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TENSOR PRODUCTS IN CONCRETE CATEGORIES

Abstract. In this paper we consider the notion of tensor multiplication in the concrete categories (by the concrete category $\mathfrak{A}$ we mean the category $\mathfrak{A}$ with fixed covariant faithful functor $U : \mathfrak{A} \to \mathbf{Ens}$). The reason of this choice is the observation of the constructions of tensor product in the categories of abelian groups, vector spaces or more generally in any variety (which are of course concrete). We modify this constructions to give the universal method of introduction the tensor multiplication in any concrete category. Moreover we are not restricted because many important categories are concrete. Our aim was the general overview on the tensor multiplication in order to apply it to objects in any category which fulfill suficient conditions. In order to do this we use the construction of tensor product via Freyd's representability theorem ([4], [1]). This allowed us to formulate the problem in the language of theory of category. The main result of this work is theorem 2 which gives the conditions sufficient to existence the tensor product in the concrete category. As an example of the nontrivial aplication of this theorem we give the proof of the existence of the tensor product in the category of compact spaces.

1. Tensor products

In all the text the symbol $\mathfrak{A}$ denotes a fixed concrete category and $\mathfrak{A}^o$ - the class of objects of $\mathfrak{A}$.

DEFINITION 1. Let $A_1, \ldots, A_n, B \in \mathfrak{A}^o$. The map $\varphi : U(A_1) \times \ldots \times U(A_n) \to U(B)$ is called the *n-morphism* in the category $\mathfrak{A}$ iff for each map

$$\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} : U(A_i) \to U(A_1) \times \ldots \times U(A_n), \qquad a_i \mapsto (a_1, \ldots, a_i, \ldots, a_n)$$

there exists a morphism

$$\gamma^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} : A_i \to B$$

such that

$$\varphi\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} = U(\gamma^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}).$$

PROPOSITION 1. *Let $A_1, \ldots, A_n, B, C \in \mathfrak{A}^o$. If $\xi : B \to C$ is an embedding and $\vartheta : U(A_1) \times \ldots \times U(A_n) \to U(B)$ is such a morphism that $U(\xi)\vartheta$ is an n-morphism, then ϑ is an n-morphism.*

Proof. Let $U(\xi)\vartheta$ be an n-morphism. Than for each $\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}$ there exists $\tau^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} : A_i \to C$ such that $U(\xi)\vartheta\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} = U(\tau^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n})$. Therefore from the fact that ξ is an embedding it follows that there exists $\vartheta^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} : A_i \to B$ such that $U(\vartheta^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}) = \vartheta\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}$. ■

PROPOSITION 2. Let $A_1,\ldots,A_n,X,Y \in \mathfrak{A}^o$. If $\varphi : U(A_1) \times \ldots \times U(A_n) \to U(X)$ is an n-morphism and $\eta : X \to Y$ is a morphism, then $U(\eta)\varphi$ is an n-morphism.

DEFINITION 2. Let $A_1,\ldots,A_n \in \mathfrak{A}^o$. By the *tensor product* of this sequence of objects we call any pair (T,τ) such that $T \in \mathfrak{A}^o$, $\tau : U(A_1)\times\ldots\times U(A_n) \to U(T)$ is an n-morphism and the following condition is fullfiled: for each $X \in \mathfrak{A}^o$ and each n-morphism $\varphi : U(A_1) \times \ldots \times U(A_n) \to U(X)$ there exists exactly one morphism $\zeta : T \to X$ such that $U(\zeta)\tau = \varphi$.

Straight from the definition of the tensor product we have

THEOREM 1. *Let $A_1,\ldots,A_n \in \mathfrak{A}^o$. Assume that (T,τ) and (T',τ') are tensor products of this sequence of objects. Then there exists exactly one isomorphism $\iota : T \to T'$, such that the diagram*

$$\begin{array}{ccc} & U(A_1) \times \ldots \times U(A_n) & \\ {}^{\tau}\swarrow & & \searrow^{\tau'} \\ U(T) & \xrightarrow[U(\iota)]{} & U(T') \end{array} \qquad (1)$$

comutes. ■

DEFINITION 3. Let $A_1,\ldots,A_n \in \mathfrak{A}^o$. One can define the covariant functor $\mathbf{C}_n : \mathfrak{A} \to \mathbf{Ens}$ as follows:

For each $X \in \mathfrak{A}^o$ $\mathbf{C}_n(X)$ is the set of all n-morphisms from $U(A_1) \times \ldots \times U(A_n)$ to $U(X)$. If $\eta : X \to Y$ is a morphism of $\mathfrak{A}$ then

$$\mathbf{C}_n(\eta) : \mathbf{C}_n(X) \to \mathbf{C}_n(Y), \qquad \mathbf{C}_n(\eta).\xi = U(\eta)\xi \;\text{ for }\; \xi \in \mathbf{C}_n(X)$$

(we use the notation $\mathbf{C}_n(\eta).\xi$ for $(\mathbf{C}_n(\eta))(\xi)$). From the proposition (2) we obtain that $\mathbf{C}_n$ is well defined.

LEMMA 1. *Let $\mathfrak{A}$ be a concrete category and let U be the faithfull functor from the definition of a concrete category. If U preserves inputs, then $\mathbf{C}_n$ preserves inputs.*

Proof. Let Γ be any diagram in $\mathfrak{A}$ with the scheme $\mathfrak{D}$ and let

$$(\tau_D : P \to \Gamma(D))_{D\in\mathfrak{D}^o}$$

be the input of this diagram. We show that the family

(2) $$(\mathbf{C}_n(\tau_D) : \mathbf{C}_n(P) \to \mathbf{C}_n\Gamma(D))_{D\in\mathfrak{D}^o}$$

is the input of $\mathbf{C}_n\Gamma$ in **Ens**. Obviously (2) is the family compatible with $\mathbf{C}_n\Gamma$. Let $(\xi_D : Z \to \mathbf{C}_n\Gamma(D))_{D\in\mathfrak{D}^o}$ be another family compatible with $\mathbf{C}_n\Gamma$. Then for each $z \in Z$ the family of n-morphisms

$$(\xi_D(z) : U(A_1) \times \ldots \times U(A_n) \to U\Gamma(D))_{D\in\mathfrak{D}^o}$$

is compatible with $U\Gamma$. Functor U preserves inputs thus

$$(U(\tau_D) : U(P) \to U\Gamma(D))_{D\in\mathfrak{D}^o}$$

is the input of $U\Gamma$ in **Ens**, therefore there exists exactly one map $\vartheta_z : U(A_1)\times \ldots \times U(A_n) \to U(P)$ such that for each $D \in \mathfrak{A}^o$ $U(\tau_D)\vartheta_z = \xi_D(z)$. We show that ϑ_z is an n-morphism. Let us take any $\pi^{U(A_i)}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}$. For each $D \in \mathfrak{D}^o$ there exists $\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n} : A_i \to \Gamma(D)$ such that $U(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n}) = \xi_D(z)\pi^{U(A_i)}_{a_1,\ldots,\hat{(a)}_i,\ldots,a_n}$. Moreover, the family $(U(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n}) : U(A_i) \to U\Gamma(D))_{D\in\mathfrak{D}^o}$ is compatible with $U\Gamma$. Thus for each $\delta \in Mor(\mathfrak{A})$

$$U(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n}\Gamma(\delta)) = U(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n})U(\Gamma(\delta)) = \\ = U(\Theta^{A_i}_{D',a_1,\ldots,\hat{a}_i,\ldots,a_n})$$

and

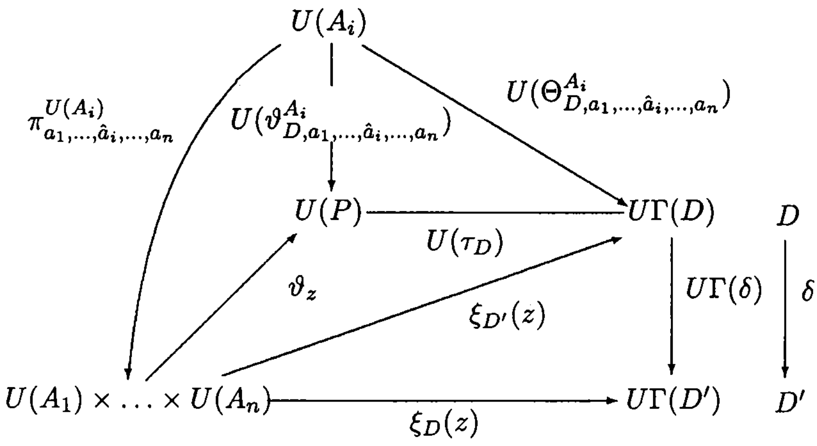
$$\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n}\Gamma(\delta) = \Theta^{A_i}_{D',a_1,\ldots,\hat{a}_i,\ldots,a_n}.$$

From this it follows that the family $(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n} : A_i \to \Gamma(D))_{D\in\mathfrak{D}^o}$ is compatible with Γ. Than there exists exactly one morphism $\vartheta^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}$: $A_i \to P$ such that, for each $D \in \mathfrak{D}^o$

$$U(\tau_D)U(\vartheta^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}) = U(\Theta^{A_i}_{D,a_1,\ldots,\hat{a}_i,\ldots,a_n}) =$$

$$= \xi_D(z)\pi^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} = U(\tau_D)\vartheta_z\pi^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n}.$$

From this and the fact, that each input is the monomorphic family we have $\vartheta_z\pi^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n} = U(\vartheta^{A_i}_{a_1,\ldots,\hat{a}_i,\ldots,a_n})$. So ϑ_z is an n-morphism. Let us define the map $\vartheta : Z \to \mathbf{C}_n(P), z \mapsto \vartheta_z$. It is easy to check that $\mathbf{C}_n(\tau_D)\vartheta = \xi_D$. In addition ϑ is defined uniquely. This follows from the fact that $U(\tau_D)$ is an input.



Let us now assume that (T, τ) is a universal pair of the functor $\mathbf{C}_n$. Then for each $X \in \mathfrak{A}^o$ and each n-morphism $\varphi \in \mathbf{C}_n(X)$ there exists exactly one morphism ϑ such that $\mathbf{C}_n(\vartheta).\tau = \varphi$ thust $U(\vartheta)\tau = \varphi$. This means that (T, τ) is the tensor product of $A_1, \ldots, A_n$. From this it is obvious that in order to proof the existence of a tensor product in a given concrete category it is enought to proof the existence of universal pair of the suitable functor $\mathbf{C}_n$. This observation allows us to formulate the following theorem.

THEOREM 2. *If the category $\mathfrak{A}$ satisfies the following conditions:*

(i) *$\mathfrak{A}$ is complete with respect to inputs,*
(ii) *every pair of morphisms has coequalizer,*
(iii) *functor U has the left adjoint,*
(iv) *functor U preserves epimorphisms,*
(v) *every injection is an embedding,*
(vi) *for each $X \in \mathfrak{A}^o$ $\{Y \in \mathfrak{A}^o : U(Y) = U(X)\}$ forms a set,*

then for each $A_1, \ldots, A_n \in \mathfrak{A}^o$ there exists the tensor product of this sequence of objects.

Proof. Let us fix $A_1, \ldots, A_n \in \mathfrak{A}$. For the proof we must show that the functor $\mathbf{C}_n$ has the universal pair. It is well known that the functor has a universal pair iff it is representable. In order to proof this we use the Freyd's theorem. From (i) $\mathfrak{A}$ is complete with respect to inputs.

From (iii) U has the left adjoint so U preserves the inputs. Thus from lemma 1 $\mathbf{C}_n$ preserves the inputs. To use Freyd's theorem we must show that the functor $\mathbf{C}_n$ has the dominating set eq., there exists a set $\mathfrak{S} \subset \mathfrak{A}^o$

such that the following condition is true

(3) $\forall(X \in \mathfrak{A}^o)\ \forall(\varphi \in \mathbf{C}_n(X))\ \exists(R_\varphi \in \mathfrak{S})\ \exists(r_\varphi \in \mathbf{C}_n(R_\varphi)),$

(4) $\exists(\xi_\vartheta \in \langle R_\varphi, X\rangle)\ \mathbf{C}_n(\xi_\varphi).r_\varphi = \varphi.$

Let $X \in \mathfrak{A}^o$ and

$$\varphi : U(A_1) \times \ldots \times U(A_n) \to U(X)$$

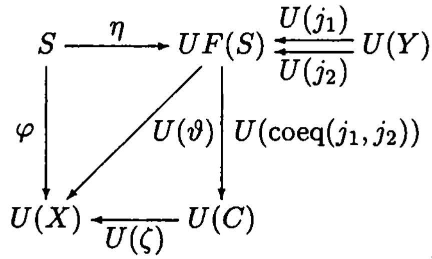
be any n-morphism. From (*iii*) we can take F the left adjoint to U. There exists a mapping (the first canonical transformation)

$$\eta : U(A_1) \times \ldots \times U(A_n) \to UF(U(A_1) \times \ldots \times U(A_n)),$$

such that there exists exactly one morphism

$$\vartheta : F(U(A_1) \times \ldots \times U(A_n)) \to X$$

such that $U(\vartheta)\eta = \varphi$. Let us take a kernell pair (j_1, j_2) of the morphism η and then a coequalizator coeq(j_1, j_2) of the pair (j_1, j_2). Then there exists exactly one morphism $\zeta : C \to X$ such that $\eta = \zeta \circ \text{coeq}(j_1, j_2)$.



Where $S = U(A_1) \times \ldots \times U(A_n)$ It is easy to check that (j_1, j_2) is the kernell pair of coeq(j_1, j_2). Since U is input preserving then $U(j_1, j_2)$ is a kernell pair of U(coeq(j_1, j_2).

We show that $U(\zeta)$ is an injection. Indeed if $c_1, c_2 \in U(C)$ and $c_1 \neq c_2$ then there exist elements $f_1, f_2 \in UF(U(A_1) \times \ldots \times U(A_n)), f_1 \neq f_2$ such that

$$U(\text{coeq}(j_1, j_2)).f_1 = c_1 \qquad U(\text{coeq}(j_1, j_2)).f_2 = c_2.$$

We obtain this from the fact that all coequalizers are epimorphisms and epimorphisms are surjections in **Ens**. $(U(j_1), U(j_2))$ is the input in **Ens** so from the general form of inputs in **Ens** $(f_1, f_2) \neq U(Y)$. Since $(U(j_1), U(j_2))$ is the kernell pair of $U(\eta)$ then $U(\vartheta).f_1 \neq U(\vartheta).f_2$. Hence

$$U(\zeta).c_1 = (U(\zeta)U(\text{coeq}(j_1, j_2))).f_1 = U(\eta).f_1 \neq$$

$$\neq U(\eta).f_2 = (U(\zeta)U(\text{coeq}(j_1, j_2))).f_2 = U(\zeta).c_2$$

from this we have that $U(\zeta)$ is an injection. Therefore from (v) ζ is an embedding.From this and proposition 1 it follows that $U(\text{coeq}(j_1, j_2))\eta$ is an n-morphism. Thus we have shown that for each n-morphism φ there exist $R_\varphi = U(C)$, $r_\varphi = \eta \circ U(\text{coeq}(j_1, j_2))$ and $\xi_\vartheta = U(\zeta)$ such that the condition 3 comes true.

For each $X \in \mathfrak{A}^o$ and any n-morphism φ there exists the object $C \in \mathfrak{A}^o$ and an n-morphism $U(\text{coeq}(j_1, j_2))\eta$ such that

$$U(\zeta)U(\text{coeq}(j_1, j_2))\eta = U(\varphi).$$

Let us now take $\mathfrak{S}$as the class of all C obtained form applying the above construction for diffrent n-morphisms ϑ. We show that in fact $\mathfrak{S}$is a set. First, the class UY of all UY from the diagram (1) is the set (of course we identify isomorphic object). Indeed, the cardinal number of each UY is limited by the cardinal number of the set $2^{UF(S)\times UF(S)}$. Thust the cardinal number of all UY (where Y is like in the diagram) is also limited and forms the set. From (vi) it follows that the class of all $Y \in \mathfrak{A}$ is the set. Now the Freyd's theorem ends the proof. ∎

It is easy to check that the vector spaces, abelian grups and more generaly any variety satisfy the sufficient conditions for existence of tensor product. More interesting example is given in section 2.

Similary, like in the category of vector spaces, one can prove that the tensor product is commutative. We can define the functor of *tensor multiplication* as follows. Let $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathfrak{A}^o$, where $\mathfrak{A}$ satysfies conditions of Theorem (2) and let $f_i : A_i \to B_i, i = 1, \ldots, n$ be any family of morphisms in $\mathfrak{A}$. Than there exists exactly one morphism $f_1 \times \ldots \times f_n : A_1 \times \ldots \times A_n \to B_1 \times \ldots \times B_n$ such that $\pi_{B_i}(f_1 \times \ldots \times f_n) = f_i \pi_{A_i}$ (where π_{A_i}, π_{B_i} are the elements of the product families of $\prod_{i=1}^n A_i$, $\prod_{i=1}^n B_i$ respectively). It can be shown that $f_1 \times \ldots \times f_n$ induces exactly one morphism $f_1 \otimes \ldots \otimes f_n : A_1 \otimes \ldots \otimes A_n \to B_1 \otimes \ldots \otimes B_n$ such that $\otimes U(f_1 \times \ldots \times f_n) = U(f_1 \otimes \ldots \otimes f_n)\otimes$. Now we can define

$$\bigotimes : Mor(\mathfrak{A} \times \ldots \times \mathfrak{A}) \to Mor(\mathfrak{A}) \qquad (f_1, \ldots, f_n) \mapsto f_1 \otimes \ldots \otimes f_n.$$

It is easy to check that $\bigotimes$ is in fact a functor.

2. Tensor products of compact spaces

Let **Comp** denote the category of compact spaces (we assume that they are Hausdorff). In this section we will prove that for any $A_1, \ldots, A_n \in$ **Comp**^{o} there exists the tensor product of this sequence of compact spaces. Category **Comp** is complete with respect to products (this follows from the Tichonow theorem) and equalizers ([3],[4]). So **Comp** is complete with respect to inputs. The form of equalizers and products (Tichonow theorem) in **Comp** and **Ens** yields the forgetful functor $U : \mathbf{Comp} \to \mathbf{Ens}$ to be input preserving.

LEMMA 2. **Comp** *is complete with respect to coequalizers.*

Proof. Let $\xi_1, \xi_2 : X \to Y$ be any morphism in **Comp**. We define the relation $\sim$ as the least equivalence relation such that $\forall(x \in X) : \xi_1(x) \sim \xi_2(x)$. We'll show that $\beta(Y/\sim)$ (where $\beta : \mathbf{Top} \to \mathbf{Comp}$ is the Čech–Stone functor [3],[4]) with the morphism $\beta \circ \mathrm{con}$, where $\mathrm{con} : Y \to Y/\sim$ and $\beta : Y/\sim \to \beta(Y/\sim)$, is the coequalizator of (ξ_1, ξ_2). It is easy to see that $\beta \circ \mathrm{con}$ is continuous and $\beta \circ \mathrm{con} \circ \xi_1 = \beta \circ \mathrm{con} \circ \xi_2$. If $\eta : Y \to Z$ is another morphism such that $\eta \circ \xi_1 = \eta \circ \xi_2$ than $\eta/\sim : Y/\sim \to Z, [y] \mapsto \eta(y)$ is unique morphism for which $(\eta/\sim) \circ \mathrm{con} = \eta$. From the fact that β is **Comp**–reflect we obtain that there exists exactly one morphism ϑ such that $\vartheta\beta = \eta/\sim$. ∎

LEMMA 3. *The interval* $[0, 1]$ *is the separator in* **Comp**.

Proof. $\xi_1, \xi_2 : A \to B$ be any morphisms in **Comp**, such that $\xi_1 \neq \xi_2$. Then there exists an element $a \in A$ for which $\xi_1(a) \neq \xi_2(a)$. Let

$$f : \{\xi_1(a), \xi_2(a)\} \to [0, 1] \quad f(\xi_1(a)) = 0, f(\xi_2(a)) = 1.$$

From Titze lemma it follows that f prolonges to the continuous map $F : B \to [0, 1]$. Of course $F\xi_1 \neq F\xi_2$. ∎

LEMMA 4. *Category* **Comp** *is localy small.*

Proof. In **Comp** monomorphisms covers injections. For each $A \in \mathbf{Comp}^o$ let

$$H_A = \{id_X : X \subset A \wedge X = cl(X) \wedge X \text{with subspace topology}\}$$

$card(H_A) \leq 2^{card(A)}$ so H_A is a set. Moreover for each $X \in \mathbf{Comp}^o$ and any injection $f : X \to A$ $f(X)$ is compact and $card(F(X)) \leq card(A)$. Thus $id_{f(X)} \in H_A$ and for continuous bijection $\tilde{f} : X \to f(X)$ $id_{f(X)}\tilde{f} = f$. ∎

LEMMA 5. *Functor* U : **Comp** $\to$ **Ens** *has the left adjoint.*

Proof. The proof follows from lemmas 3,4 and special Freyd's theorem ([4]). ∎

It is obvious that **Comp** satisfies the conditions (iv) and (vi) of theorem 2. From the above results we have the following theorem

THEOREM 3. *Let* $A_1, \ldots, A_n \in \mathbf{Comp}^o$ *be any objects. Then there exists the tensor product in* **Comp** *of this sequence of objects.*

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