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# A REPRESENTATION OF SYMMETRIC IDEMPOTENT AND ENTROPIC GROUPOIDS

The aim of this paper is to describe the structure of groupoids satisfying the following axioms:

(S) $(x \cdot y) \cdot y = x$ (Symmetry),
(I) $x \cdot x = x$ (Idempotence),
(E) $(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t)$ (Entropicity).

A groupoid $(G, \cdot)$ satisfying the above identities will be called an SIE-groupoid.

SIE-groupoids are modes, i.e. they are idempotent and entropic algebras as defined in [RS]. The investigations presented here belong to the recently developed theory of groupoid modes. ( See e.g. [JK] and references there, [LM], [RS], [RR], [R1], [R2].)

On the other hand, the class SIE of all SIE-groupoids is contained in the more general class of groupoids satisfying (S), (I) and the right distributive law. In [P1] and [P2] such groupoids were called symmetric groupoids. Motivation for studying these groupoids comes from differential geometry and knot theory. The groupoids were investigated in connection with symmetric spaces (see e.g. [L] or [Ki]) and with reference to algebraic topology (see [E]). They also have been used to characterise knots in [J]. Joyce called SIE-groupoid abelian involutary quandle.

Finally some SIE-groupoids are so-called P-groupoids. The latter have been defined in [K] and used to decompose complete undirected graphs into disjoint closed paths. An application of such groupoids in graph theory was given in [R2] as well.

This paper investigates the structure of SIE-groupoids by means of some decompositions, together with a corresponding construction method for recovering the groupoids from their decompositions. SIE- groupoids are closely related to abelian groups. To show this relationship, we define on each fibre of the described decomposition an abelian group structure. Then we are concerned with the method of building SIE- groupoids from a family of abelian groups indexed by a set with a left zero band structure. The syntactical approach used for the structure theorem for these groupoids will be then a starting point for description of all subdirectly irreducible SIE-groupoids in each nontrivial subvariety of SIE in a next paper.

In the first section we recall the basic notations and properties of SIE-groupoids. For more detailed information and for proofs of results we refer the reader to [R1] and [R2]. Some technical lemmas are presented in Section 2. The Decomposition Theorem is proved in Section 3. The congruence relation decomposing an SIE-groupoid into classes forming abelian groups is given in this section as well. To examine such decompositions further one needs to know more about the way in which abelian groups are put together. This is the concern of Section 4. In this Section an explicit method for constructing SIE-groupoids from abelian groups is given. This is done by means of a special construction called the AG-sum. In the last section the free SIE-groupoids in subvarieties of the variety of all SIE-groupoids are described as AG-sums.

## 1. Preliminaries

In this section we review without proofs some results concerning SIE-groupoids. For the proofs we refer the reader to [R1] and [R2]. We use the following convention:

(1.1) $$x_1 \dots x_n := (x_1 \dots x_{n-1}) \cdot x_n,$$

for every $n \geq 2$, and

$$xy \cdot zt := (x \cdot y) \cdot (z \cdot t).$$

The product (1.1) is called left associated.

As was shown in [R1], each SIE-groupoid satisfies the following identities:

($\mathrm{D_l}$) $x \cdot yz = xy \cdot xz$ (Distributive laws),

($\mathrm{D_r}$) $xy \cdot z = xz \cdot yz$

($\mathrm{P_a}$) $xy \cdot x = x \cdot yx$ (Partial associative law),

(N) $x \cdot yz = xzyz$,

(A) $xyzt = xtzy$.

By (N) each SIE-groupoid word can be represented in a left associated form.

The following, basic example of SIE-groupoids was given in [R1].

1.2. Example. Let $(G, +, 0)$ be an abelian group. We define a binary operation $\cdot$ on the set $G$ by $x \cdot y := 2y - x$. Then $(G, \cdot)$ is an SIE-groupoid.

We denote this groupoid by *Core*(G,+). (See [B], where *Cores* for Moufangs loops were defined.)

The variety of all SIE-groupoids is denoted by SIE. Let $Z$ denote the set of integers, and let $Q_n \subset Z^n$ be the subset of $Z^n$ consisting of all sequences $(l_1, \dots, l_n) \in Z^n$ such that at most one entry $l_i$ is an odd natural number. It is obvious that $(Q_n, \cdot)$ is a subgroupoid of $Core(Z^n, +)$. Moreover, as was shown by Joyce [J], the SIE-groupoid $(Q_n, \cdot)$ is isomorphic to the free SIE-groupoid $F_{\mathrm{SIE}}(x_0, \dots, x_n)$ on the generators $x_0, x_1, \dots, x_n$. The free SIE-groupoid on two generators $x$, $y$ is isomorphic to the groupoid $Core(Z, +)$. For more detailed information concerning free SIE-groupoids we refer the reader to [R1], [R2], [J], [LM]. In [LM], the authors considered an infinite sequence of words

$$(1.3) \qquad \begin{cases} \omega_0(x,y) = x, \ \omega_1(x,y) = y, \\ \omega_i(x,y) = \omega_{i-2}(x,y)\omega_{i-1}(x,y) & \text{for } i \geq 2, \\ \omega_{i-2}(x,y) = \omega_i(x,y)\omega_{i-1}(x,y) & \text{for } i \leq 1. \end{cases}$$

It was proved in [LM] that the words $\omega_i(x,y)$, where $i$ is an arbitrary integer, give a standard form for words in $F_{\mathrm{SIE}}(x,y)$.

The following propositions were proved in [R1], [R2] and [LM].

1.4. PROPOSITION. *The following identities are satisfied in the variety* SIE *for all integers $k, r, l$*:

$$(1.5) \qquad \omega_k(x,y) = \omega_{-k+1}(y,x),$$

$$(1.6) \qquad \omega_k(x,y)\omega_r(x,y) = \omega_{2r-k}(x,y),$$

$$(1.7) \qquad \omega_k(\omega_r(x,z), \omega_r(y,t)) = \omega_r(\omega_k(x,y), \omega_k(z,t)),$$

$$(1.8) \qquad \omega_k(\omega_l(x,y), \omega_r(x,y)) = \omega_{kr+(1-k)l}(x,y). \qquad \blacksquare$$

1.9. PROPOSITION. *If the identity $\omega_n(x,y) = \omega_0(x,y)$ holds in a variety of* SIE-*groupoids, then the identity $\omega_{n+i}(x,y) = \omega_i(x,y)$ holds as well.* ∎

In what follows we abbreviate $\omega_i(x,y)$ to $\omega_i$ if no confusion can arise. The following claims, which are easily established using Propositions 1.4 and 1.9, are to be used several times.

1.10. COROLLARY. [R2] *For all integers $r$ and $k$ the following identities hold in* SIE:

$$(1.11) \qquad \omega_{-r}(x,y) = \omega_r(x,y)\omega_0(x,y),$$

$$(1.12) \qquad \omega_{2r}(x,y) = \omega_0(x,y)\omega_r(x,y),$$

$$(1.13) \qquad \omega_r(xz, yt) = \omega_r(x,y)\omega_r(z,t),$$

$$(1.14) \qquad \omega_r(xz, y) = \omega_r(x,y)\omega_r(z,y),$$

$$\omega_r(x, yt) = \omega_r(x, y)\omega_r(x, t), \tag{1.15}$$

$$\omega_k(x, \omega_r(x, y)) = \omega_{kr}(x, y). \tag{1.16}$$

■

1.17. COROLLARY. [LM] *Suppose that the identity $\omega_n = \omega_0$ holds in a variety of* SIE*-groupoids. Then for each integer $k$, the identity $\omega_{kn} = \omega_0$ also holds.* ■

Let $N = (N, \leq)$ denote the lattice of all natural numbers with respect to the partial order $\leq$ defined by $k \leq n$ iff $k$ divides $n$. We will denote by $N^+$ the lattice $(N \cup \{\infty\}, \leq_N)$ in which the element $\infty$ is greater than all elements of $N$. As was proved in [R1], the lattice $L(\underline{\text{SIE}})$ of all subvarieties of the variety $\underline{\text{SIE}}$ of all SIE-groupoids is isomorphic to the lattice of $N^+$. The variety corresponding to $\infty$ is $\underline{\text{SIE}}$. The variety corresponding to a natural number $n$ is just the variety of SIE-groupoids defined by identity $\omega_n(x, y) = x$. We will denote this variety by $V_n$.

It was proved in [R2] that if $n > 1$ is an odd natural number, then each SIE-groupoid in $V_n$ is polynomially equivalent to an abelian group satisfying the identity $nx = 0$. Moreover, every groupoid $(G, \cdot)$ from $V_n$ is a quasigroup, and is equal to the groupoid $Core(G, +)$ for some abelian group $(G, +)$ satisfying the identity $nx = 0$. Those subvarieties of the variety $\underline{\text{SIE}}$ which contain some quasigroups are described in [R2] as well. It was proved there that the varieties $V_{2^s}$ and $V_n$ for $s > 0$ and an odd natural number $n > 1$, are independent (see [GLP]). Hence their join $V_{2^s} \vee V_n = V_{2^s n}$ is equal to the direct product $V_{2^s} \times V_n = V_{2^s n}$.

## 2. Some technical lemmas

A standard form of words in the free SIE-groupoid $F_{\text{SIE}}(x_0, \ldots, x_n)$ on $n+1$ generators was described in [R1]. For each element $\underline{r} = (2r_1, \ldots, r_j, \ldots, 2r_n)$ of $Q_n$, $\omega(\underline{r}; x_0, \ldots, x_n)$ was defined as follows

$$\omega(\underline{0}; x_0, \ldots, x_n) := x_0 = \omega_0 \quad \text{and}$$

$$\omega(2r_1, \ldots, r_j, \ldots, 2r_n; x_0, \ldots, x_n) \tag{2.1}$$

$$:= \begin{cases} \omega_{r_j}\omega_0\omega_{r_1}\omega_0 \ldots \omega_{r_{j-1}}\omega_0\omega_{r_{j+1}} \cdots \omega_0\omega_{r_n} & \text{if } r_j \text{ is odd,} \\ \omega_0\omega_{r_1}\omega_0 \ldots \omega_{r_j/2}\omega_0 \ldots \omega_0\omega_{r_n} & \text{otherwise,} \end{cases}$$

where $\omega_{r_j} := \omega_{r_j}(x_0, x_j)$, for each $1 \leq j \leq n$.

In the sequel we will write briefly $\omega(2r_1, \ldots, r_j, \ldots, 2r_n)$ if no confusion can arise and we will abbreviate $\omega(2r_1, \ldots, r_j, \ldots, 2r_n)$ to $\omega_{\underline{r}}$ for $\underline{r} = (2r_1, \ldots, r_j, \ldots, 2r_n)$ in $Q_n$. As was noted in [R1],

$$\begin{aligned} \omega_0\omega_{r_1}\omega_0 \ldots \omega_0\omega_{r_n} &= \omega_{2r_1}\omega_0 \ldots \omega_0\omega_{r_n} \\ &= \omega_{2r_j}\omega_0\omega_{r_1}\omega_0 \ldots \omega_0\omega_{r_{j-1}}\omega_0\omega_{r_{j+1}}\omega_0 \ldots \omega_{r_n} \end{aligned}$$

for each $1 \le j \le n$. So it follows immediately that for each $(r_1, \dots, r_n)$ in $Q_n$ and for every permutation $\alpha$ of the set $\{1, \dots, n\}$, the identity

$$(2.2)\quad \omega(r_1, \dots, r_n; x_0, x_1, \dots, x_n) = \omega(r_{\alpha(1)}, \dots, r_{\alpha(n)}; x_0, x_{\alpha(1)}, \dots, x_{\alpha(n)})$$

is satisfied in a variety of SIE-groupoids. Moreover, it is obvious that the identity

$$(2.3)\quad \omega(r_1, \dots, r_n; x_0, x_1, \dots, x_n) = \omega(r_1, \dots, r_n, 0; x_0, x_1, \dots, x_n, x_{n+1})$$

is satisfied in $\underline{\text{SIE}}$ for arbitrary $(r_1, \dots, r_n) \in Q_n$. (See [R1]) The following proposition was also proved in [R1].

2.4. PROPOSITION. *In the free* SIE*-groupoid* $F_{\text{SIE}}(x_0, x_1, \dots, x_n)$ *on the generators* $x_0, x_1, \dots, x_n$ *each further element may be expressed in the standard form* $\omega_{\underline{r}}$ *for some* $\underline{r}$ *in* $Q_n$. *Moreover, the identity*

$$(2.5)\quad \omega_{\underline{r}}\omega_{\underline{k}} = \omega_{2\underline{k}-\underline{r}}$$

*holds in* $\underline{\text{SIE}}$, *for all* $\underline{r}, \underline{k} \in Q_n$. ∎

Definition (2.1) generalises (1.2). Moreover, as was shown in [R1], the words $\omega_{\underline{r}}(x_0, \dots, x_n)$, where $\underline{r}$ is an arbitrary element of $Q_n$, give a standard form of words in $F_{\text{SIE}}(x_0, \dots, x_n)$. The identity (2.2) shows how this expression depends on the order of the generators $x_0, \dots, x_n$. Notice that the generator $x_0$ plays a special role in Definition 2.1. The next proposition allows us to replace $x_0$ by any other $x_i$.

2.6. PROPOSITION. *The identity*

$$(2.7)\quad \omega(k_1, \dots, k_n; x_0, \dots, x_n)$$
$$= \omega\Big(k_1, \dots, k_{i-1}, 1 - \sum_{j=1}^{n} k_j, k_{i+1}, \dots, k_n; x_i, x_1, \dots, x_{i-1}, x_0, x_{i+1}, \dots, x_n\Big)$$

*holds in the variety* $\underline{\text{SIE}}$ *for arbitrary* $(k_1, \dots, k_n)$ *in* $Q_n$ *and* $i = 1, \dots, n$.

Proof. By Definition (2.1) and (2.2), it is sufficient to prove (2.7) in the case that $i = 1$ and that all $k_j$ for $j = 2, \dots, n$ are even. The proof of this is by induction on $n$, the case $n = 1$ just being (1.5). First, it should be shown that (2.7) holds for $n = 2$ and $k_1 = 0$. In this case the identity (2.7) has the following form

$$(2.8)\quad \omega(0, k_2; x_0, x_1, x_2) = \omega(1 - k_2, k_2; x_1, x_0, x_2).$$

The identity (2.8) holds in $\underline{\text{SIE}}$ for every integer $k_2$. The easy proof of this, by induction on $k_2$ using (1.3), (2.3) and (2.5), will be omitted.

Assuming (2.7) to hold for $n$, we shall prove it for $n+1$. Let $(k_1, \ldots, k_n, 2k_{n+1})$ be in $Q_{n+1}$. From (2.1), (2.3) and (2.5) we deduce that

$$\begin{aligned}\omega(k_1, \ldots, k_n, 2k_{n+1}; x_0, x_1, \ldots, x_n, x_{n+1}) \\ &= \omega(k_1, \ldots, k_n; x_0, x_1, \ldots, x_n)\omega_0\omega_{k_{n+1}}(x_0, x_{n+1}) \\ &= \omega(k_1, \ldots, k_n; x_0, x_1, \ldots, x_n)\omega_0\omega(0, k_{n+1}; x_0, x_1, x_{n+1}).\end{aligned}$$

Next, applying the induction hypothesis, (1.5), (2.8) and (2.5), one obtains

$$\begin{aligned}\omega(k_1, \ldots, k_n, 2k_{n+1}; x_0, x_1, \ldots, x_n, x_{n+1}) \\ &= \omega\Big(1 - \sum_{j=1}^{n} k_j, k_2, \ldots, k_n; x_1, x_0, x_2, \ldots, x_n\Big)\omega_1(x_1, x_0) \\ &\quad \cdot \omega(1 - k_{n+1}, k_{n+1}; x_1, x_0, x_{n+1}) \\ &= \omega\Big(1 - \sum_{j=1}^{n} k_j - 2k_{n+1}, k_2, \ldots, k_n, 2k_{n+1}; x_1, x_0, \ldots, x_n\Big),\end{aligned}$$

which ends the proof. ∎

## 3. Decomposition of SIE-groupoids into unions of orbits

In this section we describe how an SIE-groupoid breaks up into smaller pieces that are indexed by a left zero band i.e. a semigroup satisfying the identity $xy = x$. Moreover, on each such piece an abelian group structure is defined. Let $(G, \cdot)$ be an SIE-groupoid. Let us define a relation $\pi$ on the set $G$ as follows:

(3.1) $a \, \pi \, b$ if and only if $a = bb_1 \ldots b_k$ for some elements $b_1, \ldots, b_k$ in $G$.

It is easy to check that $\pi$ is an equivalence relation. Moreover, the following lemma is true.

3.2. LEMMA. *The relation $\pi$ is a congruence relation of an* SIE-*groupoid $(G, \cdot)$. The equivalence classes are subgroupoids of $(G, \cdot)$ and the quotient $(G/\pi, \cdot)$ is a left zero band.*

Proof. Let $a, b, c, d$ be elements of $G$ such that $a = bb_1 \ldots b_k$ and $c = dd_1 \ldots d_r$ for some $b_1, \ldots, b_k, d_1, \ldots, d_r$ from $G$. By idempotence we can assume that $r = k$. Then by entropicity,

$$ac = bb_1 \ldots b_k \cdot dd_1 \ldots d_k = bd \cdot b_1 d_1 \cdot \ldots \cdot b_k d_k.$$

So $\pi$ is a congruence relation of the SIE-groupoid $(G, \cdot)$. The classes are subgroupoids of $(G, \cdot)$, because congruence classes of any idempotent algebra are subalgebras. Moreover, for any $a, b$ from $G$, $a \, \pi \, ab$, so the quotient $(G/\pi, \cdot)$ is a left zero band. ∎

3.3. LEMMA. *The congruence relation $\pi$ is the least congruence relation on an* SIE*-groupoid $(G,\cdot)$ such that the quotient $(G/\pi,\cdot)$ is a left zero band.*

Proof. For $\Phi\in \mathrm{Con}(G,\cdot)$, assume that the quotient $(G/\Phi,\cdot)$ is a left zero semigroup. Then, for any $x,y$ in $G$, $x\ \Phi\ xy$. Suppose now that $a\ \pi\ b$. This means that $a=bb_1\dots b_k$, for some $b_1,\dots,b_k$ from $G$. Since $b\ \Phi\ bb_1\ \Phi\ bb_1b_2\ \Phi\ \dots\ \Phi\ bb_1\dots b_k=a$, $a\ \Phi\ b$ and $\pi\subset\Phi$. ■

For an arbitrary element $g$ from the set $G$, the $\pi$-class containing $g$ will be called the $g$-orbit and denoted by $G_g$. By the definition of $\pi$ we obtain $G_g:=\{ga_1\dots a_k\mid a_1,\dots,a_k\in G\}$. It follows from Lemma 3.2, that the congruence relation $\pi$ decomposes an SIE-groupoid $(G,\cdot)$ into a disjoint sum of orbits, and that the quotient $(G/\pi,\cdot)$ is a left zero band. Let $\{g_i\mid i\in I\}$ be a set of generators of the SIE-groupoid $(G,\cdot)$. There is at least one generator $g_i$ from the set $\{g_i\mid i\in I\}$ in each orbit $G_g$. Indeed, if $g=g_{i_1}g_{i_2}\dots g_{i_p}$ for some $i_1,\dots,i_p$ from $I$, then $gg_{i_p}g_{i_{p-1}}\dots g_{i_2}=g_{i_1}$ so $g_{i_1}$ belongs to the orbit $G_g$. Thus, by Lemma 3.2, $G_g=G_{g_i}=:G_i$. Let $J$ be the set of indices of the chosen generators. The groupoid $(G,\cdot)$ decomposes into a disjoint sum of orbits $G_i$, for $i\in J$. Moreover, each $(G_i,\cdot)$ is a subgroupoid of $(G,\cdot)$. Throughout this section $G_i$ will denote the orbit of $g_i$.

The following lemma describes a standard form for elements belonging to one orbit.

3.4. LEMMA. *Pick an arbitrary generator $g_{i_0}$ from the set of generators $\{g_i\mid i\in I\}$. Then*

$$G_{i_0}=\{g\in G\mid g=\omega(2r_1,\dots,2r_k;g_{i_0},g_{i_1},\dots,g_{i_k},\ \text{for}\ i_1,\dots,i_k\in I-\{j_0\},\ \text{and}\ r_1,\dots,r_k\in Z\}.$$

Proof. If an element $g$ from the set $G$ has the required form, then by Definition (2.1),

$$g=g_{i_0}\omega_{r_1}(g_{i_0},g_{i_1})g_{i_0}\dots g_{i_0}\omega_{r_k}(g_{i_0},g_{i_k}),$$

so that $g$ belongs to $G_{i_0}$. Now let $g\in G_{i_0}$. It means that $g=g_{i_0}a_1\dots a_p$ for some $a_1,\dots,a_p\in G$. Since (2.1) describes a standard form of elements in the free SIE-groupoid, there exists a finite subset $\{i_1,\dots,i_k\}$ of $I$ such that $a_i=\omega(r_{i,1},\dots,r_{i,k};g_{i_0},g_{i_1},\dots,g_{i_k})$ for some integers $r_{i,1},\dots,r_{i,k}$ of which at most one is odd, and for $i=1,\dots,p$. Hence, $g_{i_0}=\omega_0(g_{i_0},g_{i_1},\dots,g_{i_k})$ so by (2.5), $g=\omega(2r_1,\dots,2r_k;g_{i_0},g_{i_1},\dots,g_{i_k})$ for some integers $r_1,\dots,r_k$. ■

Now we can define an abelian group structure on each orbit. Let $a,b$ be elements of an orbit $G_i$, $i\in I$. By Lemma (3.4) and Definition (2.1),

$$a=\omega(2r_1,\dots,2r_k;g_i,g_{i_1},\dots,g_{i_k})=g_i\omega_{r_1}(g_i,g_{i_1})g_i\dots g_i\omega_{r_k}(g_i,g_{i_k})$$

and

$$b = \omega(2p_1, \dots, 2p_k; g_i, g_{i_1}, \dots, g_{i_k}) = g_i\omega_{p_1}(g_i, g_{i_1})g_i \dots g_i\omega_{p_k}(g_i, g_{i_k})$$

for some $i_1, \dots, i_k \in I - \{i\}$ and integers $r_1, \dots, r_k, p_1, \dots, p_k$.

Let us define a + b as follows:

$$(3.5) \qquad a + b := \omega(2r_1 + 2p_1, \dots, 2r_k + 2p_k; g_i, g_{i_1}, \dots, g_{i_k}) = g_i\omega_{r_1+p_1}(g_i, g_{i_1})g_i \dots g_i\omega_{r_k+p_k}(g_i, g_{i_k}).$$

We should prove that the definition of $a+b$ does not depend on the presentation of the elements $a$ and $b$ in the standard form. We use the following lemma.

3.6. LEMMA. *For arbitrary elements $a_0, a_1, \dots, a_n$ in $(G, \cdot)$ and arbitrary integers $r_1, \dots, r_n, p_1, \dots, p_n$, the equation $\omega(2r_1, \dots, 2r_k; a_0, a_1, \dots, a_n) = \omega(2p_1, \dots, 2p_n; a_1, \dots, a_n)$ holds if and only if $\omega(2r_1 - 2p_1, \dots, 2r_k - 2p_k; a_0, a_1, \dots, a_n) = \omega_0 = a_0$.*

Proof. This follows, by easy computation, from the axioms of SIE-groupoids and (2.5). ■

3.7. PROPOSITION. *The result of the operation + on elements a and b from the orbit $G_i$ does not depend on the presentation of a and b in the standard form.*

Proof. Let us suppose that

$$a = \omega(2r_1, \dots, 2r_k; g_i, g_{i_1}, \dots, g_{i_k}) = \omega(2r'_1, \dots, 2r'_k; g_i, g_{i_1}, \dots, g_{i_k})$$

and

$$b = \omega(2p_1, \dots, 2p_k; g_i, g_{i_1}, \dots, g_{i_k}) = \omega(2p'_1, \dots, 2p'_k; g_i, g_{i_1}, \dots, g_{i_k}).$$

By Lemma 3.6,

$$\omega(2r_1 - 2r'_1, \dots, 2r_k - 2r'_k; g_i, g_{i_1}, \dots, g_{i_k}) = g_i = \omega(2p'_1 - 2p_1, \dots, 2p'_k - 2p_k; g_i, g_{i_1}, \dots, g_{i_k}).$$

Applying Lemma 3.6 once more, we obtain

$$\omega(2r_1 - 2r'_1 - 2p'_1 + 2p_1, \dots, 2r_k - 2r'_k - 2p'_k + 2p_k; g_i, g_{i_1}, \dots, g_{i_k}) = g_i$$

Consequently

$$\omega(2r_1 + 2p_1, \dots, 2r_k + 2p_k; g_i, g_{i_1}, \dots, g_{i_k}) = \omega(2r'_1 + 2p'_1, \dots, 2r'_k + 2p'_k; g_i, g_{i_1}, \dots, g_{i_k})$$

Therefore the operation + is well defined. ■

3.8. REMARK. *The identity*

$$(3.9) \qquad a + b = ag_i\omega_{p_1}(g_i, g_{i_1})g_i \dots g_i\omega_{p_k}(g_i, g_{i_k})$$

*holds in the orbit* $G_i$, *for* $a = \omega(2r_1, \dots, 2r_k; g_i, g_{i_1}, \dots, g_{i_k})$ *and* $b = \omega(2p_1, \dots, 2p_k; g_i, g_{i_1}, \dots, g_{i_k})$.

Proof. It follows easily from (2.3) and (2.5). ∎

3.10. THEOREM. *On each orbit* $G_i$ *in an* SIE*-groupoid* $(G, \cdot)$ *the operation* $+$ *defines the structure of an abelian group. Moreover,* $(G_i, \cdot) = Core(G_i, +)$.

Proof. It easy to see that the operation $+$ is associative and commutative. Let

$$a = \omega(2r_1, \dots, 2r_k; g_i, g_{i_1}, \dots, g_{i_k}).$$

Since $g_i = \omega(0, \dots, 0; g_i, g_{i_1}, \dots, g_{i_k})$, the relation (3.5) yields $a + g_i = a$. Thus the element $g_i$ plays the role of the identity element in $(G_i, +)$. Moreover, if we put

$$-a := \omega(-2r_1, \dots, -2r_k; g_i, g_{i_1}, \dots, g_{i_k}) \tag{3.11}$$

then $a + (-a) = g_i$. Therefore $(G_i, +, g_i)$ is an abelian group. Since $ab = 2b - a$, for $a, b \in G_i$, it is obvious that $(G_i, \cdot) = Core(G_i, +)$. ∎

The groupoid $Core(G, +)$ and the abelian group $(G, +)$ are closely related. By induction it is easy to prove that the identity $\omega_k(x, y) = ky - (k-1)x$ is satisfied in every groupoid $Core(G, +)$, for each natural number $k$ (see [R2]). Therefore, if $G_i$ is an orbit in an SIE-groupoid $(G, \cdot)$, then the identity

$$\omega_k(g_i, y) = ky \tag{3.12}$$

holds in the groupoid $Core(G_i, +)$, for an each natural number $k$. Recall that each $(G, \cdot)$ in $V_n$ for an odd natural number $n > 1$ is a quasigroup [R2]. So for arbitrary elements $a, b$ in $G$ there exists an element $c$ in $G$ such that $a = bc$. This means that the congruence $\pi$ is trivial on $(G, \cdot)$ and "decomposes" $(G, \cdot)$ into just one class consisting of all elements of $G$. The groupoid $(G, \cdot)$ is equal to $Core(G, +)$ and the group $(G, +)$ satisfies the identity $nx = 0$ (see [R2]). More generally, one obtains the following:

3.13. COROLLARY. *If an* SIE*-groupoid* $(G, \cdot)$ *with orbits* $(G_i, \cdot)$ *belongs to the variety* $V_{2n}$ *then each abelian group* $(G_i, +)$ *satisfies the identity* $nx = 0$.

Proof. At first we will show that the groupoid $(G_i, \cdot)$ satisfies the identity $\omega_n(x, y) = x$. Let $a$ and $b$ be elements of the orbit $G_i$. Then $a = bb_1 \dots b_k$, for some $b_1, \dots, b_k$ from $G$. Hence, by (1.15), idempotence and (1.12) $\omega_n(b, a) = \omega_n(b, bb_1 \dots b_k) = \omega_n(b, b)\omega_n(b, b_1) \dots \omega_n(b, b_k) = b$. Thus the groupoid $(G_i, \cdot)$ satisfies the identity $\omega_n(x, y) = x$. Since $(Gi, \cdot) = Core(G_i, +)$, it follows by (3.12) that the group $(G_i, +)$ satisfies the identity $nx = 0$. ∎

3.14. REMARK. *Each* SIE*-groupoid* $(G,\cdot)$ *in the variety* $V_{2^s k}$ *for* $s>0$, *but not in the variety* $V_p$ *for* $p<2^s k$, *is decomposed into at least two classes.*

Proof. Since $(G,\cdot)$ is not in $V_{2^{s-1}k}$, there are elements $g_0, g_1$ in $G$ such that $\omega_{2^{s-1}k}(g_0,g_1)\neq g_0$. We will prove that the orbits $G_0$ and $G_1$ are disjoint. Let us suppose the contrary. Then $g_1\in G_0$. There is $\mathcal{G}:=\{g_i \mid i\in I\}$, a set of generators of the SIE-groupoid $(G,\cdot)$ such that $g_0,g_1\in\mathcal{G}$. By Lemma 3.4 $g_1=\omega(2r_1,\ldots,2r_n;g_i,g_{i_1},\ldots,g_{i_n})$ for some integers $r_1,\ldots,r_n$ and some $i_1,\ldots,i_n$ from $I$. Hence

$$\omega_{2^{s-1}}(g_0,\omega_k(g_0,g_1))=\omega_{2^{s-1}}(g_0,\omega_k(g_0,\omega(2r_1,\ldots,2r_n;g_i,g_{i_1},\ldots,g_{i_n}))).$$

From (1.16) it follows that

$$\omega_{2^{s-1}}(g_0,\omega_k(g_0,g_1))=\omega_{2^{s-1}k}(g_0,g_1)\neq g_0.$$

On the other hand, one may deduce using (2.1), (1.16), (1.15), (1.12) and Corollary 1.17 that

$$\begin{aligned}\omega_{2^{s-1}}&(g_0,\omega_k(g_0,\omega(2r_1,\ldots,2r_n;g_i,g_{i_1},\ldots,g_{i_n})))\\ &=\omega_{2^{s-1}}(g_0,\omega_k(g_0,g_0\omega_{r_1}(g_0,g_{i_1})g_0\cdots g_0\omega_{r_n}(g_0,g_{i_n})))\\ &=\omega_{2^{s-1}}(g_0,g_0\omega_{kr_1}(g_0,g_{i_1})g_0\cdots g_0\omega_{kr_n}(g_0,g_{i_n}))\\ &=g_0\omega_{2^{s-1}}(g_0,\omega_{kr_1}(g_0,g_{i_1}))g_0\cdots g_0\omega_{2^{s-1}}(g_0,\omega_{kr_n}(g_0,g_{i_n}))\\ &=g_0\omega_{2^{s-1}kr_1}(g_0,g_{i_1})g_0\cdots g_0\omega_{2^{s-1}kr_n}(g_0,g_{i_n})=g_0.\end{aligned}$$

Whence $G_0\cap G_1=\emptyset$. ∎

3.15. EXAMPLE. Let us consider the SIE-groupoid $(G,\cdot)$ given by the table:

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a$ | $a$ | $c$ | $a$ | $c$ | $b$ | $d$ |
| $b$ | $d$ | $b$ | $d$ | $b$ | $a$ | $c$ |
| $c$ | $c$ | $a$ | $c$ | $a$ | $d$ | $b$ |
| $d$ | $b$ | $d$ | $b$ | $d$ | $c$ | $a$ |
| $e$ | $f$ | $f$ | $f$ | $f$ | $e$ | $e$ |
| $f$ | $e$ | $e$ | $e$ | $e$ | $f$ | $f$ . |

It is easy to check that the groupoid $(G,\cdot)$ belongs to the variety $V_8$. The congruence relation $\pi$ decomposes $(G,\cdot)$ into two orbits: $G_a=\{a,b,c,d\}$ and $G_e=\{e,f\}$. Moreover, the group $(G_a,+,a)$ is isomorphic to $Z_4$, the cyclic group of order 4 and the group $(G_e,+,e)$ is isomorphic to $Z_2$, the cyclic group of order 2. Consequently, the groupoid $(G_a,\cdot)$ is isomorphic to the groupoid $Core(Z_4,+)$ and belongs to $V_4$. Likewise, the groupoid $(G_e,\cdot)$ is isomorphic to the groupoid $Core(Z_2,+)$ and belongs to $V_2$, which is the variety LZ of left zero bands. ∎

## 4. The structure theorem

In this section we will show that every SIE-groupoid $(G,\cdot)$ may be represented by means of a certain construction from reducts of abelian groups and left zero band.

4.1. DEFINITION. Let $I$ be a nonempty set. For each $i$ in $I$ let an abelian group $(G_i,+,g_i)$ be given. For each pair $(i,j)$ in $I^2$ let $h^i_j : G_i \to G_j$ be a mapping satisfying

(i) $$h^i_i(a) = 2a = a + a, \quad \text{for } a \in G_i,$$

(ii) $$h^i_n(-a + h^j_i(b)) = 2h^j_n(b) - h^i_n(a), \quad \text{for } a \in G_i,\ b \in G_j.$$

We can define a groupoid structure on the disjoint union $G$ of $G_i$, $i \in I$, by

(4.2) $$a \cdot b := -a + h^j_i(b), \quad \text{where } a \in G_i,\ b \in G_j.$$

Then evidently each $(G_i,\cdot)$ is a subgroupoid of $(G,\cdot)$. Moreover, if $a,b$ are in $G_i$, then $ab = -a + 2b$. Whence $(G_i,\cdot) = Core(G_i,+)$. It is obvious that if $a \in G_i$, $b \in G_j$, then $ab \in G_i$. Thus if we define $\cdot$ on the set $I$ by $i \cdot j := i$, then $(I,\cdot)$ is a left zero semigroup and the mapping

$$f : G \to I, \quad a \to i, \quad \text{for } a \in G_i,$$

is a homomorphism. The groupoid $(G,\cdot)$ is said to be *the sum of the abelian groups $(G_i,+,g_i)$ over the left zero semigroup $(I,\cdot)$ by the mappings $h^i_j$*, or more briefly the *AG-sum of the $(G_i,+,g_i)$*. ■

4.3. THEOREM. *A groupoid $(G,\cdot)$ is an* SIE*-groupoid if and only if it is a sum of abelian groups $(G_i,+,g_i)$ over a left-zero band $(I,\cdot)$ by some mappings $h^i_j$.*

Proof. /⇒/ Let us suppose that a set $\{g_i \mid i \in J\}$ generates an SIE-groupoid $(G,\cdot)$. Then there is a subset $I$ of $J$ such that $G$ is the disjoint sum of orbits $G_i$, where $i \in I$. Theorem 3.10 shows that on each orbit $G_i$ we have an abelian group structure $(G_i,+,g_i)$ defined by (3.5). Moreover $(G_i,\cdot) = Core(G_i,+)$. Let us define the mappings $h^i_j : G_i \to G_j$, for arbitrary $i,j \in I$, as follows:

(4.4) $$h^i_j(a) := g_j a \quad \text{for } a \in G_i.$$

Then by (1.4) and (3.12), $h^i_i(a) = g_i a = \omega_2(g_i,a) = 2a$, which gives (i). In order to prove (ii), let $a \in G_i$ and $b \in G_j$ for some $i,j \in I$. By Lemma 3.4, (2.2) and (2.3),

$$a = \omega(2r_1,\dots,2r_k; g_i,g_j,g_n,g_{i_1},\dots,g_{i_{k-2}})$$

and

$$b = \omega(2p_1,\dots,2p_k; g_j,g_i,g_n,g_{i_1},\dots,g_{i_{k-2}})$$

for some $i_1, \dots, i_{k-2} \in J - \{i, j, n\}$ and integers $r_1, \dots, r_k, p_1, \dots, p_k$. Then

$$h_n^i(-a + h_i^j(b)) = \text{/by (4.4)/}$$
$$h_n^i(-a + g_i\omega(2p_1, \dots, 2p_k; g_j, g_i, g_n, g_{i_1}, \dots, g_{i_{k-2}})) = \text{/by (2.5)/}$$
$$h_n^i(-a + \omega(4p_1 - 1, 4p_2, \dots, 4p_k; g_j, g_i, g_n, g_{i_1}, \dots, g_{i_{k-2}})) = \text{/by (2.7)/}$$
$$h_n^i\Big( - a + \omega\Big(2 - \sum_{s=1}^{k} 4p_s, 4p_2, \dots, 4p_k; g_i, g_j, g_n, g_{i_1}, \dots, g_{i_{k-2}}\Big)\Big) =$$
$$\text{/by (3.5), (3.11) and (4.4)/}$$
$$g_n\omega\Big(2 - 2r_1 - \sum_{s=1}^{k} 4p_s, 4p_2 - 2r_2, \dots, 4p_k - 2r_k; g_i, g_j, g_n, g_{i_1}, \dots, g_{i_{k-2}}\Big) =$$
$$\text{/by (2.5)/}$$
$$\omega\Big(4 - 4r_1 - \sum_{s=1}^{k} 8p_s, 8p_2 - 4r_2 - 1, \dots, 8p_k - 4r_k; g_i, g_j, g_n, g_{i_1}, \dots, g_{i_{k-2}}\Big) =$$
$$\text{/by (2.7)/}$$
$$\omega\Big(4 - 4r_1 - \sum_{s=1}^{k} 8p_s, \sum_{s=1}^{k} 4r_s + 8p_1 - 2, 8p_3 - 4r_3, \dots, 8p_k - 4r_k;$$
$$g_n, g_j, g_i, g_{i_1}, \dots, g_{i_{k-2}}\Big).$$

On the other hand,

$$2h_n^j(b) - h_n^i(a) = \text{/by (4.4)/}$$
$$2g_n b - g_n a = \text{/by (2.5)/}$$
$$2\omega(4p_1, 4p_2 - 1, 4p_3, \dots, 4p_k; g_j, g_i, g_n, g_{i_1}, \dots, g_{i_{k-2}})$$
$$-\omega(4r_1, 4r_2 - 1, 4p_3, \dots, 4r_k; g_i, g_j, g_n, g_{i_1}, \dots, g_{i_{k-2}}) = \text{/by (2.7)/}$$
$$2\omega\Big(4p_1, 2 - \sum_{s=1}^{k} 4p_s, 4p_3, \dots, 4p_k; g_n, g_i, g_j, g_{i_1}, \dots, g_{i_{k-2}}\Big)$$
$$-\omega\Big(4r_1, 2 - \sum_{s=1}^{k} 4r_s, \dots, 4r_k; g_n, g_j, g_i, g_{i_1}, \dots, g_{i_{k-2}}) = \text{/by (3.5)/}$$
$$\omega\Big(8p_1, 4 - \sum_{s=1}^{k} 8p_s, 8p_3, \dots, 8p_k; g_n, g_i, g_j, g_{i_1}, \dots, g_{i_{k-2}}\Big)$$
$$-\omega\Big(4r_1, 2 - \sum_{s=1}^{k} 4r_s, \dots, 4r_k; g_n, g_j, g_i, g_{i_1}, \dots, g_{i_{k-2}}) = \text{/by (2.7)/}$$
$$\omega\Big(4 - \sum_{s=1}^{k} 8p_s, 8p_1, 8p_3, \dots, 8p_k; g_n, g_j, g_i, g_{i_1}, \dots, g_{i_{k-2}}\Big)$$

$$-\omega\Big(4r_1, 2-\sum_{s=1}^{k} 4r_s, \ldots, 4r_k; g_n, g_j, g_i, g_{i_1}, \ldots, g_{i_{k-2}}\Big) = \text{/by (3.5) and (3.11)/}$$

$$\omega\Big(4 - 4r_1 - \sum_{s=1}^{k} 8p_s, \sum_{s=1}^{k} 4r_s + 8p_1 - 2, 8p_3 - 4r_3, \ldots, 8p_k - 4r_k; \\ g_n, g_j, g_i, g_{i_1}, \ldots, g_{i_{k-2}}\Big).$$

Therefore, $h_n^i(-a + h_i^j(b)) = 2h_n^j(b) - h_n^i(a)$ for $a \in G_i$, $b \in G_j$, which is the desired conclusion. It remains to show that the binary operation $\cdot$ on the SIE-groupoid $(G, \cdot)$ can be defined by (4.2). Let $a \in G_i$, $b \in G_j$. By Lemma 3.4 we can assume that

$$a = \omega(2r_1, \ldots, 2r_k; g_i, g_j, g_{i_1}, \ldots, g_{i_{k-1}})$$

and

$$b = \omega(2p_1, \ldots, 2p_k; g_j, g_i, g_{i_1}, \ldots, g_{i_{k-1}}).$$

From (4.4) and (2.7) it follows that

$$-a + h_i^j(b) = -a + g_i b$$

$$= \omega\Big(2 - 2r_1 - \sum_{s=1}^{k} 4p_s, 4p_2 - 2r_2, \ldots, 4p_k - 2r_k; g_i, g_j, g_{i_1}, \ldots, g_{i_{k-1}}\Big).$$

Similarly, by (2.7) and (2.5) we have

$$ab = \omega(2r_1, \ldots, 2r_k; g_i, g_j, g_{i_1}, \ldots, g_{i_{k-1}})$$

$$\cdot\omega\Big(1 - \sum_{s=1}^{k} 2p_s, 2p_2, \ldots, 2p_k; g_i, g_j, g_{i_1}, \ldots, g_{i_{k-1}}\Big)$$

$$= \omega\Big(2 - 2r_1 - \sum_{s=1}^{k} 4p_s, 4p_2 - 2r_2, \ldots, 4p_k - 2r_k; g_i, g_j, g_{i_1}, \ldots, g_{i_{k-1}}\Big)$$

which gives $ab = -a + h_i^j(b)$, and the proof of $/\Rightarrow/$ is complete.

$/\Leftarrow/$ Assume that a groupoid $(G, \cdot)$ is an AG-sum of abelian groups $(G_i, +, g_i)$, $i \in I$, by some mappings $h_i^j$. The groupoid $(G, \cdot)$ is easily seen to be idempotent and symmetric. In order to get entropicity we take $a \in G_i$, $b \in G_j$, $c \in G_k$, $d \in G_r$ for $i, j, k, r \in I$. Now entropicity of the groupoid $(G, \cdot)$ follows easily by (4.2) and (4.1(ii)). Indeed, applying (4.2) and (4.1(ii)), one obtains

$$ab \cdot cd = (-a + h_i^j(b)) \cdot (-c + h_k^r(d)) = -(-a + h_i^j(b)) + h_i^k(-c + h_k^r(d)) = \\ = a - h_i^j(b) - h_i^k(c) + 2h_i^r(d) = ac \cdot bd,$$

and the proof is complete. ∎

Note that condition (ii) in Definition 4.1 means that $h_n^i(ab) = h_n^i(a)h_n^j(b)$ for $a \in G_i$, $b \in G_j$. In particular, the mapping $h_j^i : G_i \to G_j$ is a homomorphism of SIE-groupoids. In general, the mapping $h_j^i$ is not a group homomorphism. Let a set $\{g_i \mid i \in I\}$ be a subset of a set $\{g_i \mid i \in J\}$ of generators of an SIE-groupoid $(G, \cdot)$. Assume that the SIE-groupoid $(G, \cdot)$ is the AG-sum of orbits $(G_i, +, g_i)$, $i \in I$ by the mappings $h_j^i$ defined by (4.4). Then we have the following proposition.

4.5. PROPOSITION. *The equality*

$$h_j^i(a - b + c) = h_j^i(a) - h_j^i(b) + h_j^i(b) \tag{4.6}$$

*holds for arbitrary* $i, j \in I$ *and* $a, b, c \in G_i$.

Proof. The proof follows easily by applying (3.4), the definitions of + and $h_j^i$, ((3.5) and (4.4)), and using (2.5), (2.7) and (3.11). ∎

Our next concern will be the structure of SIE-groupoids in the nontrivial subvarieties of the variety SIE. Before stating the result to be proved, let us note that the structure of SIE-groupoids in $V_{2^s k}$ for an odd natural number $k > 1$, is especially simple, and was described in [R2]. SIE-groupoids in $V_k$ are exactly the groupoids $Core(G, +)$ for abelian groups $(G, +)$ satisfying the identity $kx = 0$. The SIE-groupoids in $V_{2^s k}$, where $s > 0$ and $k > 1$, are the direct products of groupoids from $V_{2^s}$ and $V_k$. Accordingly, we are left with the task of describing the structure of SIE-groupoids in $V_{2^s}$. The following proposition yields information about SIE-groupoids which are AG-sums of abelian groups satisfying certain identities.

4.7. PROPOSITION. *Let an* SIE-*groupoid* $(G, \cdot)$ *be an AG-sum of abelian groups* $(G_i, +, g_i)$ *satisfying the identity* $nx = 0$ *for a natural number* $n$. *Then* $(G, \cdot)$ *is in the variety* $V_{2n}$.

Proof. From (3.12), it follows that each SIE-groupoid $(G_i, \cdot) = Core(G_i, +)$ belongs to the variety $V_n$. Let $a, b \in G$. Since the elements $a$ and $ab$ are in the same orbit $G_i$, $\omega_n(a, ab) = a$. According to (1.16), we have $\omega_{2n}(a, b) = \omega_n(a, ab) = a$, which is just the statement of the proposition. ∎

Combining Proposition 4.7 with Corollary 3.13 we obtain the following theorem.

4.8. THEOREM. *An* SIE-*groupoid* $(G, \cdot)$ *belongs to the variety* $V_{2n}$ *if and only if* $(G, \cdot)$ *is an AG-sum of abelian groups* $(G_i, +, g_i)$, $i \in I$, *satisfying the identity* $nx = 0$.

Theorem 4.8 generalises the structure theorems for groupoids in $V_4$ given in [Pł] and [RR].

## 5. An application

The construction given in Theorem 4.8 enables us to describe the free SIE-groupoid $F_{2n}(X)$ on the set $X = \{x_i \mid i \in I\}$ of free generators in the variety $V_{2n}$. Note that in [R2] it was proved that if $k$ is an odd natural number then in the variety $V_k$ the free SIE-groupoid $F_k(X)$ is isomorphic to the groupoid $Core(FA_k(X), +)$, where $(FA_k(X), +)$ denotes the free abelian group satisfying the identity $kx = 0$, generated by $X - \{x_0\}$ for arbitrary $x_0 \in X$. From Lemma 3.4, (2.2) and (1.12) one can conclude that each element in the orbit $F_{2n}(X)_i$ of $x_i$ in $F_{2n}(X)$ is equal to

$$x_i\omega_{r_i}(x_i, x_{i_1})x_i \ldots x_i\omega_{r_k}(x_i, x_{i_k}) \quad \text{for } i_1, \ldots, i_k \in I \text{ and } 0 \leq r_1, \ldots, r_k < n.$$

Each element of the free abelian group $FA_n(X)$ satisfying the identity $nx = 0$ and generated by $X - \{x\}$ is equal to

$$r_1x_{i_1} + \ldots + r_kx_{i_k} \quad \text{for } i_1, \ldots, i_k \in I - \{i\} \text{ and } 0 \leq r_1, \ldots, r_k < n.$$

It is easy to see that the mapping $f : F_{2n}(X)_i \to FA_n(X)$ defined by

$$x_i \mapsto 0$$

$$x_i\omega_{r_i}(x_i, x_{i_1})x_i \ldots x_i\omega_{r_k}(x_i, x_{i_k}) \mapsto r_1x_{i_1} + \ldots + r_kx_{i_k}$$

is a group isomorphism. Since for arbitrary generators $x_i, x_j$ in $X$ one has $\omega_n(x_i, x_j) \neq x_i$, it may be proved in much the same way as Remark 3.14 that the orbits $F_{2n}(X)_i$ and $F_{2n}(X)_j$ are disjoint. Summing up, we have thus proved the following theorem.

5.1. THEOREM. *Let $n$ be a natural number. The free* SIE-*groupoid $F_{2n}(X)$ on the set $X$ in the variety $V_{2n}$ is the $AG$-sum of $|X|$ copies of free abelian groups satisfying the identity $nx = 0$ on $|X| - 1$ generators.*

The Structure Theorem proved in this paper will be a starting point for describing all subdirectly irreducible SIE-groupoids in each nontrivial subvariety of the variety SIE.

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