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AN EXTENSION OF A THEOREM OF FLETT

Abstract. In the present paper we deal with Flett's Mean Value Theorem for n times differentiable functions. We prove Flett's Formula on a base of the Taylor Theorem.

1. Introduction

In [1] T.M.Flett gave a generalization of the Lagrange Mean Value Theorem as follows:

THEOREM 1.1. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f'(a) = f'(b)$. Then there exists a point $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = (\eta - a)f'(\eta).$$

In 1996 T. Riedel and P. K. Sahoo [6] removed the boundary assumption on the derivative, that is $f'(a) = f'(b)$. Their result reads as follows

THEOREM 1.2. *Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Then there exists a point $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = (\eta - a)f'(\eta) - \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}(\eta - a)^2.$$

There exist many other extensions of the Flett Theorem. Let us mention D.H. Trahan [7], who replaced equality of derivatives by some inequalities. Also the theorem was adapted for symmetrically differentiable functions by T.V.Lakshminarasimhan [2]. Then S.G.Wayment [7] worked out a generalization of an integral mean value theorem as an analog of Flett's Theorem of differential calculus. Another extension was developped by T.Riedel, R.C.Davitt, R.C.Powers and P.K.Sahoo [5] for holomorphic functions. More details can be found in T.Riedel and P.K.Sahoo's book [6]. All of these generalizations concern once differentiable functions.

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Since the original Flett's result reminds the Lagrange Mean Value Theorem we were motivated to find an extension that would be related to Theorems 1.2 and 1.1 similarly as the Taylor Theorem is related to the Lagrange MVT. Let us add that Zs.Páles asked for a possibility of such an extension during the 35th International Symposium on Functional Equations held in Graz, Austria in 1997 (oral communication). An answer to his question is contained in the following.

2. Generalization of Flett's Theorem

Let us prove two lemmas.

In the sequel by 0 times differentiable function we mean continuous function, $f^{(0)} := f$, and we adopt the convention $\sum_{k=1}^{0} a_k = 0$. Let us prove a technical lemma:

LEMMA 2.1. *Let n be a nonnegative integer, assume that $f : [a,b] \to \mathbb{R}$ is n times differentiable and define $g : (a,b] \to \mathbb{R}$ by*

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

Then g is n times differentiable, for all $x \in (a,b]$ we have

$$g^{(n)}(x) = (-1)^n n! \frac{f(x) - f(a) + \sum_{k=1}^{n} (-1)^k \frac{1}{k!} f^{(k)}(x)(x-a)^k}{(x-a)^{n+1}}, \tag{1}$$

$$g^{(n)}(x) = \frac{f^{(n)}(x)}{x-a} - n\frac{g^{(n-1)}(x)}{x-a}, \quad x \in (a,b], \tag{2}$$

and, if moreover $f^{(n+1)}(a)$ exists, we have

$$\lim_{x \to a} g^{(n)}(x) = \frac{1}{n+1} f^{(n+1)}(a). \tag{3}$$

Proof. Observe that (2) immediately follows from (1). To prove (1) we proceed by induction. For $n = 0$ equality (1) is just the definition of g. Before we continue the induction argument let us define $\varphi_n : (a,b] \to \mathbb{R}$ by

$$\varphi_n(x) = f(x) - f(a) + \sum_{k=1}^{n} (-1)^k \frac{1}{k!} f^{(k)}(x)(x-a)^k.$$

Let us note that if f is $(n+1)$ times differentiable then φ_n is differentiable and a simple calculation shows that

$$\varphi_n'(x) = (-1)^n \frac{1}{n!} f^{(n+1)}(x)(x-a)^n, \ x \in (a,b]. \tag{4}$$

Now, assume that (1) holds for some $n \geq 0$ and suppose that f is $(n+1)$ times differentiable. Then g is $(n+1)$ times differentiable and we have for $x \in (a, b]$, using induction hypothesis and (4),

$$\begin{aligned}
g^{(n+1)}(x) &= (g^{(n)})'(x) \\
&= (-1)^n n! \frac{1}{(x-a)^{2n+2}} [(x-a)^{n+1} \varphi_n'(x) - (n+1)(x-a)^n \varphi_n(x)] \\
&= (-1)^n n! \frac{1}{(x-a)^{n+2}} \left[(-1)^n \frac{1}{n!} f^{(n+1)}(x)(x-a)^{n+1} - (n+1)\varphi_n(x) \right] \\
&= (-1)^{n+1}(n+1)! \frac{1}{(x-a)^{n+2}} \\
&\quad \times \left[\varphi_n(x) + (-1)^{n+1} \frac{1}{(n+1)!} f^{(n+1)}(x)(x-a)^{n+1} \right] \\
&= (-1)^{n+1}(n+1)! \frac{\varphi_{n+1}(x)}{(x-a)^{n+2}},
\end{aligned}$$

which ends the proof of (1) in view of the definition of φ_n.

To prove (3), observe that for $n = 0$ this follows from the existence of $f'(a)$. Suppose that (3) holds for some $n \geq 0$ and assume that f is $(n+1)$ times differentiable while $f^{(n+2)}(a)$ exists. Using (2) we have

$$\begin{aligned}
g^{(n+1)}(x) &= \frac{f^{(n+1)}(x)}{x-a} - (n+1) \frac{g^{(n)}(x)}{x-a} \\
&= \frac{f^{(n+1)}(x) - f^{(n+1)}(a)}{x-a} - (n+1) \frac{g^{(n)}(x) - \frac{1}{n+1} f^{(n+1)}(a)}{x-a} \\
&= \frac{f^{(n+1)}(x) - f^{(n+1)}(a)}{x-a} \\
&\quad - \frac{(-1)^n (n+1)! \varphi_n(x) - f^{(n+1)}(a)(x-a)^{n+1}}{(x-a)^{n+2}}.
\end{aligned}$$

This first quotient on the right-hand side tends to $f^{(n+2)}(a)$ as $x \to a$, and applying to the second one de l'Hôpital rule we get in view of (4), that right-hand

$$\begin{aligned}
&\lim_{x \to a} \frac{(-1)^n (n+1)! \varphi_n(x) - f^{(n+1)}(a)(x-a)^{n+1}}{(x-a)^{n+2}} \\
&\qquad = \lim_{x \to a} \frac{(-1)^n (n+1)! \varphi_n'(x) - (n+1) f^{(n+1)}(a)(x-a)^n}{(n+2)(x-a)^{n+1}} \\
&\qquad = \lim_{x \to a} \frac{n+1}{n+2} \frac{f^{(n+1)}(x) - f^{(n+1)}(a)}{x-a} = \frac{n+1}{n+2} f^{(n+2)}(a).
\end{aligned}$$

Thus we get

$$\lim_{x\to a} g^{(n+1)}(x) = \frac{1}{n+2} f^{(n+2)}(a)$$

which ends the proof.

Suppose that $n \in \mathbb{N} \cup \{0\}$ and let $f : [a,b] \to \mathbb{R}$ be an n times differentiable function such that $f^{(n+1)}(a)$ exists. In view of Lemma 2.1 we see that $g_n : [a,b] \to \mathbb{R}$ given by

$$g_n(x) = \begin{cases} g^{(n)}(x), & \text{if } x \in (a,b], \\ \frac{1}{n+1} f^{(n+1)}(a), & \text{if } x = a, \end{cases} \tag{5}$$

where $g(x) = \frac{f(x)-f(a)}{x-a}$, is well defined and continuous at a.

Let us prove now an extension of the Flett Theorem for functions of higher regularity.

LEMMA 2.2. *Let $n \in \mathbb{N}$ and assume that $f : [a,b] \to \mathbb{R}$ is n times differentiable with $f^{(n)}(a) = f^{(n)}(b)$. Then there exists a point $\eta \in (a,b)$ such that*

$$f(\eta) - f(a) = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!} f^{(k)}(\eta)(\eta - a)^k. \tag{6}$$

Proof. Consider the function g_{n-1} (cf.(5)) and observe that in view of (1) the equality (6) is equivalent to the fact that

$$g'_{n-1}(\eta) = g^{(n)}(\eta) = 0.$$

Therefore it remains to show that $g^{(n)}$ vanishes on at least one point from (a,b). Suppose the contrary that $g^{(n)}(x) \neq 0$ for every $x \in (a,b)$. Since $g^{(n)}$ has Darboux property as the derivative of $g^{(n-1)}$ we can assume without loss of generality that

$$g^{(n)}(x) > 0 \quad \text{for every } x \in (a,b). \tag{7}$$

Then g_{n-1} as a continuous function has to be strictly increasing and in particular, taking into account (2) we get

$$\frac{1}{n} f^{(n)}(a) = g_{n-1}(a) < g_{n-1}(b) = \frac{1}{n} f^{(n)}(b) - \frac{1}{n}(b-a) g^n(b).$$

But, in view of our assumptions it follows that

$$g^{(n)}(b) < 0.$$

This however yields a contradiction in view of (7) and Darboux property of $g^{(n)}$, and the proof is completed.

Now we present the forementioned generalization of Theorem 1.2 for n times differentiable functions.

THEOREM 2.3. *Let $f : [a,b] \to \mathbb{R}$ be an n times differentiable function. Then there exists a point $\eta \in (a,b)$ such that*

$$(8) \qquad f(\eta) - f(a) = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}f^{(k)}(\eta)(\eta-a)^k + (-1)^n\frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(\eta-a)^{n+1}.$$

Proof. Let us define a function $h : [a,b] \to \mathbb{R}$ by

$$(9) \qquad h(x) = f(x) - \frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(x-a)^{n+1}.$$

Our assumptions on f imply that h is n times differentiable in $[a,b]$. Now, differentiating h n times we obtain

$$(10) \qquad h^{(k)}(x) = f^{(k)}(x) - \frac{1}{(n-k+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(x-a)^{n-k+1}$$

for every $k = 1,\dots,n$. In particular we have $h^{(n)}(a) = h^{(n)}(b) = f^{(n)}(a)$ and applying Lemma 2.2 to the function h we get

$$(11) \qquad h(\eta) - h(a) = \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k!}h^{(k)}(\eta)(\eta-a)^k.$$

It is a simple matter to obtain (8) by putting (9) and (10) in (11), which completes the proof.

Let us note that writing $-(a-\eta)$ instead of $(\eta-a)$ in (8) and computing $f(a)$ we get the following formula resembling Taylor's formula for n times differentiable functions

$$(12) \qquad f(a) = \sum_{k=0}^{n}\frac{f^{(k)}(\eta)}{k!}(a-\eta)^k + \frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(a-\eta)^{n+1}.$$

Since $\eta \in (a,b)$ we can write $\eta = (1-t)a+tb$ for some $t \in (0,1)$ and therefore Theorem 2.3 can be rephrased in the following way (cf.(12)):

THEOREM 2.3′. *Let $f : [a,b] \to \mathbb{R}$ be an n times differentiable function. Then there exists a $t \in (0,1)$ such that*

(13)

$$f(a) = \sum_{k=0}^{n}\frac{t^k f^{(k)}(a+t(b-a))}{k!}(a-b)^k + \frac{t^{n+1}}{(n+1)!}(f^{(n)}(a)-f^{(n)}(b))(a-b)^n.$$

The above result can be easily extended to the case of real valued functions defined in normed spaces. Namely we have

THEOREM 2.4. *Let D be a nonempty and open subset of a normed space X and let $f : D \to \mathbb{R}$ be an n times differentiable function. If $\boldsymbol{a}, \boldsymbol{b} \in D$, $\boldsymbol{a} \neq \boldsymbol{b}$ and the segment*

$$\overline{\boldsymbol{ab}} = \{t\boldsymbol{a} + (1-t)\boldsymbol{b} : t \in [0,1]\} \subset D$$

then there exists a $t \in (0,1)$ such that

$$f(a) = \sum_{k=0}^{n} \frac{t^k}{k!} d^k_{a+t(b-a)} f((\boldsymbol{a}-\boldsymbol{b})^k) + \frac{t^{n+1}}{(n+1)!}[d^n_{\boldsymbol{a}} f - d^n_{\boldsymbol{b}} f]((\boldsymbol{a}-\boldsymbol{b})^n). \tag{14}$$

In the formula (14) we denote the value of k-th order differential of f at the point x taken at the k-tuple $y^k = (y, \dots, y) \in X^k$.

Proof. It is an corollary from the fact that $F : [0,1] \to \mathbb{R}$ defined by

$$F(t) = f(\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}))$$

satisfies the assumptions of Theorem 2.3' in $[0,1]$, and

$$F^k(t) = d^k_{a+t(b-a)} f((\boldsymbol{a}-\boldsymbol{b})^k)$$

for every $t \in [0,1]$ and $k = 1, \dots, n$.

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