DEMONSTRATIO MATHEMATICA
Vol. XXXII No 1 1999

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SOME PROPERTIES OF POLYNOMIAL-NORMAL DISTRIBUTIONS ASSOCIATED WITH HERMITE POLYNOMIALS

This paper considers a class of densities formed by taking the product of nonnegative polynomials and normal densities. We investigate some relations of these densities with Hermite polynomials. We construct a set of polynomials orthogonal with respect to the polynomial-normal density (PND). We invesigate the distribution of sums of independent random variables (r.v.) with PND. We construct a stochastic process such that the one-dimensional density of this process is PND.

1. Introduction

We consider a class of densities formed by taking the product of nonnegative polynomials and normal densities. This class is called by Evans and Swartz [1] polynomial-normal densities (PND). For abbreviation we use the symbol PND not only for densities but also for the class containing the r.v.'s. with densities PND. We shall investigate the class PND using Hermite polynomials. It is known that every polynomial can be represented as a linear combination of Hermite polynomials. Thus we will consider the r.v. X with the density given by one of the following equivalent forms

$$(1.1)\qquad \begin{aligned} f(x) &= \sum_{n=0}^{2l} \frac{1}{(2\pi)^{1/2}c} h_n H_n(x/c)\exp(-x^2/2c^2) \\ &= p_{2l}(x)\exp(-x^2/2c^2), \end{aligned}$$

where p_{2l} is a nonnegative polynomial of the degree $2l$ ($l \geq 0$); c and h_n are parameters ($c > 0$) and H_n is the Hermite polynomial defined by the

Key words and phrases: Hermite polynomial, Gram-Charlier series, orthogonality.

formula

$$H_n(x) = \exp(x^2/2)\frac{d^n}{dx^n}\exp(-x^2/2).$$

We are going to extend some properties of the normal distribution to the polynomial-normal distribution. In particular, we will extend the following properties: the expectation of the Hermite polynomial in a centered normal r.v. X is equal zero; i.e.

$$E[H_n(X)] = 0, \quad n \geq 1. \tag{1.2}$$

The set $(H_n(X),\ n \geq 0)$ forms a sequence of the orthogonal r.v's if X is a centred normal r.v., that means

$$E(H_m(X)H_n(X)) = 0 \quad \text{for } n \neq m. \tag{1.3}$$

We will also prove that the sum of independent r.v.'s with PND densities has also a PND density with suitably added parameters.

In the last chapter we construct a stochastic process such that the one-dimensional densities of this process are polynomial- normal densities.

Hermite polynomials in Gaussian r.v.'s and in Gaussian processes are important tools in various probability problems; see for example monographs: Kallianpur [4], Hida et al. [2].

Conditions (1.2), (1.3) have the following statistical aspects: If condition (1.2) holds for $n > n_0$ then the Gram-Charlier series reduces to the n_0 components; therefore the difficult problem of the convergence of this series automaticly disapears. The orthogonality condition (1.3) expresses the fact that $H_n(\cdot)$, $H_m(\cdot)$ are uncorrelated random variables.

A wide discussion concerning the applications, the theoretical properties, the statistical inference and the comparisons with Gram–Charlier approximation of the polynomial-normal densities is given by Ewans and Swartz [1]. These densities provide a rich class of distributions that can be used in modeling when faced with non-normal characteristics such as the skewness and the multimodality.

The general problems connected with the Gram–Charlier series, its convergence, the approximations by the "truncated" Gram–Charlier expansions are widely discussed in various statistical books; see for example Jonson, Kotz, Balakrishnan [3]. The present paper yields some new slight contribution to the applications of Gram–Charlier expansions.

In chapter 5 we construct an example of a conditionally-Gaussian process. This construction is based on the results of the previous chapter. The application of conditionally-Gaussian processes in the optimal non-linear filtering is presented for example by Liptser and Shiryayew [5].

In the present paper we investigate some properties of Hermite polynomials in r.v.'s. In some sense the converse problem was considered by

Plucińska [7]. In this paper Hermite polynomials are characterized by the probabilistic methods, more precisely by the martingale property. Thus this paper gives an example of an application of probabilistic methods to a deterministic problem.

2. The orthogonality

We suppose that the r.v. X has the PND distribution given by (1.1). We are going to prove the following propositions.

PROPOSITION 1. *Let n_1, n_2, l be given natural numbers such that $n_1 \geq 0$, $n_2 > 0$, $l \geq 0$ and X has the density* (1.1). *Then*

$$H = (H_{n_1+(2l+n_2)n}(X/c),\ n \geq 0)$$

is a sequence of orthogonal r.v.'s and

$$E((H_n(X)) = 0 \quad \text{for } n > 2l. \tag{2.1}$$

PROPOSITION 2. *If for a random variable Y*

$$E(H_n(Y)) = E(H_n(X)), \quad n \geq 1, \tag{2.2}$$

then $Y \sim X(Y, X$ are identically distributed).

Proof of Proposition 1. Without the loss of the generality we put $c = 1$. Let $n > m$, $k_n > k_m$. For every elements H_{k_n}, H_{k_m} of the sequence H the difference of their indices satisfies the inequality $k_n - k_m = n_1 + (2l + n_2)n + -(n_1 + (2l + n_2)m) > 2l$. Then by formula (6.4) we have

$$\begin{aligned} E(H_{k_m}(X)H_{k_n}(X)) &= \int_{-\infty}^{+\infty} H_{k_m}(x)H_{k_n}(x)f(x)\,dx \\ &= (2\pi)^{-1/2}\int_{-\infty}^{+\infty} H_{k_m}(x)H_{k_n}(x)\sum_{r=0}^{2l} h_r H_r(x)\exp(-x^2/2)\,dx = 0 \end{aligned} \tag{2.3}$$

since $k_m + r \leq k_m + 2l < k_n$.

Formula (2.1) follows from (2.3) for $k_m = 0$. ∎

Proof of Proposition 2. We must only to show that the moment problem has the unique solution (see for example Shiryaew [10]). Without the loss of the generality we put $c = 1$. For given l and for $r \leq l$ all the moments $E(X^{2r})$ are bounded by a constant c_l. Then by formulas (2.1), (6.1) (6.4) we get for $r > l$

$$(2.4)\quad E(X^{2r}) = E\Big(\sum_{k=0}^{r} \frac{(2r)!}{(r-k)!(2k)!2^{r-k}} H_{2k}(X)\Big)$$
$$= \sum_{k=0}^{l} \frac{(2r)!}{(r-k)!(2k)!2^{r-k}} E(H_{2k}(X))$$
$$= \sum_{k=0}^{l} \frac{(2r)!}{(r-k)!(2k)!2^{r-k}} \sum_{s=0}^{k} \frac{(-1)^s(2k)!}{(2k-2s)!2^s} E(X^{2k-2s})$$
$$\leq c_l \sum_{k=0}^{l} \frac{(2r)!}{(r-k)!(2k)!2^{r-k}} \sum_{s=0}^{k} \frac{(-1)^s}{s!(k-s)!2^k}$$
$$= c_l \sum_{k=0}^{l} \frac{(2r)!}{(r-k)!(2k)!2^{r-k}} \leq c_l 2^r (2r-1)!!.$$

By (2.4)

$$\lim_{r\to\infty} \frac{1}{2r} E(X^{2r}))^{1/2r} \leq \lim_{r\to\infty} \left(\frac{(2r-1)!!2^r}{(2r)^{2r}} c_l\right)^{1/2r} \leq 2.$$

Then the moment problem has the unique solution. ■

It is well known that the orthogonality measure (the weight function) for the set of Hermite polynomials is unique. But even for infinite subsets of this set the orthogonality measure may not be unique; for such subsets the Three Term Reccurence Formula does not hold. By Proposition 1 we get various weight functions for given subset (for given l) of the set of Hermite polynomials.

Special case of Propositions 1 - 2 was considered by Plucinska (1998) for the Generalized Maxwell distribution (Genaralized Maxwell distribution is a special case of (1.1) where $p_{2l}(x) = cx^{2l}$, c is a constant).

3. Characteristic functions. Sums

Ewans and Swartz (1994) have shown that the sum of independent r.v.'s with polynomial-normal distributions has also PND. We are going to give more explicit formulas for the distribution of sums using methods of characteristic functions and Hermite polynomials. Moreover Proposition 4 can be treated as a generalization of the addition property of Gaussian r.v.'s (generalization if we put in (3.4) $l_1 = \ldots = l_N > 0$).

First we are going to find the characteristic function.

PROPOSITION 3. *If the distribution is given by* (1.1) *then the characteristic function has the following form*

$$\phi(\xi) = \sum_{n=0}^{2l} h_n (i\xi c)^n \exp(-\xi^2 c^2/2). \tag{3.1}$$

Proof. First let us notice that for every $r \geq 0$, $n \geq 0$

$$\int_{-\infty}^{+\infty} y^{2r} H_{2n+1}(y) \exp(-y^2/2)\, dy = \int_{-\infty}^{+\infty} y^{2r+1} H_{2n+1}(y) \exp(-y^2/2)\, dy = 0. \tag{3.2}$$

In virtue of (6.2) and the orthogonality of Hermite polynomials with respect to the Gaussian measure we have for $r < n$

$$\int_{-\infty}^{+\infty} y^r H_n(y) \exp(-y^2/2)\, dy = 0. \tag{3.3}$$

Taking into account the evaluation (2.4) and formulas (3.1)–(3.3) and (6.2) we get

$$\begin{aligned}
\phi(\xi) &= \sum_{r=0}^{\infty} \frac{(i\xi)^r}{r!} E(X^r) \\
&= \sum_{r=0}^{\infty} \frac{(i\xi c)^r}{r!} \int_{-\infty}^{+\infty} y^r \sum_{n=0}^{2l} h_n H_n(y) [1/\sqrt{2\pi}] \exp(-y^2/2)\, dy \\
&= \sum_{n=0}^{l} h_{2n} \sum_{r=0}^{\infty} \frac{(i\xi c)^{2r}}{(2r)!} \int_{-\infty}^{+\infty} y^{2r} H_{2n}(y) [1/\sqrt{2\pi}] \exp(-y^2/2)\, dy \\
&\quad + \sum_{n=0}^{l-1} h_{2n+1} \sum_{r=0}^{\infty} \frac{(i\xi c)^{2r+1}}{(2r+1)!} \int_{-\infty}^{+\infty} y^{2r+1} H_{2n+1}(y) [1/\sqrt{2\pi}] \exp(-y^2/2)\, dy \\
&= \sum_{n=0}^{l} h_{2n} \sum_{r=n}^{\infty} (i\xi c)^{2r} \frac{1}{(r-n)! 2^{r-n}} + \sum_{n=0}^{l-1} h_{2n} \sum_{r=n}^{\infty} (i\xi c)^{2r+1} \frac{1}{(r-n)! 2^{r-n}} = \\
&= \sum_{n=0}^{l} h_{2n} (i\xi c)^{2n} \sum_{r=n}^{\infty} \frac{(i\xi c)^{2r-2n}}{(r-n)! 2^{r-n}} \\
&\quad + \sum_{n=0}^{l-1} h_{2n+1} (i\xi c)^{2n+1} \sum_{r=n}^{\infty} \frac{(i\xi c)^{2r-2n}}{(r-n)! 2^{r-n}} \\
&= \sum_{n=0}^{2l} h_n (i\xi c)^n \exp(-\xi^2 c^2/2). \quad \blacksquare
\end{aligned}$$

The coefficients h_n can be found by the well known method of expanding a function in series of Hermite polynomials

$$h_n = \frac{1}{n} \int_{-\infty}^{+\infty} H_n(x/c) f(x)\, dx = \frac{1}{n!} E(H_n(X/c)).$$

The distribution given by (1.1) depends on $2l+2$ parameters: l, c, $h_1, \dots, h_{2l}$. The parameters h_n can be evidently expressed by the moments of X. Then we can say that the distribution (1.1) depends on parameters: $l, c, E(X^n)$, $n \leq 2l$.

The distribution given by (1.1) will be denoted by PND$(l, c, E(X^n)$; $n \leq 2l)$ or equivalently PND$(l, c, h_n;\ n \leq 2l)$.

When we will consider some sets of r.v's we will write double indices $h_{n,x}$ instead of h_n.

PROPOSITION 4. *If $X_1, \dots, X_N$ are independent r.v.'s and the distribution of X_j is PND$(l_j, c_j, E(X_j^r); r \leq 2l_j), j = 1.,,,.N$ then the distribution of $S_N = X_1 + \dots + X_N$ is*

(3.4) $\mathrm{PND}(l_1 + \dots + l_N, (c_1^2 + \dots + c_N^2)^{1/2}, E(S_N^r);\ r \leq 2(l_1 + \dots + l_N)).$

The distribution of S_N can be written in an equivalent form

$$\mathrm{PND}(l_1 + \dots + l_N, (c_1^2 + \dots + c_N^2)^{1/2}, h_{r,S_N};\ r \leq 2(l_1 + \dots + l_N))$$

where $h_{r,S_N} = \frac{1}{r!} E[H_r((c_1^2+, \dots + c_N^2)^{-1/2} S_N)]$.

Proof. Let ϕ_{X_j} be the characteristic function of the r.v. X_j. We shall use the method of the mathematical induction. First let us consider $N = 2$. Then by Proposition 3 and formula (6.6) the product of the characteristic functions has the following form

$$\begin{aligned}
\phi_{X_1}(\xi)\phi_{X_2}(\xi) &= \Big(\sum_{r=0}^{2l_1} (i\xi c_1)^r h_{r,X_1} \exp(-\xi^2 c_1^2/2)\Big) \\
&\quad \times \Big(\sum_{r=0}^{2l_2} (i\xi c_2)^r h_{r,X_2} \exp(-\xi^2 c_2^2/2)\Big) \\
&= \Big(\sum_{r=c}^{2(l_1+l_2)} (i\xi)^r \sum_{j=0}^{r} c_1^j c_2^{r-j} h_{j,X_1} h_{r-j,X_2}\Big) \exp\Big[-\frac{1}{2}\xi^2(c_1^2 + c_2^2))\Big] \\
&= \sum_{r=0}^{2(l_1+l_2)} (i\xi)^r E\Big(\sum_{j=0}^{r} \frac{c_1 c_2}{j!(r-j)!} H_j(X_1/c_1) H_{r-j}(X_2/c_2)\Big) \\
&\quad \times \exp\Big[-\frac{1}{2}(\xi^2(c_1^2 + c_2^2))\Big]
\end{aligned}$$

$$= \sum_{r=0}^{2(l_1+l_2)} \frac{1}{r!}[i\xi(c_1^2+c_2^2)^{1/2}]^r E\left[H_r\left(\frac{X_1+X_2}{(c_1^2+c_2^2)^{1/2}}\right)\right]$$

$$\times \exp\left[-\frac{1}{2}(\xi^2(c_1^2+c_2^2))\right]$$

$$= \sum_{r=c}^{2(l_1+l_2)} [i\xi(c_1^2+c_2^2)^{1/2}]^r h_{r,X_1+X_2} \exp\left[-\frac{1}{2}(\xi^2(c_1^2+c_2^2))\right].$$

Thus we have shown that $X_1 + X_2$ has the distribution

$$\text{PND}(l_1+l_2, (c_1^2+c_2^2)^{1/2}, h_{r,X_1+X_2};\ r \le 2(l_1+l_2)). \tag{3.5}$$

Now we consider arbitrary N. Let

$$S_N = X_1 + \ldots + X_N = (X_1 + \ldots + X_{N-1}) + X_N = S_{N-1} + X_N.$$

We assume that the distribution of S_{N-1} is

$$\text{PND}(l_1 + \ldots + l_{N-1}, (c_1^2 + \ldots + c_{N-1}^2)^{1/2}, h_{r,X_1+\ldots+X_{N-1}};$$
$$r \le 2(l_1 + \ldots + l_{N-1})).$$

We repeat the reasoning of the first part of the proof, now for the sum of the two components: $S_{N-1} + X_N$. This sum has the distribution of the form (3.5) with suitable parameters. These parameters are evidently such that we get the distribution of the form (3.4). ■

4. Examples

EXAMPLE 1. We are going to find the distribution of the sum $X_1 + X_2$ of independet r.v.'s X_1, X_2 when $l_1 = l_2 = 1$. Let us put $C^2 = c_1^2 + c_2^2$.

By Proposition 4

$$\phi_{X_1+X_2}(\xi) = \sum_{r=0}^{4} h_{r,X_1+X_2}(i\xi C)^r \exp(-\xi^2 C^2/2)).$$

By Proposition 2 we get $h_{r,X_j} = 0$ for $r > 2, j = 1, 2$. Therefore by formula (6.6) we have

$$\begin{aligned}
Ch_{1,X_1+X_2} &= c_1 h_{1,X_1} + c_2 h_{1,X_2},\\
C^2 h_{2,X_1+X_2} &= c_1^2 h_{2,X_1} + c_2^2 h_{2,X,2} + 2c_1c_2 h_{1,X_1} h_{1,X_2},\\
C^3 h_{3,X_1+X_2} &= c_1 c_2^2 h_{1,X_1} h_{2,X_2} + c_1^2 c_2 h_{1,X_2} h_{2,X_1},\\
C^4 h_{4,X_1+X_2} &= c_1^2 h_{2,X_1} c_2^2 h_{2,X_2}.
\end{aligned}$$

Thus the characteristic function is given by the following formula

$$\begin{aligned}
\phi_{X_1+X_2}(\xi) = {} & [\exp(-\xi^2 C^2/2)][1 + i\xi(c_1 h_{1,X_1} + c_2 h_{1,X_2})\\
& + \xi^2(c_1^2 h_{2,X_1} + c_2^2 h_{2,X_2} + 2c_1c_2 h_{1,X_1} h_{1,X_2}) +
\end{aligned}$$

$$+ (i\xi)^3(c_1c_2^2h_{1,X_1}h_{2,X_2} + c_1^2c_2h_{1,X_2}h_{2,X_1})$$
$$+ \xi^4(c_1^2h_{2,X_1}c_2^2h_{2,X_2})].$$

Evidently

$$\phi_{X_1+X_2}(\xi) = \Phi_{X_1}(\xi)\Phi_{X_2}(\xi).$$

EXAMPLE 2. In the beginning of the present paper we assumed that p_{2l} is a nonnegative polynomial. There are various methods of the choice of parameters of the polynomial p_{2l} which ensure this fact. We give here a very short statistical condition.

Let us consider PND$(1,1,h_1,h_2)$. Then the density has the following form

$$f(x) = (2\pi)^{-1/2}[1 + h_1x + h_2H_2(x)]\exp(-x^2/2)$$
$$= (2\pi)^{-1/2}[h_2x^2 + h_1x + 1 - h_2]\exp\Big(-\frac{1}{2}x^2\Big).$$

Therefore the polynomial is nonnegative iff

$$0 \le h_2 \le 1, \quad h_1^2 \le 4h_2(1-h_2), \tag{4.1}$$

In other words by (4.1) and the definition of h_1, h_2 the moments must satisfy the inequalities

$$1 \le E(X^2) \le 3, \qquad [E(X)]^2 \le [E(X^2)-1][3-E(X^2)]. \tag{4.2}$$

These intervals for the moments ensure that the polynomial is nonnegative. Moreover it could be interesting to mention that for the arbitrary moment of the even order we have the following evaluation

$$(2n-1)!! \le E(X^{2n}) \le (2n+1)!!, \quad n \ge 1.$$

Now we are going to find a nonnegative polynomial p_4. We use the results of Example 1. Therefore the polynomial

$$p_4(x) = -\frac{1}{(2\pi)r1/2c}\sum_{n=0}^{4} h_nH_n(x/C)$$

is nonnegative if

$$Ch_1 = c_1h_{1,X_1} + c_2h_{1,X_2},$$
$$C^2h_2 = c_1^2h_{2,X_1} + c_2^2h_{2,X,2} + 2c_1c_2h_{1,X_1}h_{1,X_2},$$
$$C^3h_3 = c_1c_2^2h_{1,X_1}h_{2,X_2} + c_1^2c_2h_{1,X_2}h_{2,X_1},$$
$$C^4h_4 = c_1^2h_{2,X_1}c_2^2h_{2,X_2}$$

and the parameters have arbitrary values satisfying the conditions:

$c_1 > 0$, $c_2 > 0$, $C^2 = c_1^2 + c_2^2$, $0 \le h_{2,X_i} \le 1$, $h_{1,X_i}^2 \le 4h_{2,X_i}(1-h_{2,X_i})$, $i = 1,2$.

We can continue our reasoning and find a nonnegative polynomial p_{2l} for arbitrary l by considering the sum $X_1 + \ldots + X_l$, where $X_1, \ldots, X_l$ are independent r.v.'s and X_i has the distribution $\mathrm{PND}(1, c_i, h_{1,X_i}, h_{2,X_i})$ with parameters satisfying the above given conditions.

5. An example of Conditionally-Gaussian process

A stochastic process $X = (X_t,\ t \geq t_0)$ is a Conditionally-Gaussian process if all the conditional distributions of the type:

$$(X_{t_n} \mid X_{t_{n-1}}, \ldots, X_{t_1}) \quad \text{for } t_0 < t_1 < \ldots < t_n;\ n > 1$$

are Gaussian. Conditionally-Gaussian processes are considered for examlpe by Liptser and Shiryaew [5].

Now we will construct a Conditionally-Gaussian process $X=(X_t,\ t\geq t_0)$ such that the one-dimensional densities of X are PND.

First let us take $l = 1$. Let X_{t_0} be a r.v. such that $E(X_{t_0}) = \alpha$, $E(X_{t_0}^2) = t_0 + \beta$ and X_{t_0} has the distribution

$$\mathrm{PND}(1, \sqrt{t_0}, \alpha, t_0 + \beta)$$

where either $\alpha = \beta = 0$, $t_0 > 0$ or $2\beta - \alpha^2 > 0$, $t_0 > (2\beta - \alpha^2)^{-1}\beta^2$. The density of X_{t_0} is given by the formula

$$f(t_0, x) = (2\pi t_0)^{-1/2}[1 - \beta/2t_0 + \alpha x/t_0 + \beta x^2/2t_0] \exp(-x^2/2t_0). \tag{5.1}$$

The characteristic function has the following form

$$\Phi(\xi, t_0) = (1 + i\xi\alpha - \xi^2\beta/2) \exp(-\xi^2 t_0/2). \tag{5.2}$$

Let $(W_t,\ t \geq t_0)$ be a Wiener process and X_{t_0}, $(W_t,\ t \geq t_0)$ be independent.

PROPOSITION 5. *The stochastic process* $X = (X_t,\ t \geq t_0) = (W_t - W_{t_0} + X_{t_0},\ t \geq t_0)$ *has the following properties*:

1^0 *X has independent Gaussian increments*,

2^0 *X is a Conditionally-Gaussian process*,

3^0 *the one-dimensional densities of X are*

$$\mathrm{PND}(1, \sqrt{t}, \alpha, t + \beta). \tag{5.3}$$

Proof. Properties 1^0 and 2^0 are evident. The r.v. $W_t - W_{t_0}$ has the density

$$\mathrm{PND}(0, \sqrt{t - t_0}, 0, t - t_0).$$

Thus by formula (3.4) the density of X_t is (5.3). ∎

Now let us denote the conditional density of $(W_{t_2} \mid W_{t_1})$ by $f_{\mathrm{cond}}(t_2, x_2 \mid t_1, x_1)$. Then evidently function (5.1) satisfies the equation

$$\int f(t_1, x_1) f_{\mathrm{cond}}(t_2, x_2 \mid t_1, x_1) dx_1 = f(t_2, x_2).$$

The problem of finding a process X such that the one-dimensional densities of X are arbitrary PND's even for $l = 1$ is not an easy one.

EXAMPLE 3. It is impossible to find a Conditionally-Gaussian process such that the one-dimensional distributions are Maxwell distributions with the density

$$\tilde{f}(t,x) = (2\pi t)^{-1/2}[x^2/_t]\exp(-x^2/_{2t}). \tag{5.4}$$

This statement follows easily from the fact that the equation

$$\int \tilde{f}(t_1,x_1)\tilde{f}_{\text{cond}}(t_2,x_2 \mid t_1,x_1)dx_1 = \tilde{f}(t_2,x_2)$$

has no solution $\tilde{f}_{\text{cond}}(t_2,x_2 \mid t_1,x_1)$, which is a Guassian density.

EXAMPLE 4. It is impossible to find a stochastic process with independent increments such that the the one-dimensional densities are Maxwell densities (5.4).

Characteristic function corresponding to (5.4)

$$\phi(t,\xi) = (1 - t\xi^2)\exp(-t\xi^2/2) \tag{5.5}$$

is indecomposable one (see Lukacs [6]). Then X_{t_2} can not be represented as the sum if two independent components

$$X_{t_2} = (X_{t_2} - X_{t_1}) + X_{t_1}.$$

Thus for the Maxwell distribution the above mentioned problem is not an easy one.It is impossible to find a stochastic process with a Maxwell one-dimensional distribution neither in the class of Conditionally-Gaussian processes nor in the class of independent increments prcesses.

Now we are going to construct a stochastic process such that the one-dimensional distributions are PND for arbitrary l.

By Proposition 5 we get immediately the following Proposition.

PROPOSITION 6. *Let the characteristic function of X_{t_0} be the product of l functions Φ*

$$\Phi_l(t_0,\xi) = \Phi^l(t_0,\xi), \tag{5.6}$$

where Φ is given by (5.2).Then $X = (X_t,\ t \geq t_0) = (W_t - W_{t_0} + X_{t_0},\ t \geq t_0)$ is a stochastic process satisfying Properties 1^0, 2^0 from Proposition 5 and with one-dimensional densities corresponding to the characteristic function (5.6).

Evidently instead of function (5.6) we can consider more general case: Φ_l is the product of l functions Φ and every function Φ depends on various parameters α_i, β_i ($\alpha_i^2 \leq 2\beta_i$), $i \leq l$. Then for suitably choosen T (depending on α_i, β_i) there exists a Conditionally-Gaussian process $X = (X_t, t \in T)$

such that the one-dimensional distributions of this process have the form (1.1) for arbitrary l. For $l = 2$ we can use the results of Example 2.

6. Appendix

For the convenience of the reader we give some formulas taken (after some easy transformations) from the book of Prudnikov et al. (1983)

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{k!(n-2k)!2^k} x^{n-2k}, \tag{6.1}$$

$$x^{2n+\epsilon} = \sum_{k=0}^{n} \frac{(2n+\varepsilon)!}{(n-k)!(2k+\epsilon)!2^{n-k}} H_{2k+\epsilon}(x), \ \epsilon = 0, 1. \tag{6.2}$$

If the segments with lenghts l, m, n form a triangle (it can be degenerate) and $l + m + n = 2, 4, 6, \dots$ then

$$(2\pi)^{-1/2} \int H_l(x) H_m(x) H_n(x) \exp(-x^2/2)\, dx = \frac{l!m!n!}{\left(\frac{l+m-n}{2}\right)!\left(\frac{l+n-m}{2}\right)!\left(\frac{m+n-l}{2}\right)!} \tag{6.3}$$

in other cases

$$(2\pi)^{-1/2} \int H_l(x) H_m(x) H_n(x) \exp(-x^2/2)\, dx = 0. \tag{6.4}$$

For arbitrary real t, u, v

$$(1+t^2)^{n/2} H^n\left(\frac{u+v}{(1+t^2)^{1/2}}\right) = \sum_{k=0}^{n} \binom{n}{k} t^k H_k(u) H_{n-k}(v). \tag{6.5}$$

LEMMA 1. *For arbitrary* $c_1 > 0$, $c_2 > 0$, $n \geq 0$

$$(c_1^2 + c_2^2)^{n/2} H_n\left(\frac{x+y}{(c_1^2+c_2^2)^{1/2}}\right) = \sum_{k=0}^{n} \binom{n}{k} c_1^k c_2^{n-k} H_k\left(\frac{x}{c_1}\right) H_{n-k} \frac{y}{c_2}. \tag{6.6}$$

Lemma 1 follows from (5.5) when we put $u = x/c_1$, $v = y/c_2$, $t = c_1(c_2)^{-1}$.

Acknowledgements. The autor wishes to express her gratitude to the Referee for valuable remarks and comments improving the previous version of this paper.

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Received July 14, 1998.