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BARYCENTRIC TRANSFORMATIONS

Abstract. We present in this note a new family of automorphisms on spaces of holomorphic functions called Barycentric Transformations. Beside the theoretical aspect of these transformations we shall use them to solve explicitly barycentric differential equations of the form

$$u^{(m)}(x) = \sum_{k=1}^{n} a_k \mathcal{M}_{p_k} u(x) + f(x)$$

where $\mathcal{M}_{p_k}$ is a barycentric transformation of order p_k.

1. Notations

We shall denote respectively by $\mathbf{C}$, $\mathbf{N}$ and $\mathbf{Z}$ the sets of complex numbers, nonnegative integers and all integers; and we set $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$, $\mathbf{N}^* = \mathbf{N}\backslash\{0\}$. For every nonnegative integer $p \geq 1$ we define the zeros $\{\omega_j\}_{j=0}^{p-1}$ of the algebraic equation $z^p + 1 = 0$ as follows

$$\omega_j = \exp\left\{\frac{i\pi}{p}(1+2j)\right\}, \quad (j = 0, 1, \ldots, p-1). \tag{1}$$

Finally we shall denote by $E_p(k)$ the real number $(-1)^{\left[\frac{k}{p}\right]}$ where $[x]$ is the largest integer $\leq x$.

2. Main results

We first introduce the notions of Barycentric Transformations in a circular domain:

DEFINITION 1: A subset A of $\mathbf{C}$ is said to be *circular* if $\lambda A = A$ for any complex number λ satisfying $|\lambda| = 1$.

DEFINITION 2: A *barycentric transformation* $M_p : H(D) \to H(D)$ of order $p \geq 1$ is defined by

$$M_p f(z) = \frac{2}{p} \sum_{j=0}^{p-1} \frac{\omega_j}{\omega_j - 1} \cdot f(\omega_j z), \qquad (\forall f \in H(D),\ \forall z \in D), \tag{2}$$

where $H(D)$ is the vector space of all analytic functions defined in a *circular* domain $D \subset \mathbf{C}$. If $p = 0$, then M_0 is the identity transformation which will be denoted later on by $\mathcal{I}d$.

We point out that for every $z \in D$ the point $M_p f(z)$ represents the barycenter of the p points $f(\omega_j z)$, $j = 0, \dots, p-1$, of the complex plane $\mathbf{C}$. As we recall that if $\{\omega_j\}_{j=0}^{p-1}$ are the p zeros of $z^p + 1 = 0$ then they satisfy the following Viete's identities

$$\sum_{j=0}^{p-1} \frac{\omega_j^{m+1}}{\omega_j - 1} = \frac{p}{2} E_p(m), \qquad (\forall m \in \mathbf{Z},\ \forall p \geq 1). \tag{3}$$

If D is a *circular* domain of $\mathbf{C}$, we denote by $\mathcal{M}(D)$ the set of all finite compositions of barycentric transformations, that is

$$\mathcal{M}(D) = \{T : H(D) \to H(D),\ \exists \{i_1, \dots, i_n\} \subset \mathbf{N},\ (n \in \mathbf{N}) : T = M_{i_1} \circ M_{i_2} \circ \dots \circ M_{i_n}\}.$$

Example 1: If $f(z) = z^m$, $m \in \mathbf{N}$, then, for every $z \in \mathbf{C}$ and $p \in \mathbf{N}^*$ we have

$$M_p f(z) = E_p(m) z^m.$$

Example 2: If $f(z) = \dfrac{1}{z-1}$ and $D = \{z \in \mathbf{C} : |z| \neq 1\}$, then

$$M_p f(z) = \frac{z^p - 1}{(z^p + 1)(z - 1)}, \quad \text{for all } z \in D \text{ and } p \geq 1.$$

Theorem 1. *Let A be the annulus $A(r_1, r_2) = \{z \in \mathbf{C} : r_1 < |z| < r_2\}$. If $f : A \to \mathbf{C}$ is given by*

$$f(z) = \sum_{n=-\infty}^{n=+\infty} a_n z^n$$

for every $z \in A$, then,

$$M_p f(z) = \sum_{n=-\infty}^{n=+\infty} a_n E_p(n) z^n, \quad \textit{for every } z \in A \textit{ and } p \geq 1. \tag{4}$$

Proof. Using the Viete's identities (3) one obtains at once

$$M_p f(z) = \frac{2}{p}\sum_{j=0}^{p-1} \frac{\omega_j}{\omega_j - 1} \cdot f(\omega_j z) = \sum_{n=-\infty}^{n=+\infty} a_n \left\{ \frac{2}{p} \sum_{j=0}^{p-1} \frac{\omega_j^{n+1}}{\omega_j - 1} \right\} z^n$$
$$= \sum_{n=-\infty}^{n=+\infty} a_n E_p(n) z^n$$

which establishes the Theorem. ■

REMARKS: 1. Theorem 1 is extremely useful chiefly for explicit evaluation of sums of power series whose coefficients alternate periodically in signs.

2. r_2 may be eventually infinite.

COROLLARY 2. *If $f \in H(A)$, then the following identity*

$$\frac{4}{p^2} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \frac{\omega_i \omega_j}{(\omega_i - 1)(\omega_j - 1)} f(\omega_i \omega_j z) = f(z) \tag{5}$$

holds for every $z \in A$ and $p \in \mathbf{N}^*$.

Proof. Taking into account the result of the foregoing theorem we have $M_p \circ M_p f(z) = f(z)$, $\forall p \geq 1$, $\forall z \in A$, which gives the desired identity (5). ■

Here are some useful properties satisfied by barycentric transformations of $\mathcal{M}(D)$:

(i) If $T \in \mathcal{M}(D)$ then $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, $\forall(\alpha, \beta) \in \mathbf{C}^2$, $\forall (f, g) \in H(D) \times H(D)$,

(ii) $(R \circ S) \circ T = R \circ (S \circ T)$, $\forall R, S, T \in \mathcal{M}(D)$,

(iii) $S \circ T = T \circ S$, $\forall S, T \in \mathcal{M}(D)$,

(iv) $S \circ S = \mathcal{I}d$, $\forall S \in \mathcal{M}(D)$.

While properties (i) and (ii) are straightforward, (iii) follows at once from the fact that $M_p \circ M_q = M_q \circ M_p$, $\forall p, q \geq 0$. The last one (*iv*) follows essentially from Corollary 2.

We deduce from the preceding properties that $(\mathcal{M}(D), \circ)$ is an abelian group with neutral element $\mathcal{I}d$ and each element of it is an automorphism of $H(D)$ onto $H(D)$.

The next theorem establishes the equivalence between a "*barycentric equation*" with m nonzero coefficients and some linear system of 2^{m-1} equations in 2^{m-1} unknown variables.

THEOREM 3. *Let $a_1, a_2, \ldots, a_m \in \mathbf{C}$ and let $T_1, T_2, \ldots, T_m \in \mathcal{M}(D)$, ($T_i \neq T_j$ if $i \neq j$), D being a circular domain of $\mathbf{C}$. If $f \in H(D)$, then the sole barycentric equation of one unknown variable φ:*

$$\sum_{i=1}^{m} a_i T_i \varphi = f \tag{6}$$

is equivalent to a linear system of 2^{m-1} equations in 2^{m-1} unknown variables.

Proof. Denote by $\mathbf{T}$ the set$\{T_1, T_2, \ldots, T_m\}$. Suppose first that $\mathcal{I}d \in \mathbf{T}$, then $\operatorname{Card} \mathcal{C}(\mathbf{T}) = 2^{m-1}$ where $\mathcal{C}(\mathbf{T})$ is the set of all finite compositions of elements of $\mathbf{T}$. Applying the 2^{m-1} elements $\{S_j\}_{j=1}^{2^{m-1}}$ of $\mathcal{C}(\mathbf{T})$ to the equation (6) we obtain a linear system of 2^{m-1} equations in the unknown variables $S_1\varphi, S_2\varphi, \ldots, S_{2^{m-1}}\varphi$. If now $\mathcal{I}d \notin \mathbf{T}$, we denote by $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{2^{m-1}}\}$ (resp. $\{O_1, O_2, \ldots, O_{2^{m-1}}\}$) the set of all compositions of even number (resp. odd) of elements of $\mathbf{T}$. Applying all the $\mathcal{E}_j$ to (6) we obtain

$$\sum_{i=1}^{m} a_i \mathcal{E}_j T_i \varphi = \mathcal{E}_j f, \qquad (j = 1, \ldots, 2^{m-1}). \tag{7}$$

It is worth to observe that for each $i = 1, \ldots, m$ we have

$$\{\mathcal{E}_j T_i, \quad j = 1, \ldots, 2^{m-1}\} \equiv \{O_k, \quad k = 1, \ldots, 2^{m-1}\}$$

and accordingly, the system (7) is once again a linear system of 2^{m-1} equations in the unknown variables $O_1\varphi, \ldots, O_{2^{m-1}}\varphi$.

Finally, if the determinant of the obtained system is nonzero then the latter possesses a unique solution, say $\varphi_1, \varphi_2, \ldots, \varphi_{2^{m-1}}$. Therefore, $S_k\varphi = \varphi_k$, $\forall k = 1, \ldots, 2^{m-1}$ from which we get the solution of (6): $\varphi = S_k \varphi_k$, $\forall k = 1, \ldots, 2^{m-1}$. ∎

EXAMPLE 3: If a, b and $c \in \mathbf{C}$ (or $\mathbf{R}$) and $f \in H(D)$, then the barycentric equation in φ:

$$aT_1\varphi(z) + bT_2\varphi(z) + cT_3\varphi(z) = f(z), \qquad (\forall z \in D),$$

(where T_1, T_2 and T_3 are in $\mathcal{M}(D)$ with $T_i \neq \mathcal{I}d$, for $i = 1, 2, 3$) has the explicit solution

$$\varphi = \frac{a^3 - ab^2 - ac^2}{\Delta} T_1 f + \frac{b^3 - ba^2 - bc^2}{\Delta} T_2 f + \frac{c^3 - ca^2 - cb^2}{\Delta} T_3 f + \frac{2abc}{\Delta} T_1 T_2 T_3 f$$

where $\Delta = a^4 + b^4 + c^4 - 2(ab)^2 - 2(bc)^2 - 2(ac)^2$ is supposed to be $\neq 0$.

Now using the notion of barycentric transformations we get the following results dealing with the average properties satisfied by analytic functions.

THEOREM 4. *If $f \in H(\mathbf{C})$, then*

$$\frac{1}{p}\sum_{k=0}^{p-1}\omega_k z\ f(\omega_k z+\zeta)=\frac{1}{2}M_p^z(zf(z+\zeta))-\frac{z}{2}M_p^z f(z+\zeta),$$

$$(\forall z,\zeta\in\mathbf{C}),\quad(\forall p\geq 1),$$

where M_p^z denotes the barycentric transformation with respect to the variable z.

Proof. It suffices to use formula (2) taken with respect to z. ■

The following theorem is a generalization of Lemma 1 ([3]) extended to entire functions rather than polynomials only.

THEOREM 5. *Let $G(z)$ be an entire function, then for every integer $p \geq 1$, we have*

$$\frac{1}{p}\sum_{j=0}^{p-1}G(\omega_j z+\zeta)=\sum_{k=0}^{+\infty}\left\{\frac{E_p(k)-E_p(k-1)}{2}\right\}\frac{G^{(k)}(\zeta)}{k!}z^k$$

for every z and ζ in $\mathbf{C}$.

Proof. For every fixed ζ we expand the function $z \longmapsto G(z+\zeta)$ in a power series about the origin $z=0$; to obtain

$$G(z+\zeta)=\sum_{k=0}^{+\infty}\frac{G^{(k)}(\zeta)}{k!}z^k,$$

then we perform the barycentric transformation of order p with respect to the variable z to get expression (8). ■

We point out that if $G(z)$ is a polynomial of degree m then identity (8) reduces to the following one

$$\frac{1}{p}\sum_{j=0}^{p-1}G(\omega_j z+\zeta)=\sum_{k=0}^{m}\left\{\frac{E_p(k)-E_p(k-1)}{2}\right\}\frac{G^{(k)}(\zeta)}{k!}z^k$$

which is Lemma 1 ([3]) taking into account the definition of the set $\{\omega_j\}_{j=0}^{p-1}$ in both versions.

COROLLARY 6. *Let $G(z)$ be an entire function; if there is a couple of integers $(m,p)\in\mathbf{N}\times\mathbf{N}^*$ such that*

$$\frac{1}{p}\sum_{j=0}^{p-1}G(\omega_j z+\zeta)=\sum_{k=0}^{m}\frac{E_p(k)-E_p(k-1)}{2}\frac{G^{(k)}(\zeta)}{k!}z^k,\qquad(\forall z,\zeta\in\mathbf{C}),$$

then G must be a polynomial of degree not exceeding n_0-1 where n_0 is the smallest integer $\geq m+1$ satisfying $E_p(n_0)\neq E_p(n_0-1)$.

Proof. We deduce from Theorem 5 that

$$\sum_{k=m+1}^{\infty} \frac{E_p(k)-E_p(k-1)}{2} \frac{G^{(k)}(\zeta)}{k!} z^k = 0, \qquad (\forall z, \zeta \in \mathbf{C})$$

from which we get $(E_p(k)-E_p(k-1))G^{(k)}(\zeta) = 0$ for all $k \geq m+1$ and all $\zeta \in \mathbf{C}$. Now if we denote by n_0 the smallest integer $\geq m+1$ for which $E_p(n_0) - E_p(n_0-1) \neq 0$ then $G^{(n_0)}(\zeta) = 0, \forall \zeta \in \mathbf{C}$. Since $G(z)$ is an entire function it then reduces to the polynomial

$$G(z) = \sum_{k=0}^{m_0} \frac{G^{(k)}(\zeta)}{k!}(z-\zeta)^k,$$

where $m_0 \leq n_0 - 1$. ∎

THEOREM 7. *A nonzero entire function $f : \mathbf{C} \to \mathbf{C}$ having for zeros $\{a_1, \ldots$ $\ldots, a_m\} \subset \mathbf{C}$ satisfies the relation*

$$\prod_{j=0}^{p-1} f(\omega_j z + \zeta) = (f(\zeta))^p \prod_{k=1}^{m} \left(1 + \left(\frac{z}{a_k - \zeta}\right)^p\right)$$

for every $z \in \mathbf{C}$ and $\zeta \in \mathbf{C} \backslash \{a_1, \ldots, a_m\}$ and for some integer $p \geq 1$ if and only if there is a polynomial $Q(z)$ of degree not exceeding $p-1$ such that

$$f(z) = (z-a_1)\ldots(z-a_m)\exp Q(z), \qquad (\forall z \in \mathbf{C}).$$

Proof. We apply Corollary 6 taking into account some average properties satisfied by entire functions. ∎

Another application of barycentric transformations is the discrete form of the Principle of the Argument in a circular domain $D \subset \mathbf{C}$:

THEOREM 8. *Let $f : D \to \mathbf{C}$ be a meromorphic function having a finite number of zeros $a_1, \ldots, a_{n^+}$ and poles $b_1, \ldots, b_{n^-}$ counted with their multiplicities. There is an entire function $g(z)$ for which the following formula*

$$(9) \qquad \frac{1}{p}\sum_{j=0}^{p-1} \omega_j z \cdot \frac{f'(\omega_j z)}{f(\omega_j z)} = n^+ - n^- - \sum_{k=1}^{n^+} \frac{a_k^p}{z^p + a_k^p} + \sum_{k=1}^{n^-} \frac{b_k^p}{z^p + b_k^p} + \frac{1}{2}M_p(zg(z)) - \frac{z}{2}M_p(g(z))$$

holds for every $p \in \mathbf{N}^*$ and $z \in D$ such that

$$|z| \notin S = \{|a_1|, \ldots, |a_{n^+}|, |b_1|, \ldots, |b_{n^-}|\}.$$

Proof. It follows from the hypotheses that $f(z)$ is representable into the form:

$$f(z) = \frac{(z-a_1)\cdots(z-a_{n^+})}{(z-b_1)\cdots(z-b_{n^-})}\exp G(z),$$

where $G(z)$ is an entire function. Thus,

$$\omega_j z \cdot \frac{f'(\omega_j z)}{f(\omega_j z)} = \sum_{k=1}^{n^+} \frac{\omega_j z}{\omega_j z - a_k} - \sum_{k=1}^{n^-} \frac{\omega_j z}{\omega_j z - b_k} + \omega_j z \cdot G'(\omega_j z),$$
$$\forall j = 0, \dots, p-1; \quad \forall z \in D : |z| \notin S.$$

Summing over the j's we obtain

$$\frac{1}{p}\sum_{j=0}^{p-1} \omega_j z \cdot \frac{f'(\omega_j z)}{f(\omega_j z)} = \sum_{k=1}^{n^+}\left\{\frac{1}{p}\sum_{j=0}^{p-1}\left(\frac{\omega_j z}{\omega_j z - a_k}\right)\right\}$$
$$-\sum_{k=1}^{n^-}\left\{\frac{1}{p}\sum_{j=0}^{p-1}\left(\frac{\omega_j z}{\omega_j z - b_k}\right)\right\} + \frac{1}{p}\sum_{j=0}^{p-1}\omega_j z \cdot g(\omega_j z)$$

where we have set $g(z) = G'(z)$. Finally, using the following identities

$$\frac{1}{p}\sum_{j=0}^{p-1} \frac{x}{\omega_j z - x} = -\frac{x^p}{x^p + z^p}, \qquad (\forall x, z \in \mathbf{C} : |x| \neq |z|)$$

and

$$\frac{1}{p}\sum_{j=0}^{p-1} \omega_j z \cdot g(\omega_j z) = \frac{1}{2}M_p(zg(z)) - \frac{z}{2}M_p(g(z)), \qquad (\forall z \in \mathbf{C})$$

we get the desired formula. ∎

The next result shows the interaction between the famous Laplace Transform and Barycentric Transformations of order $p \geq 1$. It may be applied to solve explicitly functional differential equations of the form

$$u^{(m)}(x) = \sum_{k=1}^{n} a_k \mathcal{M}_{p_k} u(x) + f(x)$$

after transforming it into an appropriate barycentric equation.

THEOREM 9. *Let $f(z)$ be a Laplace-transformable entire function; then there exists a positive real number $R > 0$ such that, for every $p \in \mathbf{N}^*$, we have*

$$(10) \qquad \mathcal{L}[M_p f](z) = -M_p(\mathcal{L}[f])(z), \qquad \text{for } \mathcal{R}e\, z > R.$$

(We recall that the Laplace transform of f is given by

$$\mathcal{L}[f](z) = \int_0^{+\infty} \exp(-zt) f(t)\, dt).$$

Proof. Since $f(z)$ is entire there is a sequence $\{a_n\}_{n=0}^{+\infty} \subset \mathbf{C}$ such that $f(t) = \sum_{n=0}^{+\infty} a_n t^n$. We associate to $f(z)$ the function

$$F(z) = \sum_{n=0}^{+\infty} \frac{n! a_n}{z^{n+1}}, \qquad \text{for } |z| > R \tag{11}$$

where R is the radius of the disk $D(o, R)$containing the poles of the function $\mathcal{L}[f](z)$ in $\mathbf{C}$. It follows from the Cauchy estimates of the coefficients of (11) that $|a_n| \leq \frac{MR^n}{n!}, \forall n \in \mathbf{N}^*$; where $M = \sup_{|z|>R} |F(z)|$. Hence

$$|f(t)| \leq \sum_{n=0}^{\infty} |a_n|\, |t|^n \leq \sum_{n=0}^{\infty} \frac{M(R|t|)^n}{n!} = M \exp(R|t|), \qquad \forall t \in \mathbf{C};$$

therefore

$$|f(t)| \leq M \exp(R|t|), \qquad \forall t \in \mathbf{C}.$$

We conclude by Theorem **1** that

$$\begin{aligned} M_p F(z) &= \sum_{n=0}^{\infty} a_n E_p(-n-1) \frac{n!}{z^{n+1}} \\ &= -\sum_{n=0}^{\infty} a_n E_p(n) \frac{n!}{z^{n+1}}, \qquad \text{for } |z| > R. \end{aligned}$$

In particular, for $\mathcal{R}e\, z > R$, we obtain

$$\begin{aligned} M_p F(z) &= -\sum_{n=0}^{\infty} a_n E_p(n) \frac{n!}{z^{n+1}} = -\sum_{n=0}^{\infty} a_n E_p(n) \mathcal{L}[t^n](z) \\ &= -\mathcal{L}\Big[\sum_{n=0}^{\infty} a_n E_p(n) t^n\Big] = -\mathcal{L}[M_p f](z). \end{aligned}$$

We observe on the other hand that

$$F(z) = \sum_{n=0}^{\infty} a_n \mathcal{L}\Big[t^n\Big](z) = \mathcal{L}\Big[\sum_{n=0}^{\infty} a_n t^n\Big] = \mathcal{L}[f](z), \qquad \text{for } \mathcal{R}e\, z > R.$$

Consequently, $\mathcal{L}[M_p f](z) = -M_p(\mathcal{L}[f])(z)$, for $\mathcal{R}e\, z > R$. ∎

APPLICATIONS:

1) Let us solve the following barycentric differential problem:

$$\begin{cases} \phi''(x) = M_3\phi(x), \qquad x > 0 \\ \phi(0) = a_0 \\ \phi'(0) = a_1 \\ a_0 \text{ and } a_1 \text{ being real constants.} \end{cases}$$

Performing the Laplace transform to the differential equation and using Theorem 9, we get the following barycentric equation

$$\mathcal{L}[\phi''](z) = z^2\mathcal{L}[\phi](z) - z\phi(0) - \phi'(0) = -M_3\mathcal{L}[\phi](z)$$

so that, setting $\psi(z) = \mathcal{L}[\phi](z)$ we get

$$z^2\psi(z) + M_3\psi(z) = a_0 z + a_1.$$

A straightforward computation shows that $\psi(z)$ yields

$$\psi(z) = \frac{a_0}{3}\left\{\frac{z}{z^2+1} + \frac{2z+2z^3}{z^4-z^2+1}\right\} + \frac{a_1}{6}\left\{\frac{1}{z-1} - \frac{1}{z+1} + \frac{4-2z}{z^2-z+1} + \frac{4+2z}{z^2+z+1}\right\}.$$

Therefore,

$$\phi(x) = a_0\left\{\frac{1}{3}\cos x + \frac{2}{3}\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}x + \frac{2}{\sqrt{3}}\sin\frac{x}{2}\sinh\frac{\sqrt{3}}{2}x\right\} + a_1\left\{\frac{1}{3}\sinh x - \frac{2}{3}\sinh\frac{x}{2}\cos\frac{\sqrt{3}}{2}x + \frac{2}{\sqrt{3}}\cosh\frac{x}{2}\sin\frac{\sqrt{3}}{2}x\right\}.$$

2) Let p and q be positive integers such that $p \geq 2$ and $q \in [p, 2p-1[$ then the initial value problem

$$\begin{cases} \phi'(x) = \dfrac{\phi(x) + M_p\phi(x)}{2} + \lambda x^q, \quad x > 0, \\ \phi(0) = a_0, \end{cases}$$

where a_0 and λ are arbitrary real constants, has for unique solution the polynomial function

$$\phi(x) = a_0\sum_{n=0}^{p}\frac{1}{n!}x^n + \frac{\lambda}{q+1}x^{q+1}.$$

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