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REMARKS ON APPLICATIONS OF RANK FUNCTIONS TO ALGEBRAIC SETS OF MATRICES

0. Introduction

A function $r : \mathbb{N} \to \mathbb{N}$ is called a *rank function* if it is weakly decreasing and satisfies convexity condition

$$r(j) + r(j+2) \geq 2r(j+1), \qquad j \in \mathbb{N}.$$

The set of all rank functions will be denoted by $\mathcal{R}$. It is partially ordered by the natural relation $\leq$, namely

$$r_1 \leq r_2 \quad \text{iff } r_1(j) \leq r_2(j) \text{ for all } j \in \mathbb{N},$$

where $r_1, r_2 \in \mathcal{R}$.

Let $\mathcal{M}_n$ stand for the set of all $(n \times n)$-matrices whose terms are elements of an algebraically closed field $\mathbb{F}$ of characteristic zero (n is a non-negative integer). We denote by I_n the unit matrix of $\mathcal{M}_n$ and by O_n its zero matrix. A non-empty set $\mathcal{E} \subseteq \mathcal{M}_n$ is called a *cone* if $\mathbb{F}\mathcal{E} \subseteq \mathcal{E}$.

For a matrix $A \in \mathcal{M}_n$ we define

$$r_A : \mathbb{N} \ni j \mapsto \operatorname{rank}(A^j) \in \mathbb{N}$$

(in particular $r_A(0) = \operatorname{rank}(A^0) = \operatorname{rank}(I_n) = n$). The integer $r_A(n)$ is called the *stable rank* of A. One can prove that a function $f : \mathbb{N} \to \mathbb{N}$ is a rank function if and only if there is a matrix $B \in \mathcal{M}_{f(0)}$ such that $f = r_B$ (see [6, Theorem 2 and Theorem 3]).

Given a set $\mathcal{E} \subseteq \mathcal{M}_n$, we shall consider

$$\mathcal{R}(\mathcal{E}) = \{r_A \mid A \in \mathcal{E}\}$$

which is a finite set and for some $\mathcal{E}$'s can be described in a very easy way (in virtue of the above characterization of rank functions). For example, if

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$\mathcal{T}_n \subseteq \mathcal{M}_n$ is the set of all upper triangular matrices, then

$$\mathcal{R}(\mathcal{T}_n) = \{r \in \mathcal{R} \mid r(0) = n\};$$

in turn, if $\mathcal{S}_n^k = \{A \in \mathcal{M}_n \mid A^k = O_n\}$, where $k \in \mathbb{N}$ is such that $k \leq n$, then

$$\mathcal{R}(\mathcal{S}_n^k) = \{r \in \mathcal{R} \mid r(0) = n,\ r(k) = 0\}.$$

Gerstenhaber's classical theorem on closures of conjugacy classes of nilpotent matrices (see [2] and [3, Theorem 3.10]) implies that if $\mathcal{E} \subseteq \mathcal{M}_n$ is a $\mathcal{GL}_n$-invariant algebraic set of nilpotent matrices, then there is a bijective correspondence between the irreducible components of $\mathcal{E}$ and the maximal elements of $\mathcal{R}(\mathcal{E})$ (throughout the text, $\mathcal{GL}_n \subseteq \mathcal{M}_n$ is the group of all invertible matrices, to be a $\mathcal{GL}_n$-invariant set means to be a set invariant under the action of $\mathcal{GL}_n$ on $\mathcal{M}_n$ by the conjugation, and the maximality of a rank function is the maximality in the sense of the ordering $\leq$ in the set $\mathcal{R}$; when dealing with notions and facts of the algebraic geometry, we always use the terminology and notations of [8]).

The purpose of this note is to generalize the above corollary of Gerstenhaber's theorem as far as it is possible in a direct way, and to indicate other basic applications of rank functions to investigations of the topological and geometric structure of algebraic sets of matrices.

1. A modification of Gerstenhaber's theorem

We shall denote by $\mathcal{O}(A)$ the conjugacy class of a matrix $A \in \mathcal{M}_n$ and write $\mathbb{F}^*$ instead of $\mathbb{F} \setminus \{0\}$.

PROPOSITION 1.1. *Let* $A, B \in \mathcal{M}_n$ *be such that* $\max\{r_A(n), r_B(n)\} \leq 1$. *Then the following conditions are equivalent*:

($\bullet$) $r_A = r_B$,

($\bullet\bullet$) $\mathbb{F}^*\mathcal{O}(A) = \mathbb{F}^*\mathcal{O}(B)$.

Proof. Implication ($\bullet\bullet$) $\Rightarrow$ ($\bullet$) is obvious (under no assumption on A, B). Let condition ($\bullet$) be satisfied. If $r_A(n) = r_B(n) = 0$, then A and B both are nilpotent and by [6, Proposition 4], $\mathcal{O}(A) = \mathcal{O}(B)$. Condition ($\bullet\bullet$) follows. If $r_A(n) = r_B(n) = 1$, then by [6, Theorem 1], there are $\tilde{A}, \tilde{B} \in \mathcal{M}_{n-1}$ and $\lambda, \mu \in \mathbb{F}^*$ such that $r_{\tilde{A}} = r_A - 1$, $r_{\tilde{B}} = r_B - 1$, $\tilde{A} \oplus \lambda \in \mathcal{O}(A)$, and $\tilde{B} \oplus \mu \in \mathcal{O}(B)$. Since $\tilde{A}$ and $\tilde{B}$ both are nilpotent, $\mathcal{O}(\tilde{A}) = \mathcal{O}(\tilde{B})$ (by condition ($\bullet$)). In consequence,

$$\tfrac{\lambda}{\mu}(\tilde{B} \oplus \mu) \in \mathcal{O}(A)$$

(because of the invariancy of conjugacy classes of nilpotent matrices under the multiplication by scalars different from zero) which yields condition ($\bullet\bullet$). ∎

We shall use bars for denoting closures of subsets of $\mathcal{M}_n$ in the Zariski topology on $\mathcal{M}_n = \mathbb{F}^{n^2}$. Given $\rho \in \mathcal{R}$ such that $\rho(\rho(0)) \leq 1$ we define $M(\rho) \in \mathcal{M}_{\rho(0)}$ to be the matrix in the Jordan canonical form such that $r_{M(\rho)} = \rho$ and $\operatorname{tr}(M(\rho)) \in \{0,1\}$ (we assume the nilpotent blocks of a matrix in the Jordan canonical form to be ordered with respect to their sizes, with the biggest one in the upper left corner of the matrix). Let us note that $M(\rho)$ contains $\rho(j-1)+\rho(j+1)-2\rho(j)$ nilpotent blocks of size j for all $j \in \mathbb{N}\setminus\{0\}$ (see [6, Remark 2]). By the *Jordan partition of a nilpotent matrix* $A \in \mathcal{M}_n$ we mean a finite sequence $(q_k)_{k=1}^n$ of non-negative integers with q_k equal to the size of kth block of the Jordan canonical form of A, whenever k is not greater that the number of blocks, and to 0, whenever k is greater than the number of blocks. The *conjugate of a partition* $(q_k)_{k=1}^n \in \mathbf{N}^n$ is defined to be the finite sequence $(p_j)_{j=1}^n \in \mathbf{N}^n$ with $p_j = \#\{k \in \mathbf{N} \mid q_k \geq j\}$.

THEOREM 1.2. *Let $A \in \mathcal{M}_n$ be such that $r_A(n) \leq 1$ and let $B \in \mathcal{M}_n$. Then the following conditions are equivalent:*

(*) $r_B \leq r_A$,
(**) $B \in \mathcal{V} := \overline{\mathbb{F}\mathcal{O}(A)}$.

Proof. It is obvious that condition (**) implies condition (*) (under no assumption on $A, B \in \mathcal{M}_n$).

We assume that condition (*) is satisfied.

If $r_A(n) = 0$, then condition (**) immediately follows, by Gerstenhaber's theorem on closures of conjugacy classes. In turn, let $r_A(n) = r_B(n) = 1$. Then by [6, Theorem 1], there are $\tilde{A}, \tilde{B} \in \mathcal{M}_{n-1}$ and $\lambda, \mu \in \mathbb{F}^*$ such that $\tilde{A} \oplus \lambda \in \mathcal{O}(A)$, $\tilde{B} \oplus \mu \in \mathcal{O}(B)$, $r_{\tilde{A}} = r_A - 1$, and $r_{\tilde{B}} = r_B - 1$. By Gerstenhaber's theorem, we obtain $\tilde{B} \in \overline{\mathcal{O}(\tilde{A})}$. This yields

$$B \in \frac{\mu}{\lambda}\overline{\mathcal{O}(A)} \subseteq \overline{\mathbb{F}\mathcal{O}(A)}$$

(because conjugacy classes of nilpotent matrices are invariant under the multiplication by scalars different from zero). Finally, let $r_A(n) = 1$ and let $r_B(n) = 0$. Then by Gerstenhaber's theorem, $B \in \overline{\mathcal{O}(M(s))}$, where

$$s = \max\{r \in \mathcal{R} \mid r \leq r_A,\ r(n) = 0\}.$$

Thus, to complete the proof it suffices to show that

$$\{B' \in \mathcal{V} \mid B' \text{ is nilpotent}\} = \overline{\mathcal{O}(M(s))}.$$

Let us observe that

$$\{B' \in \mathcal{V} \mid B' \text{ is nilpotent}\} = \{B' \in \mathcal{V} \mid \operatorname{tr}(B') = 0\}.$$

In consequence,

$$(\circ) \qquad \dim\{B' \in \mathcal{V} \mid B' \text{ is nilpotent}\} = \dim \mathcal{V} - 1.$$

We set $\tilde{r} = r_A - 1$ and observe that by Proposition 1.1,

$$\mathbf{F}^*\mathcal{O}(A) = \{A' \in \mathcal{M}_n \mid r_{A'} = r_A\}.$$

Applying [1, Lemma 1.1] to $\mathbf{F}^*\mathcal{O}(A)$, we obtain

$$(\circ\circ) \qquad \dim \mathcal{V} = n^2 - \operatorname{codim}_{\mathcal{M}_{n-1}} \overline{\mathcal{O}(M(\tilde{r}))}$$

(because $\{A'' \in \mathcal{M}_{n-1} \mid r_{A''} = \tilde{r}\} = \mathcal{O}(M(\tilde{r}))$). By [5, 1.3], we get

$$(\circ\circ\circ) \qquad \operatorname{codim}_{\mathcal{M}_{n-1}} \overline{\mathcal{O}(M(\tilde{r}))} = \sum_{j=1}^{n-1} p_j^2,$$

as well as

$$(\diamond\circ) \qquad \dim \overline{\mathcal{O}(M(s))} = n^2 - 1 - \sum_{j=1}^{n-1} p_j^2,$$

where $(p_j)_{j=1}^{n-1}$ is the conjugate of the Jordan partition of $M(\tilde{r})$. Concluding $(\circ)$, $(\circ\circ)$, $(\circ\circ\circ)$, and $(\diamond\circ)$, we obtain

$$\dim\{B' \in \mathcal{V} \mid B' \text{ is nilpotent}\} = \dim \overline{\mathcal{O}(M(s))}.$$

Since $\{B' \in \mathcal{V} \mid B' \text{ is nilpotent}\} \subseteq \overline{\mathcal{O}(M(s))}$, the desired equality holds. The proof is complete. ∎

2. Consequences of the modification of Gerstenhaber's theorem

We set

$$\mathcal{C}_n^1 = \{\mathcal{E} \subseteq \mathcal{M}_n \mid \mathcal{E} \text{ is } \mathcal{GL}_n\text{-invariant}, \ \mathbb{F}^*\mathcal{E} \subseteq \mathcal{E}, \ \max_{A \in \mathcal{E}} r_A(n) \leq 1\}$$

and

$$\mathcal{R}_n^1 = \{r \in \mathcal{R} \mid r(0) = n, \ r(n) \leq 1\}.$$

Let us recall that a set $\mathcal{E} \subseteq \mathcal{M}_n$ is *constructible* if it is the set-theorical sum of a finite family of sets which all are locally closed in the Zariski topology on $\mathcal{M}_n$.

As an immediate consequence of Proposition 1.1 we obtain

PROPOSITION 1.2. *The maps*

$$\mathcal{C}_n^1 \ni \mathcal{E} \mapsto \mathcal{R}(\mathcal{E}) \subseteq \mathcal{R}_n^1$$

and

$$\mathcal{R}_n^1 \supseteq \mathcal{Q} \mapsto \bigcup_{r \in \mathcal{Q}} \mathbb{F}^*\mathcal{O}(M(r)) \in \mathcal{C}_n^1$$

are mutually inverse bijections which preserve inclusions and which are compatible with set-theorical operations. Moreover, all the elements of $\mathcal{C}_n^1$ are constructible sets.

Now, by Proposition 2.1 and Theorem 1.2, we directly obtain

THEOREM 2.2. *For $\mathcal{E} \in \mathcal{C}_n^1$ the following conditions are equivalent:*

(*) $\mathcal{E}$ is an algebraic set,

(**) if $r \in \mathcal{R}$ is such that $r(0) = n$ and there is $\tilde{r} \in \mathcal{R}(\mathcal{E})$ such that $r \leq \tilde{r}$, then $r \in \mathcal{R}(\mathcal{E})$.

In particular, $\mathcal{R}(\overline{\mathcal{E}}) = \{r \in \mathcal{R}_n^1 \mid \text{there is } \tilde{r} \in \mathcal{R}(\mathcal{E}) \text{ such that } r \leq \tilde{r}\}$.

For $\mathcal{E} \subseteq \mathcal{M}_n$ we define $\mathcal{Z}(\mathcal{E})$ to be the set of all maximal elements of $\mathcal{R}(\mathcal{E})$ and $\mathcal{F}(\mathcal{E})$ to be the family of all irreducible components of $\overline{\mathcal{E}}$.

As a simple consequence of the irreducibility of conjugacy classes of matrices, Theorem 1.2, and the fact that for each $r \in \mathcal{R}(\mathcal{E})$ there is an $r_0 \in \mathcal{Z}(\mathcal{E})$ such that $r \leq r_0$ ($\mathcal{E} \subseteq \mathcal{M}_n$ is an arbitrary set), we obtain the announced generalization of the corollary of Gerstenhaber's theorem. In a possibly detailed and computationally effective way, it may be formulated as follows.

THEOREM 2.3. *If $\mathcal{E} \in \mathcal{C}_n^1$, then the formulae*

$$\mathcal{Z}(\mathcal{E}) \ni r \mapsto \overline{\mathbb{F}\mathcal{O}(M(r))} \in \mathcal{F}(\mathcal{E})$$

and

$$\mathcal{F}(\mathcal{E}) \ni \mathcal{W} \mapsto \max \mathcal{R}(\mathcal{W}) \in \mathcal{Z}(\mathcal{E})$$

correctly define mutually inverse bijective maps.

COROLLARY 2.4. *For $\mathcal{E} \in \mathcal{C}_n^1$ the following conditions are equivalent :*

(*) $\overline{\mathcal{E}}$ is an irreducible set,

(**) there is the greatest element in $\mathcal{R}(\mathcal{E})$.

Moreover, if (**) is satisfied, then $\overline{\mathcal{E}} = \overline{\mathbb{F}\mathcal{O}(M(s))}$, where $s = \max \mathcal{R}(\mathcal{E})$.

One should notice that in the case where stable ranks do not exceed one, each $\mathcal{GL}_n$-invariant irreducible algebraic cone (i.e. the closure of the cone over a single conjugacy class) is a normal set (see [1], [7, Corollary 3.15], and Theorem 1.2).

The above considered family $\mathcal{C}_n^1$ may seems to be not a quite natural object of studying, but it is the simplest and actually the largest (as we shall see in the sequel) family of $\mathcal{GL}_n$-invariant sets $\mathcal{E} \subseteq \mathcal{M}_n$ satisfying condition $\mathbb{F}^*\mathcal{E} \subseteq \mathcal{E}$ which contains all the $\mathcal{GL}_n$-invariant sets of nilpotent matrices and such that rank functions suffice to characterize the geometry and the topology of its elements.

3. Some applications

Applying rank functions one can give quite nice proofs of some results on the irreducibility of $\mathcal{GL}_n$-invariant algebraic cones.

The first one of such results is rather known but nowhere stated in an explicit way (see [4] for the orbits degeneration diagrams; the result can be also deduced from these diagrams).

THEOREM 3.1. (i) *Let $\mathcal{X} \subseteq \mathcal{M}_n$ be a $\mathcal{GL}_n$-invariant algebraic set of nilpotent matrices. Then $\mathcal{X}$ is irreducible whenever $n \leq 5$.*

(ii) *If $n \geq 6$, then there is a reducible $\mathcal{GL}_n$-invariant algebraic subset of $\mathcal{M}_n$ which consists of nilpotent matrices.*

Proof. It is very easy to see that the ordering $\leq$ is linear in the set

$$\{r \in \mathcal{R} \mid r(0) = n,\ r(n) = 0\}$$

whenever $n \leq 5$ (it suffices to draw an appropriate picture). Assertion *(i)* immediately follows from Corollary 2.4 and from the fact that all the $\mathcal{GL}_n$-invariant algebraic sets of nilpotent matrices are cones.

If $n \geq 6$, then rank functions $r_1, r_2 : \mathbb{N} \to \mathbb{N}$ defined by the formulae

$$r_1(j) = \begin{cases} n, & \text{if } j = 0, \\ 3, & \text{if } j = 1, \\ 0, & \text{if } j \geq 2, \end{cases} \quad \text{and} \quad r_2(j) = \begin{cases} n, & \text{if } j = 0, \\ 2, & \text{if } j = 1, \\ 1, & \text{if } j = 2, \\ 0, & \text{if } j \geq 3, \end{cases}$$

are not comparable. In consequence, $\overline{\mathcal{O}(M(r_1))} \cup \overline{\mathcal{O}(M(r_2))} \subset \mathcal{M}_n$ is a two-components $\mathcal{GL}_n$-invariant algebraic set of nilpotent matrices (see Theorem 2.3). ■

The second result on the irreducibility of $\mathcal{GL}_n$-invariant cones is the following.

PROPOSITION 3.2. *Let $A, B \in \mathcal{M}_n$ be such that* $\max\{r_A(n), r_B(n)\} \leq 1$. *Then $\mathcal{W} = \overline{\mathbb{F}\mathcal{O}(A)} \cap \overline{\mathbb{F}\mathcal{O}(B)}$ is an irreducible algebraic set.*

Proof. By Theorem 1.2,

$$\mathcal{R}(\mathcal{W}) = \{r \in \mathcal{R}_n^1 \mid r \leq r_A,\ r \leq r_B\}.$$

Let

$$s(j) = \max\{r(j) | r \in \mathcal{R}(\mathcal{W})\},\ j \in \mathbb{N}.$$

The above defined function $s : \mathbb{N} \to \mathbb{N}$ is a rank function (see [6, Proposition 5]). Moreover, it evidently is the supremum of $\mathcal{R}(\mathcal{W})$ (with respect to the ordering $\leq$ in the set $\mathcal{R}$). By definition, $s \in \mathcal{R}(\mathcal{W})$. The assertion follows by Corollary 2.4. ■

4. On sets of matrices whose stable ranks may exceed one

Unfortunately, if a $\mathcal{GL}_n$-invariant algebraic cone contains matrices of stable ranks greater than one, then in general there are no so direct connections between its topology and geometry and the associated set of rank functions. Let us see some irregularities which appear in this case.

For $A \in \mathcal{M}_n$ and $k \in \mathbb{N} \setminus \{0\}$ such that $k \leq n$ we define $\mathrm{s}_k(A)$ to be the sum of all the size k principal minors of A. It is obvious that $\mathrm{s}_k : \mathcal{M}_n \to \mathbb{F}$ is a $\mathcal{GL}_n$-invariant and homogeneous polynomial in n^2 variables.

THEOREM 4.1. *Let $n, k \in \mathbb{N}$ be such that $n \geq k \geq 2$ and let $r : \mathbb{N} \to \mathbb{N}$ be a rank function satisfying conditions $r(0) = n$ and $r(n) = k$. Then there are $A_1, A_2 \in \mathcal{M}_n$ such that $r_{A_1} = r_{A_2} = r$, $A_1 \notin \overline{\mathbb{F}\mathcal{O}(A_2)}$, and $A_2 \notin \overline{\mathbb{F}\mathcal{O}(A_1)}$.*

Proof. We define

$$A_j = M(r - k) \oplus I_{k-1} \oplus j,$$

where $j = 1, 2$. By definition, $A_j \in \mathcal{M}_n$ and $r_{A_j} = r$.

Let $\varphi_j : \mathcal{M}_n \to \mathbb{F}$ be defined by the formula

$$\varphi_j(A) = j(\mathrm{tr}(A))^k - (k + j - 1)^k \mathrm{s}_k(A), \ A \in \mathcal{M}_n, \ j = 1, 2.$$

It is obvious that φ_1 and φ_2 both are $\mathcal{GL}_n$-invariant and homogeneous polynomials in n^2 variables. Furthermore,

$$\varphi_j(A_j) = 0, \ j = 1, 2,$$

while

$$\varphi_2(A_1) = 2k^k - (k+1)^k \neq 0$$

and

$$\varphi_1(A_2) = (k+1)^k - 2k^k.$$

Thus $A_1 \notin \overline{\mathbb{F}\mathcal{O}(A_2)}$ and $A_2 \notin \overline{\mathbb{F}\mathcal{O}(A_1)}$. The proof is complete. ■

COROLLARY 4.2. *Let $n, k \in \mathbb{N}$, and $r \in \mathcal{R}$ satisfy the assumptions of Theorem 4.1. Then there is a* reducible *$\mathcal{GL}_n$-invariant algebraic cone $\mathcal{S} \subset \mathcal{M}_n$ such that r is the greatest element of $\mathcal{R}(\mathcal{S})$.*

One should notice that in the case where stable ranks do not exceed one each irreducible $\mathcal{GL}_n$-invariant algebraic cone is the closure of the cone over a single conjugacy class while if $n \geq k \geq 2$, then the determinantal variety

$$\mathcal{H}_n^k = \{A \in \mathcal{M}_n \mid \mathrm{rank}(A) \leq k\}$$

which is an irreducible $\mathcal{GL}_n$-invariant algebraic cone fails to be the closure of the cone over a single conjugacy class.

To end the section let us consider

EXAMPLE 4.3. We set

$$\mathcal{E} = \mathbb{F}^*\mathcal{O}(K_2 \oplus 1 \oplus 2) \cup \mathbb{F}^*\mathcal{O}(O_2 \oplus 1 \oplus 1) \cup \overline{\mathbb{F}\mathcal{O}(K_3 \oplus 1)} \subset \mathcal{M}_4,$$

where $K_j \in \mathcal{M}_j$ is the nilpotent Jordan block of size $j = 2, 3$. By Theorem 1.2,

$$\mathcal{R}(\mathcal{E}) = \{r \in \mathcal{R} \mid r(0) = 4,\ r \leq 2 + r_{K_2}\},$$

but $\mathcal{E}$ is not an algebraic set because $O_2 \oplus 1 \oplus 2 \in \overline{\mathbb{F}\mathcal{O}(K_2 \oplus 1 \oplus 2)} \setminus \mathcal{E}$.

On the other hand, the straight line $\mathcal{L} = \mathbb{F}I_n \subseteq \mathcal{M}_n$ is a $\mathcal{GL}_n$-invariant algebraic cone but if $n \geq 2$, then $\mathcal{R}(\mathcal{L}) = \{r_{O_n}, n\}$ does not satisfy condition (**) of Theorem 2.2.

Unless, the above examples do not imply that rank functions cannot be helpful in studying arbitrary sets of square matrices.

5. A general inequality

Even if $\mathcal{E} \subseteq \mathcal{M}_n$ satisfies no special geometric or rank conditions there is a connection between $\mathcal{F}(\mathcal{E})$ and $\mathcal{Z}(\mathcal{E})$, namely

THEOREM 5.1. *If $\mathcal{E} \subseteq \mathcal{M}_n$, then*

$$\#\mathcal{Z}(\mathcal{E}) \leq \#\mathcal{F}(\mathcal{E}).$$

Proof. We assume $\mathcal{Z}(\mathcal{E}) \neq \emptyset$ and set

$$\mathcal{E}_r = \overline{\mathcal{E}} \cap \{A \in \mathcal{M}_n \mid r_A \leq r\}$$

for an $r \in \mathcal{Z}(\mathcal{E})$. Then all the $\mathcal{E}_r$'s are non-empty algebraic sets.

Since for each $\rho \in \mathcal{R}(\mathcal{E})$ there is an $r \in \mathcal{Z}(\mathcal{E})$ such that $\rho \leq r$, we have

$$\bigcup\{\mathcal{E}_r \mid r \in \mathcal{Z}(\mathcal{E})\} = \overline{\mathcal{E}}.$$

The incomparability of the elements of $\mathcal{Z}(\mathcal{E})$ yields

$$\mathcal{E}_{r_0} \not\subset \bigcup\{\mathcal{E}_r \mid r \in \mathcal{Z}(\mathcal{E}),\ r \neq r_0\},\ r_0 \in \mathcal{Z}(\mathcal{E}).$$

Thus for an arbitrary $r_0 \in \mathcal{Z}(\mathcal{E})$ there is $\mathcal{W} \in \mathcal{F}(\mathcal{E})$ such that $\mathcal{W} \subseteq \mathcal{E}_{r_0}$ and

$$\mathcal{W} \not\subset \bigcup\{\mathcal{E}_r \mid r \in \mathcal{Z}(\mathcal{E}),\ r \neq r_0\}.$$

The assertion follows. ■

COROLLARY 5.2. *If $\mathcal{E} \subseteq \mathcal{M}_n$ is such that $\overline{\mathcal{E}}$ is an irreducible set, then there is the greatest element in $\mathcal{R}(\mathcal{E})$.*

The above simple observations seem to be quite useful in studying the geometry of sets of square matrices.

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