + \frac{K_1}{\tilde{\rho}_2 L}\right) \tag{25}$$

or $K_1 = 0$. Then $\tau_4 = \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}\tilde{\rho}_2}{K_1+\tilde{\rho}_2 L})$. From the inequality (25) and the condition $\tau_4 \leq \frac{\mathbb{A}\tilde{\rho}_2}{K_1+\tilde{\rho}_2 L}$ we have

$$\|S(f)(x)\| < \int_0^\infty (K_1 + \tilde{\rho}_2 L)e^{-\mathbb{A}t}\|x\|dt \leq \kappa\|x\| \quad \text{for} \quad x \in B_\tau,\ f \in \bar{\mathcal{B}}^*_\kappa. \tag{26}$$

By the inequalities (24) or (26) we obtain $\|S(f)\|_* \leq \kappa$ for $f \in \bar{\mathcal{B}}^*_\kappa$. Therefore S is a well-defined mapping and $S : \bar{\mathcal{B}}^*_\kappa \to \bar{\mathcal{B}}^*_\kappa$.

If (25) holds then from above considerations it follows, by (26), that $\|S(f)\|_* < \kappa$ for $f \in \bar{\mathcal{B}}^*_\kappa$. Hence $S : \bar{\mathcal{B}}^*_\kappa \to \mathcal{B}^*_\kappa \subset \bar{\mathcal{B}}^*_\kappa$.

Next we shall prove that S is a contraction. Let $f_1, f_2 \in \bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_\tau, Y)$. Then, by the condition (ii), we have the estimate

$$\begin{aligned}\|S(f_1)(x) - S(f_2)(x)\| &\leq L\|f_1 - f_2\|_*\tau \int_0^\infty e^{-(\alpha+\mathbb{A})t}\|x\|dt \\ &\leq \frac{L\tau}{\alpha+\mathbb{A}}\|f_1 - f_2\|_*\|x\| \quad \text{for} \quad x \in B_\tau,\ f_1, f_2 \in \bar{\mathcal{B}}^*_\kappa\end{aligned}$$

implying

$$\|S(f_1) - S(f_2)\|_* \le \frac{L\tau}{\alpha + \mathbb{A}}\|f_1 - f_2\|_* \quad \text{for} \quad f_1, f_2 \in \bar{\mathcal{B}}^*_\kappa,$$

where, by the inequality $\frac{\mathbb{A}\tilde{\rho}_2}{K_1+\tilde{\rho}_2 L} \le \frac{\alpha+\mathbb{A}}{L}$ and the assumption on τ_4, the coefficient $\frac{L\tau}{\alpha+\mathbb{A}}$ is less than 1. Therefore S is a contraction.

By the Banach fixed point theorem, there exists exactly one mapping $f_\tau \in \bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_\tau, Y)$ which is a fixed point of the mapping S and so is a solution of the problem (10); in addition, if the condition (25) is fulfilled then, $f_\tau = S(f_\tau)$ belongs to the open ball $\mathcal{B}^*_\kappa$. From the uniqueness of the solution we obtain the equality: $f_\tau = f_{\tau'}\,|_{B_\tau}$ for $\tau < \tau' < \tau_4$. The searched solution $f \in \bar{\mathcal{B}}^*_\kappa \subset C^{0,1}_*(B_{\tau_4}, Y)$ coincide with the function f_τ, i. e., $f(x) = f_\tau(x)$ for $x \in B_\tau$, $\tau < \tau_4$, on every ball B_τ and besides the condition $\|f\|_* \le \kappa$ is fulfilled. ∎

Remark. For $K_2 = 0$ the thesis of Theorem 4 is another version of Theorem 2 with the theoretically larger domain of the solution.

Example 2. Let us consider the function F of the form

$$F(x,y) := G(x, y + y_0) \quad \text{for} \quad x \in B_{\rho_1} \quad \text{and} \quad y \in B_{\rho_2},$$

where $y_0 \in Y$ is a certain constant and $G \in L(X, Y; Y)$ is a bilinear and continuous operator from $X \times Y$ into the Banach space Y. The following inequalities take place

$$\|G(x, y + y_0)\| \le \|G\|\|x\|(\|y\| + \|y_0\|) \le \|G\|(\tilde{\rho}_2 + \|y_0\|)\|x\|. \tag{27}$$

The assumptions of Theorem 2 are fulfilled if $K := \|G\|(\tilde{\rho}_2 + \|y_0\|)$, $L := \|G\|$.

Therefore the Cauchy problem

$$\nabla_h f(x) + Af(x) = G(x, f(x) + y_0) \quad \textit{for} \quad x \in B_{\rho_1},\ f(0) = 0 \tag{28}$$

(i. e., the problem (10), in the case when $F(x,y) = G(x, y + y_0), G \in L(X, Y; Y)$ and $y_0 \in Y$ is a certain constant) has exactly one solution f in the ball $\mathcal{B}_{\tilde{\rho}_2} \subset C(B_{\tau_2}, Y)$, where

$$\tau_2 = \min(r_1, \tilde{\rho}_1, \frac{\mathbb{A}/\|G\|}{1 + \|y_0\|/\tilde{\rho}_2})$$

(if $\mathbb{A}=\alpha+a$ or $\mathbb{A}=\alpha-\|A\|$ and α, r_1 denote the constants from Assumption 1).

Example 3. Let us consider the Cauchy problem (28). Notice, that for every $\lambda \in [0,1]$ we have

$$\begin{aligned} \|G(x, y + y_0)\| &\le \|G(x,y)\| + \|G(x, y_0)\| \\ &\le \lambda\|G\|\|x\|\|y\| + (1-\lambda)\|G\|\|x\|\|y\| + \|G\|\|x\|\|y_0\| \\ &\le \|G\|(\lambda\tilde{\rho}_2 + \|y_0\|)\|x\| + \|G\|(1-\lambda)\tilde{\rho}_1\|y\| \end{aligned} \tag{28'}$$

(compare the inequality (28)). The assumptions of Theorem 4 are fulfilled if $K_1 := \|G\|(\lambda\tilde{\rho}_2 + \|y_0\|)$, $K_2 := \|G\|(1-\lambda)\tilde{\rho}_1$, $L := \|G\|$. The analysis of the constant $\tau_4 = \tau_4(\lambda)$, used in the thesis of Theorem 4, leads to the conclusion that $\max_{\tau\in[0,1]} \tau_4(\lambda) = \tau_4(1) = \tau_2$.

3. The Cauchy problem for the generalized differential equations of first order with singularity in a point zero, in C^1 class

Let X, Y be real Banach spaces, U be an open subset of X.

Let $C^1(B_\tau, Y)$ denotes a space of the mappings $f : B_\tau \to Y$, with $B_\tau \subset X$, of the class C^1 on B_τ, bounded together with its first derivative on B_τ, with the norm

$$\|f\|_1 = \max_{x\in B_\tau}(\|f(x)\|, \|Df(x)\|). \tag{29}$$

The space $C^1(B_\tau, Y)$ is complete ([8], p. 226).

By $\mathcal{B}^1_\rho \subset C^1(B_\tau, Y)$ we shall denote the ball with centre at zero and radius $\rho > 0$ considered in the space $C^1(B_\tau, Y)$.

We shall introduce the following assumption

ASSUMPTION 2. *Let $h : B_{\rho_1} \to X$, with $B_{\rho_1} \subset X$, be a regular mapping of the class C^2 and let $C > 0$ be a constant such that* (1) *holds. We assume that α, β are fixed constants from $(0, C)$, $v : [0,\infty)\times B_{r_2} \to X$, where $r_2 \in (0, \rho_1)$, is a natural transformation generated by the mapping h and (3), (5) hold for $x \in B_{r_2}$.*

For the mapping $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, and for the operator $A \in L(Y, Y)$ fulfilling one of the conditions:

1) $A = aI_Y$ for $a > -(\alpha + \min(\alpha, \beta))$, where I_Y is the identity operator on Y,

2) $\|A\| < \alpha$,

we consider the Cauchy problem

$$Df(x)h(x) + Af(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{\rho_1}, \quad f(0) = 0 \tag{30}$$

under the Assumption 2.

We introduce the notation

$$\mathbb{A} = \alpha - \|A\| \quad \text{or} \quad \mathbb{A} = \alpha + a.$$

LEMMA 4. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$ be a mapping of the class C^1 such that for certain constants $\tilde{\rho}_1 \in (0, \rho_1], \tilde{\rho}_2 \in (0, \rho_2]$ and $M > 0$ we have*

(i) $F(0,0) = 0$

(ii) $\|DF(x,y)\| \le M\|x\|$ *for* $(x, y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$.

Then for $\tau_5 = \min(r_2, \tilde{\rho}_1)$ (*where* r_2 *is a constant from Assumption 2 and for any function* $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_5}, Y)$ *two following conditions are equivalent*:

I. f *is the solution of the Cauchy problem* (30)

II. *the following equality takes place*:

$$f(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \tag{31}$$

for $x \in B_{\tau_5}$, *where the above integral is absolutely convergent.*

Proof. Let $\tau_5 = \min(r_2, \tilde{\rho}_1)$. By (i), (ii), we have

$$\|F(x,y) - F(0,0)\| \le (\|x\| + \|y\|) \sup \|DF(\cdot,\cdot)\| \le M(\|x\| + \|y\|)\|x\|$$

for $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$.

Therefore for any function $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_5}, Y)$ and for both forms of the mapping A we obtain

$$\|e^{-At} F(v(t,x), f(v(t,x)))\| \le M e^{-(\alpha + \mathbb{A})t}(1 + \tilde{\rho}_2)\|x\|^2 \quad \text{for } t \in [0,\infty), x \in B_{\tau_5}. \tag{32}$$

Let $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_5}, Y)$ be the solution of problem (30). By Lemma IX.7, from [6] we get the equality

$$-\frac{\partial}{\partial t}[e^{-At} f(v(t,x))] = e^{-At} F(v(t,x), f(v(t,x))) \quad \text{for} \quad t \in [0,\infty), \ x \in B_{\tau_5}$$

which, together with initial condition $v(0,x) = x$, is equivalent to

$$e^{-At} f(v(t,x)) - f(x) = -\int_0^t e^{-A\tau} F(v(\tau,x), f(v(\tau,x)))d\tau \quad \text{for} \quad t \in [0,\infty), \ x \in B_{\tau_5}. \tag{33}$$

Now consider the case when $A = aI_Y, a > -\alpha$, or $\|A\| < \alpha$. Then $\mathbb{A} > 0$. From the assumptions that $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_5}, Y)$ it follows that

$$f(x) = Df(0)(x) + o(x), \quad \text{where} \quad \lim_{x \to 0} \frac{\|o(x)\|}{\|x\|} = 0.$$

Therefore, we have the estimate

$$\|e^{-At} f(v(t,x))\| \le e^{-\mathbb{A}t}\Big(\|Df(0)\| + \frac{\|o(v(t,x))\|}{\|v(t,x)\|}\Big)\|x\| \quad \text{for} \quad t \in [0,\infty), \ x \in B_{\tau_5}. \tag{34}$$

Since $\lim_{t\to\infty} v(t,x) = 0$ from the inequality (34), due to the condition $f(0) = 0$ and the form of the mapping A, we conclude that $\lim_{t\to\infty} e^{-At} f(v(t,x)) = 0$. By (33) we obtain

$$f(x) = \int_0^\infty e^{-A\tau} F(v(\tau, x), f(v(\tau, x)))d\tau \quad \text{for} \quad x \in B_{\tau_5},$$

where the above integral is absolutely convergent by (32).

Now we shall show that every solution $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_5}, Y)$ of the above integral equation is a solution of the problem (30). By Lemma 1, we have $v(t_1, v(t,x)) = v(t_1 + t, x)$ for $t_1, t \in [0, \infty)$ and $x \in B_{\tau_5}$. Hence

$$f(v(t,x)) = e^{At} \int_t^\infty e^{-A\tau} F(v(\tau, x), f(v(\tau, x)))d\tau \quad \text{for} \quad t \in [0, \infty), x \in B_{\tau_5}.$$

Differentiating above equation with respect to t, we obtain

$$\frac{\partial}{\partial t} f(v(t,x)) = Ae^{At} \int_t^\infty e^{-A\tau} F(v(\tau, x), f(v(\tau, x)))d\tau - F(v(t,x), f(v(t,x)))$$

for $t \in [0, \infty)$, $x \in B_{\tau_5}$.

Therefore, by (31), we have

$$Df(x)h(x) + Af(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{\tau_5}.$$

Now, it is sufficient to show that f(0)= 0. By Lemma 1, we have $v(t, 0) = 0$ for $t \in [0, \infty)$ and the assumptions (i), (ii) lead to $F(0, y) = 0$ for $y \in \mathcal{B}_{\tilde{\rho}_2}$. Therefore

$$\int_0^\infty e^{-At} F(v(t, 0), f(v(t, 0)))dt = 0$$

and hence $f(0) = 0$. Consequently, $f \in C^1(B_{\tau_5}, Y)$ is a solution of the problem (30).

Let now $A = aI_Y$ for $-(\alpha + \min(\alpha, \beta)) < a \leq -\alpha$. By (i), (ii), we have

$$\|e^{-at} D_x F(v(t,x), f(v(t,x)))\| \leq M(1 + \tilde{\rho}_2) e^{-(\beta + \mathbb{A})t} \|x\| \quad \text{for} \quad t \in [0, \infty), x \in B_{\tau_5}. \tag{35}$$

Define the function

$$g(x) := \int_0^\infty e^{-a\tau} F(v(\tau, x), f(v(\tau, x)))d\tau \quad \text{for} \quad t \in [0, \infty), \ x \in B_{\tau_5}.$$

Since $\alpha + \mathbb{A} > 0$ and $\beta + \mathbb{A} > 0$, from (32), (33) it follows that $g \in C^1(B_{\tau_5}, Y)$. By Lemma 1, we have $v(t_1, v(t,x)) = v(t_1 + t, x)$ for $t, t_1 \in [0, \infty), x \in B_{\tau_5}$. Hence

$$g(v(t,x)) = e^{at} \int_t^\infty e^{-a\tau} F(v(\tau, x), f(v(\tau, x)))d\tau \quad \text{for} \quad t \in [0, \infty), \ x \in B_{\tau_5}.$$

Differentiating above equation with respect to t, we obtain

$$\frac{\partial}{\partial t} g(v(t,x)) = ae^{at} \int_t^\infty e^{-a\tau} F(v(\tau, x), f(v(\tau, x)))d\tau - F(v(t,x), f(v(t,x)))$$

for $t \in [0,\infty)$, $x \in B_{r_5}$. Therefore, by the form of the mapping g, for $t=0$ we have

$$Dg(x)h(x) + ag(x) = F(x, f(x)) \quad \text{for} \quad x \in B_{r_5}.$$

Naturally $g(0) = 0$. Simultaneously the function f is the solution of the problem (30) and so the function $\bar{g}(x) := f(x) - g(x)$ is the solution of the equation

$$D\bar{g}(x)h(x) + a\bar{g}(x) = 0 \quad \text{with the condition} \quad \bar{g}(0) = 0.$$

For $x \in B_{r_5}$ the function $\bar{G}(t) := \bar{g}(v(t,x))$ is the solution of the equation

$$\bar{G}' + a\bar{G} = 0 \quad \text{with the condition} \quad \lim_{t\to\infty} \bar{G}(t) = 0.$$

Since for $a < 0$ the function $\bar{G} \equiv 0$ is the unique solution of the above equation, from our earlier consideratins and from (34), (35) it follows that for $-(\alpha + \min(\alpha,\beta)) < a \leq -\alpha$ the function f given by the formula (31) is the unique solution of the problem (30). ■

THEOREM 5. *Let $F : B_{\rho_1} \times B_{\rho_2} \to Y$, with $B_{\rho_1} \times B_{\rho_2} \subset X \times Y$, be the mapping of the class C^1 such that for certain constants $\tilde{\rho}_1 \in (0,\rho_1], \tilde{\rho}_2 \in (0,\rho_2]$ and $M, W > 0$ the following conditions take places*

(i) *$F(0,y) = 0$ for $y \in B_{\tilde{\rho}_2}$,*
(ii) *$\|DF(x,y)\| \leq M\|x\|$ for $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$,*
(iii) *$\|DF(x,y_1) - DF(x,y_2)\| \leq W\|y_1 - y_2\|\|x\|$ for $x \in B_{\tilde{\rho}_1}$, $y_1, y_2 \in B_{\tilde{\rho}_2}$.*

Then for $r_6 = \min(r_2, \tilde{\rho}_1, \frac{(\alpha+\mathbb{A})\tilde{\rho}_2}{M\tilde{\rho}_1}, \frac{\alpha+\mathbb{A}}{W\tilde{\rho}_1}, \frac{(\beta+\mathbb{A})\tilde{\rho}_2}{M(1+\tilde{\rho}_2)}, \frac{\beta+\mathbb{A}}{M+W(1+\tilde{\rho}_2)})$ (where $\mathbb{A} = \alpha + a$ or $\mathbb{A} = \alpha - \|A\|$ and α, β, r_2 denote the constants from Assumption 2) the problem (30) has exactly one solution f in the ball $\mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{r_6}, Y)$.

Proof. Let

$$r_6 = \min\left(r_2, \tilde{\rho}_1, \frac{(\alpha+\mathbb{A})\tilde{\rho}_2}{M\tilde{\rho}_1}, \frac{\alpha+\mathbb{A}}{W\tilde{\rho}_1}, \frac{(\beta+\mathbb{A})\tilde{\rho}_2}{M(1+\tilde{\rho}_2)}, \frac{\beta+\mathbb{A}}{M+W(1+\tilde{\rho}_2)}\right).$$

For every $r < r_6$ we shall prove the existence and uniqueness of the solution $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_r, Y)$ of the integral equation

$$f(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_r$$

equivalent to the problem (30) by Lemma 4. On this purpose, using the Banach fixed point theorem, consider a mapping S defined on the closed ball $\bar{\mathcal{B}}^1_{\tilde{\rho}_2} \subset C^1(B_r, Y)$ by

$$S(f)(x) = \int_0^\infty e^{-At} F(v(t,x), f(v(t,x)))dt \quad \text{for} \quad x \in B_r, \; f \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2}.$$

Every solution of the problem (30) is a fixed point of the mapping S. Notice, that by (i), (ii), we have $\|F(x,y)\| = \|F(x,y) - F(0,y)\| \leq \|x\| \sup \|D_1F(\cdot,y)\| \leq M\|x\|^2$ for $(x,y) \in B_{\tilde{\rho}_1} \times B_{\tilde{\rho}_2}$.

From the above inequalities for any function $f \in \mathcal{B}^1_{\tilde{\rho}_2}$ and for both forms of the mapping A we obtain

(36) $\|e^{-At}F(v(t,x), f(v(t,x)))\| \leq M\|e^{-At}\|\|v(t,x)\|^2 \leq Me^{-(\alpha+\mathbb{A})t}\|x\|^2$

for $t \in [0,\infty)$, $x \in B_\tau$.

Consequently, from the assumption (i) and the inequality (36) we have

$$\|S(f)(x)\| \leq \int_0^\infty \|e^{-At}F(v(t,x), f(v(t,x)))\|dt$$

$$\leq M\int_0^\infty e^{-(\alpha+\mathbb{A})t}\|x\|^2 dt = \frac{M\tilde{\rho}_1\|x\|}{\alpha+\mathbb{A}}$$

for $x \in B_\tau$. Therefore from the definition τ_6 we obtain

$$\|S(f)\| \leq \frac{M\tilde{\rho}_1\tau}{\alpha+\mathbb{A}} < \tilde{\rho}_2 \quad \text{for} \quad f \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2}.$$

Notice, that for both forms of the mapping A we have

(37) $\|e^{-At}D_xF(v(t,x), f(v(t,x)))\| \leq M(1+\tilde{\rho}_2)e^{-(\beta+\mathbb{A})t}\|x\|$

for $t \in [0,\infty), x \in B_\tau$ (compare inequality (35)).

If $f \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2} \subset C^1(B_\tau, Y)$, then $\|Df(x)\| \leq \tilde{\rho}_2$ for $x \in B_\tau$. Consequently, by (ii), the inequality (37) and Theorem 135 in [8], we obtain

$$\|DS(f(x))\| \leq \frac{M(1+\tilde{\rho}_2)\|x\|}{\beta+\mathbb{A}} \quad \text{for} \quad x \in B_\tau.$$

From the definition of τ_6 it follows that

$$\|DS(f)\| \leq \frac{M(1+\tilde{\rho}_2)\tau}{\beta+\mathbb{A}} < \tilde{\rho}_2 \quad \text{for} \quad f \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2}.$$

Hence S is well-defined mapping and $S : \bar{\mathcal{B}}^1_{\tilde{\rho}_2} \to \mathcal{B}^1_{\tilde{\rho}_2} \subset \bar{\mathcal{B}}^1_{\tilde{\rho}_2}$.

We shall prove that S is a contraction. Let $f_1, f_2 \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2} \subset C^1(B_\tau, Y)$. Then, by (i), (iii), we have

$$\|F(x,y_1) - F(x,y_2)\| \leq W\|x\|^2\|y_1 - y_2\| \quad \text{for} \quad x \in B_{\tilde{\rho}_1}, y_1, y_2 \in B_{\tilde{\rho}_2}.$$

Hence

$$\|S(f_1)(x) - S(f_2)(x)\| \leq \int_0^\infty We^{-(\alpha+\mathbb{A})t}\tilde{\rho}_1\|x\|\|f_1(v(t,x)) - f_2(v(t,x))\|dt$$

for $x \in B_\tau$. Therefore

(38) $\|S(f_1) - S(f_2)\| \leq \dfrac{W\tilde{\rho}_1\tau}{\alpha+\mathbb{A}}\|f_1 - f_2\| \quad \text{for} \quad f_1, f_2 \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2}.$

Since $\|Df_1(v(t,x))\| \le \tilde{\rho}_2$ for $t \in [0,\infty)$ and $x \in B_\tau$, by (ii), (iii), the formula (29) and the identity $a_1b_1 - a_2b_2 = (a_1-a_2)b_1 + (b_1-b_2)a_2$, we obtain

$$\|D_xF(v(t,x), f_1(v(t,x))) - D_xF(v(t,x), f_2(v(t,x)))\| \le e^{-(\alpha+\beta)t}\|x\|[W(1+\tilde{\rho}_2)+M]\|f_1-f_2\|_1.$$

Hence

$$\|DS(f_1(x)) - DS(f_2(x))\| \le \int_0^\infty e^{-(\beta+\mathbb{A})t}[W(1+\tilde{\rho}_2)+M]\|x\|\|f_1-f_2\|_1$$

for $x \in B_\tau$. Therefore

$$(39) \quad \|DS(f_1)-DS(f_2)\| \le \frac{[W(1+\tilde{\rho}_2)+M]\tau}{\beta+\mathbb{A}}\|f_1-f_2\|_1 \quad \text{for } f_1, f_2 \in \bar{\mathcal{B}}^1_{\tilde{\rho}_2}.$$

Consequently, from (38), (39) and the definition of τ_6 it follows that the mapping S is a contraction.

By the Banach theorem, there exists exactly one mapping $f_\tau \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_\tau, Y)$ which is a fixed point of the mapping S and so is a solution of the problem (30); in addition $f_\tau = S(f_\tau)$ belongs to an open ball $\mathcal{B}^1_{\tilde{\rho}_2}$. From the uniqueness of the solution we obtain the equality $f_\tau = f_{\tau'}\,|_{B_\tau}$ dla $\tau < \tau' < \tau_6$. The searched solution $f \in \mathcal{B}^1_{\tilde{\rho}_2} \subset C^1(B_{\tau_6}, Y)$ coincide with the function f_τ on every ball B_τ, i. e., $f(x) = f_\tau(x)$ for $x \in B_\tau, \tau < \tau_6$. ■

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