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ON THE LOCATION OF CRITICAL POINTS OF SOME COMPLEX POLYNOMIALS

Abstract. Let $\mathcal{P}(a,n)$ be the set of all complex polynomials of degree n which have all their roots in the closed unit disk and one fixed root at a, $0 \le a \le 1$. In this paper we show that for $n \ge 3$ all critical points of the polynomial $f(z) = (z^{n-1}+1)(z-a)$ lie outside the set $\mathcal{K}(a,n)$ consisting of all b such that for some c the polynomial $p(z) = (z-b)^n - c$ belongs to $\mathcal{P}(a,n)$. Hence we infer that minimal sets satisfying the Sendov property (i.e. containing at least one critical point of each $p \in \mathcal{P}(a,n)$) exist but they are not unique.

1. Introduction

In this paper $D(z_0;r)$ will denote the disk $\{z : |z-z_0| \le r\}$ and $D = D(0;1)$ will be the unit disk in the complex plane $\mathbf{C}$.

Over thirty years ago Sendov conjectured that if all the zeros of the polynomial

$$p(z) = \prod_{\nu=1}^{n}(z-z_\nu), \qquad (n \ge 2),$$

lie in the unit disk D then for each z_ν the disk $D(z_\nu;1)$ contains at least one zero of $p'(z)$. This assertion has not been proved in general so far, but since it was published in 1967 many partial results have been obtained (for references see [6, 7, 8]). In particular the following theorem due to Bojanov, Rahman and Szynal [1] holds.

THEOREM A. *If* $p(z) = \prod_{\nu=1}^{n}(z-z_\nu)$ *has all its zeros in* D *then each of the disks* $D(z_\nu,(1+|z_1z_2\dots z_n|)^{1/n})$, $\nu = 1,2,\dots,n$, *contains at least one zero of* $p'(z)$.

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So the disk slightly larger then $D(z_\nu; 1)$ (and asymptotically tending to $D(z_\nu; 1)$) has the desired property. This result suggests the following question.

Find a possibly large set which is contained in the disk $D(z_\nu; 1)$ and does not possess the property mentioned above.

In this paper we describe such a set and we draw some conclusions concerning somewhat more general problem due to Goodman, Rahman and Ratti [4].

2. Notation and statement of the result

Let $\mathcal{P}(a,n)$ be the set of all complex polynomials of degree n which have all their roots in the closed unit disk D and at least one root at a, $0 \le a \le 1$.

A set S will be said to have the Sendov property if it contains at least one critical point of each $p \in \mathcal{P}(a,n)$. For example, from Theorem A it easily follows that the disk $D(a; (1+a)^{1/n})$ has the Sendov property, and written in this notation, Sendov Conjecture asserts that the smaller disk $D(a; 1)$ has it, too.

Let us define $\mathcal{K}(a,n)$ as the set which consists of all $b \in D$ such that for some $c \in \mathbf{C}$ the polynomial $p(z) = (z-b)^n - c$ belongs to the family $\mathcal{P}(a,n)$. We have

$$(1) \qquad \mathcal{K}(a,n) = \left\{ b \in D : \left[b + e^{\frac{2k\pi i}{n}}(a-b) \right] \in D \ \text{ for } \ k = 1, \ldots, n \right\}.$$

From elementary geometric observations it follows that the set $\mathcal{K}(a,n)$ is closed and contains the ellipse $\{z : |z| + |z-a| \le 1\}$ (for details we refer to [2]).

It is obvious that $\mathcal{K}(a,n)$ is the smallest set (in the sense of inclusion) which pretends to possess the Sendov property. The main goal of this paper is to show that this is not true, which is an immediate consequence of the following result.

PROPOSITION 1. *For any $0 < a \le 1$ and $n \ge 3$ all critical points of the polynomial $f(z) = (z^{n-1}+1)(z-a)$ lie outside the set $\mathcal{K}(a,n)$.*

Finally let us remark, that for $n = 6, 8, 10$ and 12 Miller [**6**] has shown that the disk $D(\frac{a}{2}, 1 - \frac{a}{2})$, which is larger then $\mathcal{K}(a,n)$, does not have the Sendov property, either.

3. Proof of Proposition 1

Let us fix $0 < a \le 1$ and $n \ge 6$. Let z_0 be any of the critical points of f and let $|z_0| = \delta$ and $\arg z_0 = \beta$. From the equality $f'(z_0) = 0$ we get

$$(2) \qquad a - z_0 = \frac{1 + z_0^{n-1}}{(n-1) z_0^{n-2}}.$$

Farther, taking into account real and imaginary parts of $f'(z_0) = 0$, if $\sin\beta \neq 0$ we obtain by easy calculations

$$\delta = \frac{n-1}{n} a \frac{\sin(n-2)\beta}{\sin(n-1)\beta} \tag{3}$$

and

$$\delta^{n-1} = \frac{\sin(n-2)\beta}{n \sin\beta}. \tag{4}$$

To show that $z_0 \notin \mathcal{K}(a,n)$ by (1) it is sufficient to find k, $1 \leq k \leq n$, such that

$$|e^{\frac{2k\pi i}{n}}(a - z_0) + z_0| > 1. \tag{5}$$

We start from the case of real critical points. For odd n the polynomial f does not have ones at all, so let n be an even number. In this case f' has the only negative root. Substituting $z_0 = -\delta$ and $k = \frac{n}{2}$ to the left hand side of (5) and using (2) we obtain

$$\left| e^{\pi i} \frac{1-\delta^{n-1}}{(n-1)\delta^{n-2}} - \delta \right| > 1 \iff \frac{1-\delta^{n-1}}{(n-1)\delta^{n-2}} + \delta > 1 \iff$$
$$\iff \frac{1-\delta^{n-1}}{(n-1)\delta^{n-2}} > 1 - \delta \iff \frac{1+\delta+\ldots+\delta^{n-2}}{(n-1)\delta^{n-2}} > 1.$$

Hence z_0 does not belong to $\mathcal{K}(a,n)$.

Now we pass to the case of non-real z_0, so we have $\sin\beta \neq 0$ where $\beta = \arg z_0$. For such critical points the following inequality holds

$$\delta = |z_0| < \frac{\alpha n - 1}{\alpha n}, \quad \text{where } \alpha = \left[\ln\frac{\pi}{2}\right]^{-1} \approx 2.2144337. \tag{6}$$

Indeed, if $|\beta| \leq \frac{\pi}{2n-3}$ or $|\pi - \beta| \leq \frac{\pi}{2n-3}$, then $|\sin(n-1)\beta| \geq |\sin(n-2)\beta|$, and using (3) we get

$$\delta \leq \frac{n-1}{n} < \frac{\alpha n - 1}{\alpha n}.$$

If β does not satisfy the above inequalities, then $|\sin\beta| > \sin\frac{\pi}{2n-3}$ and taking into account (4) we obtain

$$\delta = \sqrt[n-1]{\frac{\sin(n-2)\beta}{n\sin\beta}} \leq \sqrt[n-1]{\frac{1}{n|\sin\beta|}} < \sqrt[n-1]{\frac{1}{n\sin\frac{\pi}{2n-3}}} \leq \frac{\alpha n - 1}{\alpha n}.$$

The last inequality holds because it is equivalent to the following one

$$\left(\frac{\alpha n - 1}{\alpha n}\right)^{1-n} \leq n \sin\frac{\pi}{2n-3},$$

which in turn is true since the sequence

$$\left(\frac{\alpha n-1}{\alpha n}\right)^{1-n}=\left[\left(1-\frac{1}{\alpha n}\right)^{-\alpha n}\right]^{\frac{n-1}{\alpha n}}$$

is increasing and tends to $e^{1/\alpha}=\frac{\pi}{2}$, while the sequence $n\sin\frac{\pi}{2n-3}$ is decreasing and has the same limit. So we have proved (6).

Now we can back to the inequality (5). To make the left hand side of it as big as possible we choose k in such a way that $|\arg e^{\frac{2k\pi i}{n}}(a-z_0)-\arg z_0|\le\frac{\pi}{n}$. Hence

$$\left|e^{\frac{2k\pi i}{n}}(a-z_0)+z_0\right|\ge\cos\frac{\pi}{n}|a-z_0|+|z_0|=\cos\frac{\pi}{n}\left|\frac{1+z_0^{n-1}}{(n-1)z_0^{n-2}}\right|+|z_0|\ge$$
$$\ge\cos\frac{\pi}{n}\frac{1-\delta^{n-1}}{(n-1)\delta^{n-2}}+\delta.$$

So to prove that $z_0\notin\mathcal{K}(a,n)$ it is sufficient to show that for $\delta<\dfrac{\alpha n-1}{\alpha n}$, where $\alpha=\left[\ln\frac{\pi}{2}\right]^{-1}$, the following inequality holds

$$\cos\frac{\pi}{n}>\frac{(n-1)(1-\delta)\delta^{n-2}}{1-\delta^{n-1}}=\frac{(n-1)\delta^{n-2}}{1+\delta+\ldots+\delta^{n-2}}.\tag{7}$$

Denote $G(n,\delta)=\dfrac{(n-1)\delta^{n-2}}{1+\delta+\ldots+\delta^{n-2}}$. Since $G(n,\delta)$ is increasing with respect to δ, the above inequality reduces to the following one

$$\cos\frac{\pi}{n}>G\left(n,\frac{\alpha n-1}{\alpha n}\right).$$

The sequence $G\left(n,\dfrac{\alpha n-1}{\alpha n}\right)$ is decreasing and tends to $[\alpha(e^{1/\alpha}-1)]^{-1}\approx$ 0.791145. On the other hand $\cos\frac{\pi}{n}$ increases and tends to 1. Moreover $G\left(6,\dfrac{6\alpha-1}{6\alpha}\right)\approx 0.8499243$ while $\cos\frac{\pi}{6}\approx 0.8660254$, so the inequality (7) holds for $n\ge 6$.

This, together with (6), completes the proof of Proposition 1 for $n\ge 6$.

The case $n=3,4$ and 5 can be proved similarly (however it needs some technicalities) or by means of numerical calculations, so we omit the proof.

4. Corollaries

In connection with the Sendov Conjecture Goodman, Rahman and Ratti [**4**] stated the problem of finding a minimal region with the Sendov property. Saff and Twomey [**9**] (and independently Goodman [**3**]) have specified this problem in the following way.

Describe a set $S(a,n)$ such that

i) $S(a,n)$ contains at least one critical point of each $p \in \mathcal{P}(a,n)$,

ii) no proper subset of $S(a,n)$ has property (i).

This problem seems to be rather difficult. To date, the only partial solution to it was given (in the case of $n = 3$) by Saff and Twomey [9].

Now, using Proposition 1, we can prove the following result.

THEOREM 1. *For any* $n \geq 2$ *and* $0 \leq a \leq 1$ *there exists a minimal set* $S(a,n)$. *For each* a *and* $n \geq 3$ *it is not unique.*

First let us formulate without a proof the following two lemmas (for detailes we refer to [2]).

LEMMA 1. *If* $X \subseteq D$ *has the Sendov property* (i) *then there exists a minimal set* $S(a,n) \subseteq X$.

LEMMA 2. *If* $z_0 \notin \mathcal{K}(a,n)$ *then there exist* $\varepsilon > 0$ *and a minimal set* $S(a,n) \subseteq D$ *such that* $S(a,n) \cap D(z_0,\varepsilon) = \emptyset$.

Let us only mention that the first lemma is an immediate consequence of the Zorn's Lemma while the other follows from the Hurwitz Theorem [5] and some topological observations.

Now we can sketch the proof of Theorem 1.

The existence of the set $S(a,n)$ follows simply from Lemma 1 and the fact that the unit disk D has the Sendov property.

To prove the second part of the statement let us fix a, $0 < a \leq 1$, and n, $n \geq 3$. Consider the polynomial $f(z) = (z^{n-1}+1)(z-a)$. We have $f \in \mathcal{P}(a,n)$ and from Proposition 1 it follows that all its critical points $\xi_1, \dots, \xi_{n-1}$ lie outside the set $\mathcal{K}(a,n)$. If $S(a,n)$ were the unique one then, in view of Lemma 2, each of $\xi_1, \dots, \xi_{n-1}$ would not belong to $S(a,n)$ and $S(a,n)$ would not be minimal. In the case $a = 0$ the reasoning is similar. Thus, for $n \geq 3$, there must be at least two minimal sets, which ends the proof.

Finally let us mention that even if in the definition of $S(a,n)$ we take into account closed and connected sets satisfying (i) the above reasoning remains valid and Theorem 1 is true.

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