DEMONSTRATIO MATHEMATICA
Vol. XXXII No 2 1999

**Anatolij Dvurečenskij, Maria Gabriella Graziano**

# ON REPRESENTATIONS OF COMMUTATIVE BCK-ALGEBRAS

**Abstract.** We introduce the relative cancellation property for commutative BCK-algebras and we study the problem of embedding of commutative BCK-algebras into Abelian lattice ordered groups. We show that if a BCK-algebra is not directed upwards, then we cannot use the method of Wyler and Baer. Anyway, supposing either union property or unitary extendibility, we can present the embedding of such a BCK-algebra into the positive cone of an Abelian lattice ordered group with universal property. Finally, some interesting examples are presented.

## 1. Introduction

BCK-algebras were introduced by Imai and Iséki [ImIs], [Ise], and they have been intensively studied by several authors ([MeJu] is an interesting monograph about BCK-algebras). A very important class of BCK-algebras is the class of commutative BCK-algebras [RoTr], and in particular, of bounded commutative BCK-algebras which are categorically equivalent to MV-algebras [Mun 1]. For MV-algebras there is a representation theorem by Mundici [Mun] via intervals $[0, u]$ with a strong unit $u$ in an Abelian lattice ordered group. In this case we have also a universal property.

If a commutative BCK-algebra is directed upwards, its representation via Abelian lattice ordered groups with universal property was made in [DvGr].

In the present paper, we shall study a representation of a class of commutative BCK-algebras which are not directed upwards, in general, via lattice ordered groups with universal property. We show that the "words" method

---

The paper has been partially supported by grant 2/4033/97 SAV, Bratislava, Slovakia, and by Ministero dell'Università e della Ricerca Scientifica e Tecnologica, and by Gruppo Nazionale per l'Analisi Funzionale e Applicazioni, Italy.

*Key words and phrases*: BCK-algebra, commutative BCK-algebra, relative cancellation property, lattice ordered group, unitary extendibility, union property.

1991 *Mathematics Subject Classification*: 06F35, 03G12.

of Baer [Bae], and applied by Wyler [Wyl] to clans, does not work in the case of commutative BCK-algebras which are not directed upwards.

More information we obtain if we shall study commutative BCK-algebras with relative cancellation property.

The paper is organized as follows. In section 2, we give some lattice-like properties of commutative BCK-algebras and we introduce the relative cancellation property which is a central notion. Some Riesz properties for commutative BCK-algebras known in lattice ordered groups are presented in Section 3. A "words" technique of Baer [Bae] and Wyler [Wyl] is presented in Section 4. Finally, we shall study union BCK-algebras (Section 5) and union BCK-algebras which are unitarily extendible [Hoo] (Section 6). For them we find a lattice ordered group representation with universal property. These results generalize the famous result of Mundici [Mun] for MV-algebras. The paper is accomplished with plenty of illustrating examples.

The relative cancellation property is a necessary condition for a representation of commutative BCK-algebras via Abelian lattice ordered groups. In generally, the problem of representation of commutative BCK-algebras with this property seems to be open, and the authors hope to solve it.

DEFINITION 1.1. A *BCK-algebra* is a non-empty set $X$ with a binary operation $*$ and with a constant element 0 such that the following axioms are satisfied: for all $x$, $y$, $z \in X$,

(BCK-1) $((x*y)*(x*z))*(z*y)=0$;
(BCK-2) $(x*(x*y))*y=0$;
(BCK-3) $x*x=0$;
(BCK-4) $x*y=0$ and $y*x=0$ imply $x=y$;
(BCK-5) $0*x=0$.

We write $X=(X;*,0)$. In $X$, there is a binary relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad x*y=0.$$

This is a partial order relation on $X$ which is called the BCK-*order*. We recall some elementary properties of BCK-algebras [MeJu, Thms I.1.2, I.1.3, I.1.4]: if $(X;*,0)$ is a BCK-algebra, then for all $x$, $y$, $z \in X$,

(a) $x \leq y$ implies $z*y \leq z*x$;
(b) $(x*y)*z=(x*z)*y$;
(c) $(x*z)*(y*z) \leq x*y$;
(d) $x \leq y$ implies $x*z \leq y*z$;
(e) $x*y \leq x$;
(f) $x*0=x$.

If we denote by $x \wedge y := y * (y * x)$, $x, y \in X$, $x \wedge y$ is a lower bound of $x$ and $y$, $x \wedge x = x$, $x \wedge 0 = 0 = 0 \wedge x$, and it is true that

$$x * (y \wedge x) = x * y, \quad x, y \in X. \tag{1.1}$$

In general, $x \wedge y \neq y \wedge x$, and a BCK-algebra is said to be *commutative* if it satisfies, for all $x$, $y \in X$,

$$x * (x * y) = y * (y * x), \quad x, y \in X. \tag{1.2}$$

In this case, $x \wedge y = y \wedge x$ is the greatest lower bound of $x$ and $y$, and $(X; *, 0)$ is a lower semilattice with respect to the BCK-order. According to [MeJu, Thm I.5.6], a BCK-algebra $X$ is commutative if and only if $x \leq y$ implies $x = y * (y * x)$.

## 2. Lattice properties of commutative BCK-algebras

We recall that it follows from [CST] that any commutative BCK-algebra is a distributive near lattice, therefore the following result holds.

THEOREM 2.1. *Let $(X; *, 0)$ be a commutative BCK-algebra, and let $x, y, z, z_1 \in X$.*

(i) *Let $x, y \leq z$. Define*

$$x \vee_z y := z * ((z * x) \wedge (z * y)). \tag{2.1}$$

*Then $x \leq x \vee_z y$, $y \leq x \vee_z y$ and, for any $z_1 \in X$ with $z_1 \geq z$, we have*

$$x \vee_z y = x \vee_{z_1} y. \tag{2.2}$$

(ii) *$x \vee y$ exists in $X$ if and only if there exists an element $z \in X$ such that $x, y \leq z$.*

(iii) *If $x \vee y$ exists in $X$, then $(z \wedge x) \vee (z \wedge y)$ exists in $X$ for any $z \in X$, and*

$$z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y). \tag{2.3}$$

(iv) *If $(X; *, 0)$ is directed upwards,[1] then $X$ is a distributive lattice with respect to the BCK-order.*

We say that a commutative BCK-algebra $(X; *, 0)$ has the *relative cancellation property* if, for $a, x, y \in X$, $a \leq x, y$ with $x * a = y * a$ imply $x = y$.

LEMMA 2.2. *Let $(X; *, 0)$ be any upwards directed commutative BCK-algebra. Then it has the relative cancellation property.*

---

[1]A poset $(X; \leq)$ is said to be *directed upwards* if given $a, b \in X$, there is an element $c \in X$ such that $a \leq c$ and $b \leq c$.

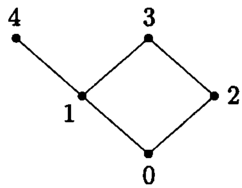
P r o o f. Suppose that there exist $c, c' \in X$ such that $a \leq c, c'$ and $c{*}a = c'{*}a$. We have to prove that $c = c'$. Let $u \in X$ be such that $c, c' \leq u$.

Then $(u * a) * (u * c) = (u * (u * c)) * a = (u \wedge c) * a = c * a$. Since $a \leq c$, we have $u * c \leq u * a$ and then $[u * (c * a)] * a = (u * a) * (c * a) = (u * a) * [(u * a) * (u * c)] = (u * a) \wedge (u * c) = u * c$.

In the same way we show that $[u * (c' * a)] * a = u * c'$. From $c * a = c' * a$, it follows $u * c' = u * c$. Hence $c = u * (u * c) = u * (u * c') = c.'$ Q.E.D.

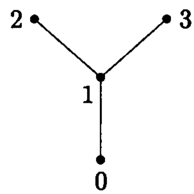
EXAMPLE 2.3. The example $B_{4-3-3}$ from [MeJu], where $*$ is given by the table (next to it there is its Hasse diagram), gives a commutative BCK-algebra which is not upwards directed, consequently it is not a lattice. It has not the relative cancellation property. Indeed, we have $1 \leq 2, 3$ and $2 * 1 = 1 = 3 * 1$ but $2 \neq 3$, consequently, $X$ cannot be embedded into the positive cone of a lattice ordered group.

| $*$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 1 | 0 | 1 |
| 3 | 3 | 1 | 1 | 0 |




EXAMPLE 2.4. The example $B_{5-2-7}$ from [MeJu], where $*$ is given by the table, gives a commutative BCK-algebra which has the relative cancellation property but which is not directed upwards.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 | 2 | 0 |
| 3 | 3 | 1 | 3 | 0 | 3 |
| 4 | 4 | 4 | 2 | 4 | 0 |




THEOREM 2.5. *Let $(X; *, 0)$ be a commutative BCK-algebra having the relative cancellation property. Define a partial binary operation $+$ with the domain $S = \{(a, b) \in X \times X :$ there exists $c \in X$ with $c \geq b$ and $a = c{*}b\}$, and we define $c = a + b$ if $c \geq b$, and $a = c * b$. Then, for all $a, b, c \in X$, we have*

(I) *$a + 0$, $0 + a$ exist for any $a \in X$, and $a + 0 = a = 0 + a$.*

(II) *$a + b$ exists if and only if $b + a$ exists, and then $a + b = b + a$ (the commutativity).*

(III) *$a + b$ and $(a + b) + c$ exist in $X$ if and only if $b + c$ and $a + (b + c)$ exist in $X$, and in this case $(a + b) + c = a + (b + c)$ (the associativity).*

(IV) *If* $a+b$ *exists in* $X$ *and if* $a_1 \leq a,\ b_1 \leq b$, *then* $a_1+b_1$ *exists in* $X$, *and* $a_1+b_1 \leq a+b$.

(V) $a=b$ *holds whenever* $u+a=u+b$ *for some* $u \in X$ (*the cancellation property*).

(VI) *If* $a+b \leq a+b'$ *then* $b \leq b'$. *Conversely, if* $b \leq b'$ *and* $a+b'$ *exists in* $X$, *then* $a+b \leq a+b'$.

(VII) *If* $a+c$ *and* $b+c$ *exist in* $X$, *then* $(a \wedge b)+c$ *exists in* $X$, *and*

$$(a+c) \wedge (b+c) = (a \wedge b)+c \tag{2.4}$$

(VIII) *Let* $u \geq x, y$. *Then* $x \vee y$ *exists in* $X$, *and*

$$(x \vee y) * y = x * (x \wedge y) = x * y, \tag{2.5}$$

*and* $(x*(x \wedge y))+y$, $(x*y)+y$ *exist in* $X$, *and*

$$x \vee y = (x*(x \wedge y))+y = (x*y)+y. \tag{2.6}$$

Proof. The relative cancellation property entails that the partial binary operation $+$ is defined correctly.

(I) and (II) are evident. To prove prove (III) suppose that $a+b$ and $(a+b)+c$ exist in $X$. Then there exist $x, y \in X$ such that $a \leq x$ and $x*a=b$, $a+b=x \leq y$ and $y*x=y*(a+b)=c$.

Since $c=y*x \leq y*a$ and $(y*a)*c=(y*a)*(y*x)=[y*(y*x)]*a= x*a=b$, then $b+c \in X$ and $b+c=y*a$.

From $a \leq y$ and $y*a=b+c$, it follows $(a+(b+c)) \in X$ and $a+(b+c)= y=(a+b)+c$.

(IV) There are $x, y$ in $X$ such that $a_1+x=a$ and $b_1+y=b$. Using commutativity and associativity, we have $a+b=(a_1+x)+(b_1+y)= (a_1+b_1)+(x+y)$ which entails the assertion.

(V) It follows easily from definition of the relative cancellation property.

(VI) There exists $c \in X$ such that $(a+b)+c=a+b'$. Using associativity and commutativity of $+$, we have $a+(b+c)=a+b'$ and $(b+c)+a=b'+a$, so that $b+c=b'$ and, consequently, $b \leq b'$. The second part follows from (IV).

(VII) Since $a+c \geq c$ and $b+c \geq c$, we have $(a+c) \wedge (b+c) \geq c$. Then using property (b), we have

$$((a+c) \wedge (b+c))*c = ((a+c)*((a+c)*(b+c)))*c = a*((a+c)*(b+c)).$$

Calculate

$$a*b = ((a+c)*c)*((b+c)*c) \leq (a+c)*(b+c),$$

when we have used property (c) of BCK-algebras, so that

$$a*((a+c)*(b+c)) \leq a*(a*b) = a \wedge b.$$

Hence,

$$((a+c)\wedge(b+c))*c \le a\wedge b. \tag{2.7}$$

Since $a \ge a\wedge b$ and $a+c$ exists in $X$, we have by (IV) that $(a\wedge b)+c$ exists in $X$. It is clear that $a+c \ge (a\wedge b)+c$ and $b+c \ge (a\wedge b)+c$, so that

$$\begin{aligned}(a+c)\wedge(b+c) &\ge (a\wedge b)+c,\\ ((a+c)\wedge(b+c))*c &\ge ((a\wedge b)+c)*c,\\ ((a+c)\wedge(b+c))*c &\ge a\wedge b,\end{aligned}$$

which in view of (2.7) entails $((a+c)\wedge(b+c))*c = a\wedge b$, and, consequently, we have (2.4).

(VIII) Due to (1.1), we have $x*(x\wedge y) = x*y$. Using Theorem 2.1 (ii), definition (2.1), and the basic properties (b), (1.1) and (1.2), we have

$$\begin{aligned}(x\vee y)*y &= (u*((u*x)\wedge(u*y))*y\\ &= (u*y)*((u*x)\wedge(u*y))\\ &= (u*y)*(u*x) = (u*(u*x))*y = x*y,\end{aligned}$$

which proves (2.5) and, consequently, (2.6). Q.E.D.

In the paper [DvGr, Thm 3.4], we have proved that any upwards directed commutative BCK-algebra has the relative cancellation property, and it can be converted via Theorem 2.5 into a commutative minimal clan. Motivated by this, we call the structure $(X;+,*,0)$ the BCK-*clan*, where $+$ is derived from Theorem 2.5, supposing that $(X;*,0)$ is a commutative BCK-algebra having relative cancellation property.

EXAMPLE 2.6. Let $G=(G;+,\le,0)$ be a lattice ordered group and let $G^+ := \{x\in G : x\ge 0\}$ be its positive cone. Define a total binary operation $*_G$ on $G^+$ via

$$x *_G y := (x-y)\vee 0, \tag{2.8}$$

for all $x,y\in G^+$. Then

$$x *_G y = x-(x\wedge y) = (x\vee y)-y, \quad x,y\in G^+, \tag{2.9}$$

and $(G^+;*_G,0)$ is an upwards directed commutative BCK-algebra having the relative cancellation property.

Let $G_0$ be a non-void subset of $G^+$ such that $x,y\in G_0$ entail $x*_G y\in G_0$. Then $(G_0;*_G,0)$ is a commutative BCK-subalgebra of $(G^+;*_G,0)$ having the relative cancellation property, which is a lower semilattice.

Our aim is to prove that some commutative BCK-algebras having relative cancellation property can be embedded onto some $(G_0;*_G,0)$ of some lattice ordered group $(G;+,\le,0)$ with $G_0\subseteq G^+$.

## 3. BCK-algebras and the Riesz decomposition property

A poset $(X;\leq)$ has the *interpolation property* (or simply has *interpolation*), iff $x,y,p,q\in X$ with $x,y\leq p,q$ imply that there exists $z\in X$ with $x,y\leq z\leq p,q$. A BCK-clan $(X;+,*,0)$ has the *Riesz decomposition property* iff $a,b,x\in X$ with $x\leq a+b$ imply that there exist $a_1,b_1\in X$ with $a_1\leq a$, $b_1\leq b$ and $x=a_1+b_1$.

It is evident that any commutative BCK-algebra $(X;*,0)$ has interpolation. Indeed, let $x,y\leq p,q$ then $x,y\leq p\wedge q\leq p,q$.

EXAMPLE 3.1. Let $\Omega\neq\emptyset$ and let $A,B$ be two non-void disjoint subsets of $\Omega$. Define $E:=\{X\subseteq\Omega : X\subseteq A \text{ or } X\subseteq B\}$. Then $(E;*,\emptyset)$, where $*$ is the set-theoretical difference, is a commutative BCK-algebra having relative cancellation property which is not directed upwards. The BCK-clan $(E;+,\setminus,\emptyset)$ is not a minimal clan but it has the Riesz decomposition property.

Similarly, it is possible to show, (see e.g. [Wyl, Thm 5.9]) that any commutative clan has the Riesz decomposition property, consequently, any BCK-clan $(X;+,*,0)$ has it whenever $X$ is directed upwards; this follows also from a more general proposition:

THEOREM 3.2. *Let $(X;*,0)$ be a commutative BCK-algebra having the relative cancellation property. Then the BCK-clan $(X;+,*,0)$ has the Riesz decomposition property.*

Proof. Suppose that $c\leq a+b$, and define $u:=a+b$ and $X_u:=\{x\in X : x\leq u\}$. Put now $v=c*(b\wedge c)$ and $a_1=a\wedge v$, so that $a_1\leq a$. On the other hand, $c*v=b\wedge c$ and if we put $b_1=c*a_1$, then $a_1,b_1$ are defined in $X_u$ and, moreover, $a_1+b_1=c$.

We must show that $b_1\leq b$. Applying (2.6) and (2.4), we obtain

$$\begin{aligned} a_1+b_1=c &\leq (a+b)\wedge(b\vee c)\\ &=(a+b)\wedge[(c*(b\wedge c))+b]\\ &=(a+b)\wedge(v+b)\\ &=(a\wedge v)+b=a_1+b, \end{aligned}$$

and thus by (VI), $b_1\leq b$. Q.E.D.

LEMMA 3.3. *Let, for $a_1,a_2,b_1,b_2\in X$, we have $a_1+a_2=b_1+b_2$, where $(X;+,*,0)$ is a BCK-clan. Then there are $c_{11},c_{12},c_{21},c_{22}\in X$ such that*

$$\begin{aligned} a_1&=c_{11}+c_{12}, & b_1&=c_{11}+c_{21},\\ a_2&=c_{21}+c_{22}, & b_2&=c_{12}+c_{22}. \end{aligned}$$

Proof. Since $a_1\leq b_1+b_2$, due to Theorem 3.2, we have that $a_1=c_{11}+c_{12}$, where $c_{11}\leq b_1$ and $c_{12}\leq b_2$. Then there are $c_{21},c_{22}\in X$ such that $c_{11}+c_{21}=b_1$ and $c_{12}+c_{22}=b_2$.

Calculate $(c_{11}+c_{12})+a_2=(c_{11}+c_{21})+(c_{12}+c_{22})$. Using associativity and cancellation (V), we have $a_2=c_{21}+c_{22}$. Q.E.D.

## 4. Construction of a derived semigroup

Throughout this section we shall suppose that $X=(X;+,*,0)$ is a BCK-clan derived from a commutative BCK-algebra $(X;*,0)$ which has the relative cancellation property. The aim of this section is a construction of an ordered Abelian semigroup in that $X$ can be embedded preserving $*$ and $+$; that will be possible, for instance, if $X$ is directed upwards.

We shall follow ideas of Wyler [Wyl] who used Baer's approach [Bae].

We define $[X]:=\bigcup_{n=1}^{\infty}X^n$. An element $A=(x_1,\dots,x_n)\in[X]$ is said to be a *word* with entries (letters) $x_1,\dots,x_n$ from the alphabet $X$, and $n$ is the length of the word $A$. Two words $(x_1,\dots,x_n)$ and $(y_1,\dots,y_m)$ are the same if $n=m$ and $x_i=y_i$ for any $i$ with $1\le i\le n$.

$[X]$ can be organized into an additive semigroup via

$$(x_1,\dots,x_n)+(y_1,\dots,y_m):=(x_1,\dots,x_n,y_1,\dots,y_m),$$

where $(x_1,\dots,x_n),\ (y_1,\dots,y_m)\in[X]$.

We call two words $A+(a,b)+B$ and $A+(c)+B$, where $A,B\in[X]$ and $a,b,c\in X$, *directly similar* if $a+b=c$. Here $A$ or $B$ or both may be the empty words. For $A,B\in[X]$, we write $A\sim B$ iff $A$ and $B$ are directly similar. Then (i) $A\sim A$, and (ii) if $A\sim B$, then $B\sim A$. Let $\simeq$ be the transitive closure of $\sim$, i.e., $A\simeq B$ iff there is a finitely many words $A_1,\dots,A_n\in[X]$ such that $A=A_1\sim A_2\sim\cdots\sim A_n=B$.

Then $\simeq$ is a congruence relation. Indeed, suppose that $A\simeq B$ and $C\simeq D$. Without loss of generality we can assume that there are two sequences of words $A_1,\dots,A_n$ and $C_1,\dots,C_m$ with $n=m$ such that $A=A_1\sim A_2\sim\cdots\sim A_n=B$ and $C=C_1\sim C_2\sim\cdots\sim C_n=D$. Then $A+C=A+C_1\sim A+C_2\sim\cdots\sim A+C_n=A+D\sim A_1+D\sim A_2+D\sim\cdots\sim A_n+D=B+D$.

Denote by $S(X)$ the quotient semigroup $[X]/\simeq$; we call it a *derived semigroup* of a BCK-algebra $(X;*,0)$. Given a word $A\in[X]$, we denote by $[A]$ the corresponding equivalence class in $S(X)$ given by $A$. Hence $[(0)]$ is the neutral element of $S(X)$, and if $(a_1,\dots,a_n)\in[X]$, we have

$$(a_1,\dots,a_n)=(a_1)+\cdots+(a_n),$$

so that

$$[(a_1,\dots,a_n)]=[(a_1)]+\cdots+[(a_n)]. \tag{4.1}$$

According to Baer [Bae], given a word $(a_1,\dots,a_n)$, inductively define a subset of $X$, $\mathcal{S}(a_1,\dots,a_n)$, as follows: (i) if $n=1$, define $\mathcal{S}(a_1)=\{a_1\}$, (ii) if $n>1$, then $a\in\mathcal{S}(a_1,\dots,a_n)$ iff there exist an integer $i$ and $b,c\in X$ with $1\le i\le n-1$, $b\in\mathcal{S}(a_1,\dots,a_i)$, $c\in\mathcal{S}(a_{i+1},\dots,a_n)$ such that $a=b+c$.

LEMMA 4.1. *Let* $(X;+,*,0)$ *be a BCK-clan.*

(i) *If* $a \in \mathcal{S}(a_1,\dots,a_n)$, *where* $n > 1$, *then there exists an integer* $i$ *with* $1 \le i < n$ *such that* $a_i + a_{i+1}$ *exists in* $X$ *and* $a \in \mathcal{S}(a_1,\dots,a_{i-1},a_i + a_{i+1},a_{i+2},\dots,a_n)$.

(ii) *If* $1 \le i < n$, $n > 1$, *and if* $a_i + a_{i+1}$ *exists in* $X$, *then* $\mathcal{S}(a_1,\dots,a_{i-1}, a_i + a_{i+1},a_{i+2},\dots,a_n) \subseteq \mathcal{S}(a_1,\dots,a_n)$.

(iii) *If* $a_1 + \dots + a_n = a$, *then* $a \in \mathcal{S}(a_1,\dots,a_n)$.

(iv) *If* $\mathcal{S}(a_1,\dots,a_n) \neq \emptyset$, *then* $a := a_1 + \dots + a_n$ *exists in* $X$, *and* $\mathcal{S}(a_1,\dots,a_n) = \{a\}$.

(v) *For any word* $A \in [X]$ *we have* $|\mathcal{S}(A)| \le 1$; *and if* $(a_1,\dots,a_n) \sim (b_1,\dots,b_{n+1})$ *and* $\mathcal{S}(a_1,\dots,a_n) \neq \emptyset$, *then* $\mathcal{S}(b_1,\dots,b_{n+1}) \neq \emptyset$.

(vi) *If* $(a_1,\dots,a_n) \sim (b_1,\dots,b_{n+1})$ *and* $\mathcal{S}(a_1,\dots,a_n) \neq \emptyset$, *then* $\mathcal{S}(a_1,\dots,a_n) = \mathcal{S}(b_1,\dots,b_{n+1})$.

Proof. (i) Let $n = 2$. If $a \in \mathcal{S}(a_1,a_2)$, then $a = a_1' + a_2'$, where $a_1' \in \mathcal{S}(a_1)$, $a_2' \in \mathcal{S}(a_2)$, and by definition, $a_1 = a_1'$, $a_2 = a_2'$, consequently $a \in \mathcal{S}(a_1+a_2)$. Let now $n = 3$ and $a \in \mathcal{S}(a_1,a_2,a_3)$. Then there are $b,c \in X$ with $a = b + c$ such that either $b \in \mathcal{S}(a_1)$ and $c \in \mathcal{S}(a_2,a_3)$ or $b \in \mathcal{S}(a_1,a_2)$ and $c \in \mathcal{S}(a_3)$. In the first case, by induction step for $n = 2$, we have $c \in \mathcal{S}(a_2 + a_3)$, consequently, $a \in \mathcal{S}(a_1, a_2 + a_3)$. Similarly for the second case. The general case of $n$ follows from complete induction.

(ii) Let $n = 2$ and let $a_1 + a_2$ exist in $X$. Then $a_1 + a_2 \in \mathcal{S}(a_1,a_2)$. Since $\mathcal{S}(a_1 + a_2) = \{a_1 + a_2\}$, we conclude $\mathcal{S}(a_1 + a_2) \subseteq \mathcal{S}(a_1,a_2)$. Let now $n = 3$. Suppose, for example, $a_1 + a_2$ exists in $X$ and let $a \in \mathcal{S}(a_1 + a_2, a_3)$. Then $a = b + c$, where $b \in \mathcal{S}(a_1 + a_2) \subseteq \mathcal{S}(a_1,a_2)$ (in view of induction step for $n = 2$) and $c \in \mathcal{S}(a_3)$. Hence $a = (a_1 + a_2) + a_3 \in \mathcal{S}(a_1,a_2,a_3)$. Similarly we exhibit the second possibility. The rest of this proof uses the complete induction and the associativity of $+$.

(iii) Let $A = (a_1,\dots,a_n)$. If $n = 1$, we have $a = a_1$ and $a_1 \in \mathcal{S}(A) = \{a_1\}$, so that $a \in \mathcal{S}(a_1)$. If $n = 2$, then $a = a_1' + a_2'$, where $a_1' \in \mathcal{S}(a_1) = \{a_1\}$ and $a_2' \in \mathcal{S}(a_2) = \{a_2\}$. Therefore, $a = a_1 + a_2 \in \mathcal{S}(a_1,a_2)$. The general case of $n$ can be proved by complete induction.

(iv) If $n = 1$, the statement holds. Let now $n = 2$, then there exists $a \in \mathcal{S}(a_1,a_2)$ such that $a = a_1 + a_2$, which proves $\mathcal{S}(a_1,a_2) = \{a_1 + a_2\}$. Let now $n = 3$ and suppose $a \in \mathcal{S}(a_1,a_2,a_3)$. Then $a = b_1 + b_2$, where either $b_1 \in \mathcal{S}(a_1)$ and $b_2 \in \mathcal{S}(a_2,a_3)$, or $b_1 \in \mathcal{S}(a_1,a_2)$ and $b_2 \in \mathcal{S}(a_3)$. In the first case we have $\mathcal{S}(a_1) = \{a_1\} = \{b_1\}$ and $b_2 = a_2 + a_3$. Using associativity of $+$, we have $a = a_1 + a_2 + a_3$ and $\mathcal{S}(a_1,a_2,a_3) = \{a\}$. In analogical way we proceed in the second case.

Similarly we obtain the general case of $n$ by complete induction.

(v) Due to (iii) and (iv) either $\mathcal{S}(A) = \emptyset$ or $|\mathcal{S}(A)| = 1$.

Let $n=1$, then $a=a_1=b_1+b_2$, so that $a\in\mathcal{S}(b_1,b_2)$. Let $n=2$. By (iv), $a=a_1+a_2$ exists in $X$, and either $a_1=b_1+b_2$ and $a_2=b_3$ or $a_1=b_1$ and $a_2=b_2+b_3$. For the first case we have $a_1\in\mathcal{S}(b_1,b_2)$ and $a_2\in\mathcal{S}(b_3)$, so that $a=a_1+a_2\in\mathcal{S}(b_1,b_2,b_3)\neq\emptyset$.

For the second case we have $a_1\in\mathcal{S}(b_1)$ and $a_2\in\mathcal{S}(b_2,b_3)$. Using again (iv), we have $a=a_1+a_2=b_1+b_2+b_3\in\mathcal{S}(b_1,b_2,b_3)$. The general case of $n$ follows from complete induction.

(vi) It follows from (v) and (iii), (iv). Q.E.D.

LEMMA 4.2. *Let* $(X;+,*,0)$ *be a BCK-clan. Then* (i) $(a_1,\dots,a_n)\simeq(b)$ *if and only if* $a_1+\cdots+a_n$ *exists in* $X$ *and equals* $b$; (ii) *if* $[(a)]=[(b)]$ *then* $a=b$.

Proof. It follows from (v) Lemma 4.1 and the results of Baer [Bae, (iv), (v) Thm 1.1] (compare also [Wyl]). Q.E.D.

LEMMA 4.3. *Let* $a$ *and* $b$ *from* $X$ *have an upper bound in* $X$. *Then*

$$[(a)]+[(b)]=[(b)]+[(a)]. \tag{4.2}$$

*In particular, if* $X$ *is directed upwards, then* $S(X)$ *is a commutative semigroup.*

Proof. Let $u\in X$ be an upper bound of $a$ and $b$. Then due to (ii) Theorem 2.1, $a\vee b$ exists in $X$. Using (2.5) and (2.6), we have

$$\begin{aligned}(a,b)&\sim(a\wedge b,a*(a\wedge b),b)=(a\wedge b,(a\vee b)*b,b)\\&\sim(a\wedge b,a\vee b)\sim(a\wedge b,(a\vee b)*a,a)\\&=(a\wedge b,b*(a\wedge b),a)\sim(b,a).\end{aligned}$$

The second assertion follows from (4.1) and (4.2). Q.E.D.

LEMMA 4.4. *Let, for* $A,B\in[X]$, *we have* $[A]+[B]=[(0)]$. *Then* $[A]=[(0)]=[B]$.

Proof. It follows easily from Lemma 4.2(i). Q.E.D.

We note that if $X$ is not directed upwards, then (4.2) can fail. Indeed, take $X$ from Example 2.4. Then the partial binary operation $+$ is given by the table below, and here we have $[(1)]+[(2)]=[(1,2)]\neq[(2,1)]=[(2)]+[(1)]$.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 3 | | | |
| 2 | 2 | | 4 | | |
| 3 | 3 | | | | |
| 4 | 4 | | | | |

The following result was proved originally by Wyler [Wyl, Thm 5.4] for commutative clans. His proof works also in our case because it does not depend on the existence of suprema in $X$, and therefore, the proof is omitted (see also Ravindran [Rav] for a special case of $X$).

LEMMA 4.5. *Let $A, B, W \in [X]$ such that $[A] + [B] = [W]$. Then there are elements $c_i', c_i'' \in X$, for $i = 1, \dots, k$, such that $W = (c_1' + c_1'', \dots, c_k' + c_k'')$, and $A \simeq (c_1', \dots, c_k')$, $B \simeq (c_1'', \dots, c_k'')$.*

If we assume that $X$ is directed upwards, then we can say more about the derived semigroup $S(X)$.

THEOREM 4.6. *Let $(X; +; *, 0)$ be an upwards directed BCK-clan. Then, for all $A, B, C \in [X]$, we have*

(i) $[A] + [B] = [B] + [A]$.
(ii) *If $[A] + [B] = [A] + [C]$, then $[B] = [C]$.*
(iii) *$\leq_X$, where $[A] \leq_X [B]$ if and only if there is an $C \in [X]$ such that $[A] + [C] = [B]$, is a partial order on $S(X)$.*
(iv) $[(a)] \wedge [(b)] = [(a \wedge b)]$, $a, b \in X$.
(v) *$(S(X); +, \leq_X, [(0)])$ is an Abelian partially ordered semigroup which is a lower semilattice and it is the generating positive cone of a lattice ordered group $G(X)$.*
(vi) *A mapping $h : X \to G(X)$ defined via*
$$h(a) := [(a)], \ a \in X,$$
*is injective and preserving $\wedge$, and $+$.*
(vii) *The couple $(G(X), h)$ has the universal property, i.e., if $g : X \to G_1$ is an order and $+$ preserving mapping into a partially ordered Abelian group $G_1$, then $g = g' \circ h$ for a uniquely determined group homomorphism of ordered Abelian groups $g' : G(X) \to G_1$.*
(viii) *If $X$ is bounded with the greatest element $1 \in X$, then $h(1)$ is a strong unit in $G(X)$ and $h$ maps $X$ onto the order interval $[[(0)], [(1)]] := \{g \in G(X) : [(0)] \leq_X g \leq_X [(1)]\}$.*[2]
(ix) *If $g \in G(X)$, $0 \leq g \leq h(x)$ for some $x \in X$, then there exists a unique $y \in X$ such that $y \leq x$ and $h(y) = g$.*

Proof. This result has been proved in [DvGr, Thm 3.5] using Wyler's ideas [Wyl], or it can follow ideas from this section. Q.E.D.

REMARK 4.7. We recall that if, for the derived group $S(X)$, (4.2) holds, then $S(X)$ is the positive generating cone of a lattice ordered group $G(X)$

[2] In this case, a commutative BCK-algebra $X$ can be converted by a natural way into an MV-algebra and conversely, and (viii) gives Mundici's famous representation theorem of MV-algebras [Mun].

as it is indicated in Theorem 4.6. Unfortunately, that is not a general case, and we do not know other conditions as upwards directness guaranteeing (4.2). Hence, we have to look for other ways for representing commutative BCK-algebras via lattice ordered groups. Such possibilities will be done in Sections 5 and 6.

The lattice ordered group $G(X)$ from the proof of the last theorem is said to be the *derived group* of $(X;+,*,0)$.

## 5. Union BCK-algebras

In this section, we introduce a family of commutative BCK-algebras which can be embedded into the positive cones of lattice ordered groups.

Let $\{(X_i;*_i,0_i)\}_{i\in I}$ be a system of BCK-algebras such that (i) $0_i=0$ for any $i\in I$; and (ii) $X_i\cap X_j=\{0\}$ for $i\neq j$, $i,j\in I$. Put $X=\bigcup_{i\in I}X_i$ and define a total binary operation $*$ on $X$ via

$$x*y=\begin{cases} x*_i y & \text{if } x,y\in X_i,\ i\in I,\\ x & \text{if } x\in X_i,\ y\in X_j,\ i\neq j,\ i,j\in I.\end{cases}$$

Then $(X;*,0)$ is a BCK-algebra called a *union BCK-algebra*, and we denote $X=\bigoplus_{i\in I}X_i$. A union BCK-algebra is commutative iff each $X_i$ is commutative. It is clear that any BCK-algebra $X=\bigoplus_{i\in I}X_i$, where $X_i=X$ and $|I|=1$. Moreover, any $X_i$ is a BCK-subalgebra of $X=\bigoplus_{i\in I}X_i$, and if $u\in X_i$, then

$$X_u:=\{x\in X: x\leq u\}\subseteq X_i.$$

Conversely, let $\{X_i\}_{i\in I}$ be a family of subsets of a BCK-algebra $(X;*,0)$ such that

(a) $X=\bigcup_{i\in I}X_i$,
(b) $X_i\cap X_j=\{0\}$,
(c) $u\in X_i$ implies $X_u\subseteq X_i$,

then each $X_i$ is a BCK-subalgebra of $X$, and $X$ is the union BCK-algebra of $\{X_i\}_{i\in I}$.

More about union BCK-algebras is in [MeJu].

We recall that if $X=\bigoplus_{i\in I}X_i$ and there are two different subalgebras $X_i$ and $X_j$ having at least two elements, then $X$ does not satisfy the condition (S).[3] Indeed, if $a\in X_i$ and $b\in X_j$ are non-zero elements, then $a*a\leq b$ and $b*a\leq b$ but there is no element $c\in X$ such that $a\leq c$ and $b\leq c$.

[3] A BCK-algebra $(X;*,0)$ satisfies *condition* (S) if given elements $a,b\in X$ there is an element $a\circ b\in X$ such that (i) $(a\circ b)*a\leq b$, and (ii) if $x*a\leq b$ then $x\leq a\circ b$. Then $(X;\circ,0,\leq)$ is a commutative po-semigroup [MeJu, ThmI.7.7]. Since $a\leq a\circ b$ and $b\leq a\circ b$, $(X;*,0)$ is directed upwards and by [DvGr], $X$ can be embedded into the positive cone of some lattice ordered group.

We say that a BCK-algebra $(X;*,0)$ has a *u.d. union property* if $X$ can be expressed as $X = \bigoplus_{i\in I} X_i$, where each $X_i$ is directed upwards. A BCK-clan $(X;+,*,0)$ has a *u.d. union property* if $(X;*,0)$ has it.

Since $\{0\}$ is trivially an upwards directed BCK-subalgebra of $X$, using Zorn's lemma we conclude that in any BCK-algebra $X$ there is a maximal upwards directed BCK-subalgebra of $X$.

Any BCK-algebra with u.d. union property has a unique decomposition $X = \bigoplus_{i\in I} X_i$, where all $X_i$ are directed upwards:

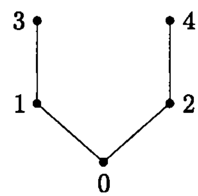
LEMMA 5.1. *Let $X = \bigoplus_{i\in I} X_i$, where each $X_i$ is directed upwards. Then, if $X = \bigoplus_{j\in J} Y_j$, where each $Y_j$ is directed upwards, for any $i \in I$, there is a unique $j \in J$ such that $X_i = Y_j$ and conversely.*

Proof. Let $u \in X_i$ be non-zero. Then there is a unique $Y_j$ such that $u \in Y_j$. Hence, $X_u \subseteq X_i$ and $X_u \subseteq Y_j$. Let now $v$ be an arbitrary element of $X_i$. There is an element $w \in X_i$ which as an upper bound of $u$ and $v$, i.e., $u, v \leq w$. Then $v \in X_w \subseteq X_i$ and since $u \in Y_j$, we conclude that $X_w \subseteq Y_j$, so that $X_i \subseteq Y_j$. By symmetrical reasoning we can show that $Y_j \subseteq X_i$. Q.E.D.

We recall that Example 2.4 gives a BCK-algebra and a BCK-clan which have u.d. union property, while the following example not.

EXAMPLE 5.2. The example $B_{5-2-8}$ from [MeJu], where $*$ is given by the table below, gives a commutative BCK-algebra which has not u.d. union property because $1 \leq 3, 4$ while 3 and 4 has no upper bound in it.

| + | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 0 | 0 |
| 2 | 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 2 | 1 | 0 | 2 |
| 4 | 4 | 1 | 4 | 1 | 0 |




LEMMA 5.3. (i) *Any commutative BCK-algebra with the u.d. union property has relative cancellation property.*

(ii) *If $X = \bigoplus_{i\in I} X_i$ is a commutative BCK-algebra, then, for $x \in X_i$ and $y \in X_j$, where $x \neq 0 \neq y$ and $i \neq j$, $x + y$ does not exist in $X$. In general, if $u + v$ exists in $X$, then $u$ and $v$ belong to the same subalgebra $X_i$.*

Proof. (i) Suppose that $a \leq x, y$ and $x * a = y * a$. If $a = 0$, then trivially $x = y$. If $a \neq 0$, then $a \in X_x \cap X_y$, so that there is a unique $X_i$ in $X = \bigoplus_{i\in I} X_i$ such that $x, y \in X_i$. Applying Lemma 2.2, we see that $x = y$.

(ii) This part follows from an observation that if $x+y$ exists in $X$, then $x, y \leq x+y$ so that $x, y \in X_{x+y}$, which means that $x$ and $y$ have to belong to the same subalgebra. Q.E.D.

LEMMA 5.4. *Let $X = \bigoplus_{i\in I} X_i$ and let $(X; +, *, 0)$ be a BCK-algebra having the u.d. union property. Let all elements of $(a_1, \dots, a_n)$ belong to a fixed $X_s$, $s \in I$. If $(a_1, \dots, a_n) \sim (b_1, \dots, b_m)$, then all elements $b_j$'s belong to $X_s$.*

*If $[(a_1, \dots, a_n)] = [(b_1, \dots, b_m)]$, then all $b_j \in X_s$, for $j = 1, \dots, m$.*

Proof. Let $(a_1, \dots, a_n)$ be directly similar with $(b_1, \dots, b_m)$. Then either some $a_i = a' + a''$ and $b_i = a'$, $b_{i+1} = a''$ and $a_k = b_k$ for $1 \leq k < i$ and $a_k = b_{k+1}$ for $i < k \leq n$, or $b_j = a_j + a_{j+1}$ and other elements coordinately coincide. In the first case $b_i, b_{i+1} \leq a_i$ so that $b_i, b_{i+1} \in X_s$ and similarly in the second one.

The rest of Lemma is now evident. Q.E.D.

We now present the main result, a representation theorem for commutative BCK-algebras having u.d. union property, which converts in some sense Example 2.6.

THEOREM 5.5. *Let $(X; *, 0)$ be a commutative BCK-algebra with the u.d. union property. Then there is a lattice ordered group $(G; +, \leq, 0)$ with the positive generating cone $G^+$ and a subset $G_0$ of $G^+$ generating $G^+$, which is a lower semilattice closed under $*_G$, such that $X$ can be injectively embedded onto $G_0$ with an embedding $h$ satisfying*

$$h(a+b) = h(a) + h(b) \tag{5.1}$$

*whenever $a+b$ exists in $X$, and*

$$h(x * y) = h(x) *_G h(y) := h(x) - (h(x) \wedge h(y)), \ \ x, y \in X. \tag{5.2}$$

Proof. (i) Suppose $X = \bigoplus_i X_i$, where each $X_i$ is an upwards directed BCK-subalgebra. Since this decomposition is due to Lemma 5.1 unique and by Lemma 5.4, $(a_1, \dots, a_n) \simeq (b_1, \dots, b_m)$ with entries $a_k \in X_i$ for any $1 \leq k \leq n$, we can construct the derived semigroups $S(X_i)$ for each $X_i$ such that $S(X_i) \subseteq S(X)$ for any $i$ with $S(X_i) \cap S(X_j) = \{[0]\}$ for $i \neq j$. According to Theorem 4.6, for each $i$, the derived group $G_i := G(X_i)$ is a lattice derived group and a mapping $h_i : X_i \to G_i$ is an embedding preserving $+$ and the order in $X_i$.

Define the direct sum $G$ of the system of groups $\{G(X_i)\}_i$, $G = G(X) := \bigoplus_i G(X_i)$. That is, $G(X)$ is the subset of the product $\prod_i G(X_i)$ consisting of all elements $(w_i)_i$, where $w_i \in X_i$, with finitely many non-zero $w_i$'s, and the addition and the ordering are by coordinates. In addition, $G(X)$ is a lattice ordered group.

The mapping $h : X \to G$ defined via $h(a) := (b_i)_i$, where $b_i = [(a)]$ if $a \in X_i$ and $b_i = [(0)]$ if $a \notin X_i$, is an injection.

(ii) It is possible to show that $h(a) \leq h(b)$ in $G$ iff $a \leq b$ in $X$, and $h(0) = [(0)]$. Indeed, let $a \leq b$ in $X$. Since $a \in X_b := \{x \in X : x \leq b\}$, both $a$ and $b$ belong to the same subalgebra $X_i$ of $X$, so that $h(a) \leq h(b)$. Conversely, let $h(a) \leq h(b)$ in $G$. By definition of $h$, both $a$ and $b$ belong to the same subalgebra $X_i$, and using (iv) Theorem 4.6, we conclude that $h_i(a) \leq h_i(b)$, so that, $a \leq b$.

(iii) Let now $c = a \wedge b$, $a, b \in X$. Then $c \in X_a \cap X_b$. If $a$ and $b$ belong to the same subalgebra, say $X_i$, then by (iv) Theorem 4.6, $h_i(c) = h_i(a) \wedge h_i(b)$, so that $h(c) = h(a) \wedge h(b)$. If $a$ and $b$ belong to different subalgebras, say $X_i$ and $X_j$, then $c = 0$ and by definition of $h$, $h(a) \wedge h(b) = [(0)] = h(c)$.

(iv) Equation (5.1) is a consequence of (ii) Lemma 5.3.

Let now $x$ and $y$ be arbitrary elements of $X$. Due to (5.1), we have

$$h(x * (x \wedge y)) = h(x) - h(x \wedge y) = h(x) - (h(x) \wedge h(y)).$$

On the other hand, we have $x * y = x * (y \wedge x) = x * (x \wedge y)$ which proves (5.2) and finishes the proof. Q.E.D.

We say that a partially ordered Abelian group $(G; +, \leq, 0)$ with a mapping $h : X \to G^+$ is a *universal group* for a BCK-clan $(X; +, *, 0)$ if (i) the positive cone $G^+$ is generating for $G$; (ii) $h(X)$ generates $G^+$; (iii) $h(x + y) = h(x) + h(y)$ whenever $x + y$ exists in $X$, $x, y \in X$, and (iv) for any partially ordered Abelian group $G_1$ and any order and + preserving mapping $g : X \to G_1$ there is a group homomorphism of ordered groups $g' : G \to G_1$ such that $g = g' \circ h$. The universal group, if it exists, is unique up to isomorphism, and $g'$ from (iv) is a unique group homomorphism of ordered semigroups with that property. We denote the universal group $G = (G, h)$.

THEOREM 5.6. *Under the conditions of Theorem* 5.5, *the group* $G(X)$ *and the embedding* $h$ *from the proof of Theorem* 5.5 *has the universal property for* $(X; +, *, 0)$, *and* $(G(X), h)$ *is a universal group for* $(X; +, *)$.

Proof. Let $g$ be a mapping from $X$ into a partially ordered Abelian group $G_1$ preserving + and the order in $X$. Then the restriction $g_i : X_i \to G_1$ of $g$ onto $X_i$ preserves + and the order in $X_i$, and due to (vii) of Theorem 4.6, there is a unique group homomorphism of ordered semigroups $g'_i : G_i(X) \to G_1$ such that $g_i = g'_i \circ h_i$. Define $g' : G \to G_1$ via

$$g'((w_i)_i) := \sum_i g'_i(w_i), \quad (w_i)_i \in G(X). \tag{5.3}$$

Since in any $(w_i)_i$ there are only finitely many non-zero elements

$w_i \in G(X_i)$, the sum on the right-hand side of (5.3) is defined well, and $g'$ is a group homomorphism of ordered groups from $G = G(X)$ into $G_1$.

Take $a \in X$. If $a = 0$, then $g'(h(0)) = g'([(0)]) = 0 = g(0)$. If $a \neq 0$, then there is a unique subalgebra $X_i$ containing $a$. Then $g'(h(a)) = g_i(h_i(a)) = g_i(a) = g(a)$.

The uniqueness of $g'$ is now clear because if $g' : G(X) \to G_1$ is a group homomorphism such that $g''(h(a)) = g(a)$, $a \in X$, then for $a \in X_i$, $g_i(a) = g(a) = g''(h(a)) = g''(\cdots, [(0)], \cdots, [(0)], h_i(a), [(0)], \cdots [(0)], \cdots) = g'(h_i(a))$. The definition of $G(X) = \bigoplus_i G(X_i)$ gives $g' = g''$.

The last statement follows from the construction of derived semigroups and derived groups. Q.E.D.

The following example shows that there is a commutative BCK-algebra which can be embedded into a lattice ordered group but not via Theorem 5.5.

EXAMPLE 5.7. Let $\Omega = \{1,2,3,4\}$, and $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$. Then $(X; \setminus, \emptyset)$, where $\setminus$ is the set-theoretical difference, is a commutative BCK-algebra having the relative cancellation property but not the u.d. union property. Therefore, for its embedding into a lattice ordered group we cannot use Theorem 5.5, but it can be embedded into a lattice ordered group, because it can be embedded into the Boolean algebra $2^\Omega$, and $2^\Omega$ into a lattice ordered group.

We recall that Example 5.7 is a special case of a positively implicative commutative BCK-algebra, i.e., $X$ satisfies $x * y = (x * y) * y$ for all $x, y \in X$. Such algebras are so-called implicative BCK-algebras. In this case (see [DvGr, Thm 6.7] and a discussion to this theorem, and [MeJu, Thm VII.2.7]), it was proved that $X$ can be embedded into a Boolean algebra of subsets, and consequently, into a lattice ordered group.

REMARK 5.8. Let a BCK-clan $(X; +, *, 0)$ be isomorphic under an isomorphism $h$ with a BCK-subclan $(G_0; +, *_G, 0)$ of the BCK-clan $(G^+; +, *_G, 0)$ of some lattice ordered group $G$ such that $(G, h)$ is a universal group for $X$. Hence, if $(G_1, h_1)$ is any universal group for $X$, then $h_1$ is injective.

REMARK 5.9. Let $Z$ be the group of all integers with the usual addition and the order. It is worth to recall that

(i) Example 2.3 cannot be embedded into any lattice ordered group.
(ii) Example 2.4 has a universal group $(Z^2, h)$, where $h(0) = (0,0)$, $h(1) = (1,0)$, $h(2) = (0,1)$, $h(3) = (2,0)$, $h(4) = (0,2)$.
(iii) Example 5.2 has a universal group $(Z^2, h)$, where $h(0) = (0,0)$, $h(1) = (0,1)$, $h(2) = (1,0)$, $h(3) = (1,1)$, $h(4) = (0,2)$.

(iv) Example 5.7 has a universal group $(Z^3, h)$, where $h(\emptyset) = (0,0,0)$, $h(\{1\}) = (1,0,0)$, $h(\{2\}) = (0,1,0)$, $h(\{3\}) = (0,0,1)$, $h(\{1,2\}) = (1,1,0)$, $h(\{2,3\}) = (0,1,1)$.

## 6. Unitarily extendible BCK-algebras

In the present section, we introduce another class of commutative BCK-algebras, which can be embedded into lattice ordered groups.

According to Hoo, [Hoo], a commutative BCK-algebra $(X;*,0)$ is said to be *unitarily extendible* if, for $e \notin X$, we can extend the commutative BCK-algebra structure on $X$ to the commutative BCK-algebra structure on $X \cup \{e\}$ with $e$ as the greatest element in $X \cup \{e\}$. It is clear that any non-zero BCK-algebra is unitarily extendible. Hoo proved the following two important statements:

THEOREM 6.1. (i) *A non-zero commutative BCK-algebra $(X;*,0)$ is unitarily extendible if and only if there is a function $f: X \setminus \{0\} \to X \setminus \{0\}$ such that $f(f(x)) = x$ and $x * f(y) = y * f(x)$ for all $x, y \in X \setminus \{0\}$.*[4]

(ii) *A bounded commutative BCK-algebra $(X;*,0)$ is unitarily extendible if and only if there exists an element $x_0 \in X \setminus \{0\}$ such that $x_0 \leq x$ for all $x \in X \setminus \{0\}$.*

We note that Example 2.3 is a case of a commutative BCK-algebra which is not unitarily extendible; Example 2.4 is a case of a BCK-algebra having u.d. union property but it is not unitarily extendible. Similarly, the set of all nonnegative integers $N = \{0,1,2,\ldots\}$ ordered by the natural way can be converted into a commutative BCK-algebra only in one way: $n * m := \max\{0, n-m\}$, $n, m \in N$, in this case it has a linear structure and it can be embedded into the lattice ordered group of all integers, but $N$ is not unitarily extendible.

EXAMPLE 6.2. Let $X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$. Then $(X; \setminus, \emptyset)$, where $\setminus$ is the set-theoretical difference, is a commutative BCK-algebra which has no u.d. union property, but $X$ is unitarily extendible.

The proofs of the following three examples follow from [Dvu, Exs 3.2–3.4], Theorem 6.1, and the definition of unitarily extendible BCK-algebras.

EXAMPLE 6.3. Let $X = [0,1)$ be the interval of real numbers ordered by the natural way. Let $g: [0,1) \to [0,\infty)$ be an increasing, continuous function with $g(0) = 0$. Define a total binary operation $*_g$ on $X$ via

$$x *_g y := g^{-1}(\max\{0,\, g(x) - g(y)\}), \quad x, y \in X. \tag{6.1}$$

[4] We recall that in this case $e * x = f(x)$ for each $x \in X \setminus \{0\}$.

Then $([0,1);*_g,0)$ is a commutative BCK-algebra with the natural order. Conversely, if $([0,1);*,0)$ is a commutative BCK-algebra with the natural order, then there is an increasing, continuous function $g:[0,1)\to[0,\infty)$ with $g(0)=0$ such that $*=*_g$, where $*_g$ is defined via (6.1). Moreover, $([0,1);*_g,0)$ is unitarily extendible if and only if $g$ is bounded; in this case the function $f$ from Theorem 6.1 is defined via $f(x)=g(1^-)-g(x)$, $x\in(0,1)$, and $([0,1);*_g,0)$ is isomorphic with $([0,1);*_{id_{[0,1)}},0)$.

EXAMPLE 6.4. Let $X=[0,\infty)$ be the interval of real numbers ordered by the natural way. Then $([0,\infty);*,0)$ is a commutative BCK-algebra with the natural order if and only if there exists an increasing, continuous function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that $*=*_g$, where $*_g$ is defined via (6.1). $([0,\infty);*_g,0)$ is unitarily extendible if and only if $g$ is bounded. In this case, $([0,\infty);*_g,0)$ is isomorphic with $([0,1);*_{id_{[0,\infty)}},0)$, and the function $f$ from Theorem 6.1 is defined via $f(x)=g(+\infty)-g(x)$, $x\in(0,\infty)$.

EXAMPLE 6.5. Let $X=[0,1]$ be the interval of real numbers ordered by the natural way. Then $([0,1];*,0)$ is a commutative BCK-algebra with the natural order if and only if there exists an increasing, continuous function $g:[0,1]\to[0,1]$ with $g(0)=0$ and $g(1)=1$ such that $*=*_g$, where $*_g$ is defined via (6.1). Any $([0,1];*_g,0)$ is bounded and isomorphic with $([0,1];*_{id_X},0)$. Moreover, $([0,1];*_g,0)$ is not unitarily extendible.

THEOREM 6.6. *Let $X=\bigoplus_{i\in I}X_i$ be a union BCK-algebra, where each $X_i$ is either directed upwards or a non-zero and unitarily extendible commutative BCK-subalgebra of $X$. Then $X$ can be converted into a BCK-clan $(X;+,*,0)$, and there is a lattice ordered group $(G;+,\le,0)$ with the positive generating cone $G^+$ and a lower semilattice $G_0$ of $G^+$, which generates $G^+$, such that $X$ can be embedded onto $G_0$ with an injective embedding $h$ satisfying* (5.1) *and* (5.2).

Proof. Let $X=\bigoplus_{i\in I}X_i$ satisfy the conditions of Theorem. Then $X$ has a relative cancellation property, and $(X;+,*,0)$ is a well-defined BCK-clan.

For any $i\in I$, let $\tilde X_i:=X_i$ if $X_i$ is directed upwards and $\tilde X_i:=X_i\cup\{e_i\}$ when $X_i$ is only unitarily extendible, where $e_i$ is the greatest element in $X_i\cup\{e_i\}$. Without loss of generality we can assume that all $e_i$ are diverse. Define the commutative union BCK-algebra $\tilde X:=\bigoplus_{i\in I}\tilde X_i$. According to Theorems 5.5 and 5.6, there exists a universal group $(G,h)$ for $(\tilde X;+,*,0)$ satisfying (5.1) and (5.2), so that $h$ is an embedding of $\tilde X$ preserving $+$ and the order in $\tilde X$. Consequently, the restriction $h_0$ of $h$ onto $X$ also satisfies (5.1) and (5.2), respectively.

We know that $h(\tilde X)$ generates the positive cone $G^+$ of $G$. We show that also $h(X)$ generates $G^+$. Take $g\in G^+$. Then $g=\sum_i h(x_i)+\sum_j h(e_j)$,

$x_i \in X_i$, where the sums $\sum_i$ and $\sum_j$ are both over finite (also empty) sets of indices. For any $e_j \in \tilde{X}_j$, there are two non-zero elements $x'_j, x''_j \in X_j$ such that $e_j = x'_j + x''_j$, where the sum is in $\tilde{X}_j$, hence $h(e_j) = h(x'_j) + h(x''_j)$. This means that any element $g \in G^+$ can be expressed as a finite sum of elements from $h(X)$. Q.E.D.

THEOREM 6.7. *Under the conditions of Theorem* 6.6, $(X; +, *, 0)$ *possesses a lattice ordered universal group.*

Proof. Take the lattice ordered group $G$ and the embedding $h$ from $\tilde{X}$ into $G$ which have been used in the proof of Theorem 6.6. Let $h_0$ be the restriction of $h$ onto $X$. We claim that $(G; h_0)$ is a universal group for $(X; +, *, 0)$ in question. For that it is necessary to verify only the condition (iv) of universal groups.

So let $G_1$ be an ordered Abelian group and let $g : X \to G_1$ be a mapping preserving $+$ and the order in $X$. We extend $g$ to $\hat{g} : \tilde{X} \to G_1$ which will preserve order and all existing sums in $\tilde{X}$. Suppose that $\tilde{X}_i = X_i \cup \{e_i\}$, where $e_i \notin X_i$ and $e_i$ is the greatest element in $\tilde{X}_i$. Take two non-zero vectors $x_1, y_1 \in X_i$. Then $x_2 := e_i * x_1$, $y_1 := e_i * y_1 \in X_i \setminus \{0,\}$ and $x_1 + x_2 = e_i = y_1 + y_2$ in $\tilde{X}_i$. We assert that $g(x_1) + g(x_2) = g(y_1) + g(y_2)$. Indeed, according to Lemma 3.3, there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in \tilde{X}_i$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, $y_2 = c_{12} + c_{22}$. It is clear that $c_{11}, c_{12}, c_{21}, c_{22} \in X_i$ and all sums in the last four equalities exist also in $X_i$.

Calculate

$$\begin{aligned} g(x_1) + g(x_2) &= g(c_{11} + c_{12}) + g(c_{21} + c_{22}) \\ &= g(c_{11}) + g(c_{12}) + g(c_{21}) + g(c_{22}) \\ &= g(c_{11}) + g(c_{21}) + g(c_{12}) + g(c_{22}) \\ &= g(c_{11} + c_{21}) + g(c_{12} + c_{22}) = g(y_1) + g(y_2). \end{aligned}$$

Then $\hat{g} : \tilde{X} \to G_1$ defined via $\hat{g}(x) := g(x)$ if $x \in X$, and $\hat{g}(e_i) := g(x_1) + g(x_2)$, if $x_1 + x_2 = e_i$, $x_1, x_2 \in X_i \setminus \{0\}$, preserves order and sums in $\tilde{X}$. Due to Theorem 5.6, $(G, h)$ is a universal group for $(\tilde{X}; +, *, 0)$, so that there exists a unique group homomorphism of ordered semigroups $g' : G \to G_1$ such that $\hat{g} = g' \circ h$. Consequently, $g = g' \circ h_0$, and $g'$ is a unique group homomorphism of ordered semigroups satisfying the equation $g = g' \circ h_0$. Q.E.D.

REMARK 6.8. We recall that Example 5.2 gives a unitarily extendible BCK-algebra which has not u.d. union property. Indeed, the function $f : X \setminus \{0\} \to X \setminus \{0\}$ in question can be chosen as follows: $f(4) = 2$, $f(3) = 1$, $f(2) = 4$, $f(1) = 3$. According to Theorem 6.7, it has a universal group which is a lattice ordered one, namely $Z^2$, as it has been shown in (iii) of Remark 5.9.

**Acknowledgements.** One of the authors, M.G.G., is grateful to Mathematical Institute of the Slovak Academy of Sciences for their hospitality during her stay in April 1997, when the work on the present paper initiated.

## References

[Bae] R. Baer, *Free sums of groups and their generalizations. An analysis of the associative law*, Amer. J. Math. 41 (1949), 706–742.

[CST] W. H. Cornish, T. Sturm and T. Traczyk, *Embedding of commutative BCK-algebras into distributive lattice BCK-algebras*, Math. Japonica 29 (1984), 309–320.

[Dvu] A. Dvurečenskij, *Measures and states on BCK-algebras*, Atti Sem. Mat. Fis. Univ. Modena (1997).

[DvGr] A. Dvurečenskij and M. G. Graziano, *Remarks on representations of minimal clans*, Tatra Mt. Math. Publications 15 (1998), 31–53.

[DvKi] A. Dvurečenskij and H. S. Kim, *On connections between BCK-algebras and difference posets*, Studia Logica 60 (1998), 421–439.

[Hoo] C. S. Hoo, *Unitarily extensions of MV and BCK-algebras*, Math. Japonica 37 (1992), 585–590.

[ImIs] Y. Imai and K. Iséki, *On axiom systems of propositional calculi*, Proc. Japan Acad. 42 (1966), 19–22.

[Ise] K. Iséki, *A special class of BCK-algebras*, Math. Seminar Notes 5 (1979), 107–119.

[MeJu] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Co., Seoul, 1994.

[Mun] D. Mundici, *Interpretation of AF $C^*$-algebras in /Lukasiewicz sentential calculus*, J. Funct. Anal. 65 (1986), 15-63.

[Mun 1] D. Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japonica, 6 (1986), 889–894.

[Rav] K. Ravindran, *On a structure theory of effect algebras*, PhD thesis, Kansas State Univer., Manhattan, Kansas, 1996, pp. 1–54.

[RoTr] A. Romanowska and T. Traczyk, *On commutative BCK-algebras*, Math. Japonica 25 (1980), 567–583.

[Wyl] O. Wyler, Clans. Compos. Math. 17 (1966/67), 172–189.

Anatolij Dvurečenskij
MATHEMATICAL INSTITUTE
SLOVAK ACADEMY OF SCIENCES
Štefánikova 49
SK-814 73 BRATISLAVA, SLOVAKIA
E-mail: dvurecen@mau.savba.sk

Maria Gabriella Graziano
DIPARTIMENTO DI MATEMATICA E APPLICAZIONI "R. CACCIOPPOLI"
UNIVERSITÀ DEGLI STUDI DI NAPOLI "FEDERICO II"
COMPLESSO UNIVERSITARIO "MONTE S. ANGELO"
Via Cintia, I-801 26 NAPOLI, ITALIA
E-mail: grazian@matna3.dma.unina.it

*Received June 17, 1998.*