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RECURRENT IMMERSIONS

Dedicated to Professor Dr. Alan West on his 65th birthday

1. Introduction

In this paper we consider the relations between immersions of recurrent type with the immersions having certain planar properties. Namely, we investigate the weak recurrent immersions $f : M \longrightarrow \mathbb{R}^{m+d}$ of m-dimensional Riemannian manifolds M into $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$. We show that if the immersion f (or the submanifold M) has P2-PNS property then it is weak 1-recurrent. We also show that if M is an isotropic submanifold in $\mathbb{R}^{m+d}$ then M is of AW(3) type if and only if M is weak 2-recurrent. Finally we consider the isotropic immersions with PP3-PNS property. We show that the isotropic submanifold has PP3-PNS property if and only if it is generalized weak 2-recurrent.

Let $f : (M,g) \longrightarrow (N,\tilde{g})$ be an isometric immersion from an m-dimensional Riemannian manifold (M,g) into $(m+d)$-dimensional Riemannian manifold $(N,\tilde{g})$. Immersions satisfying certain conditions imposed on the second fundamental form h were investigated by many authors. For instance, if the tensor field $\bar{\nabla} h$ vanishes on M then the immersion f is said to be a parallel immersion. A parallel immersion is sometimes called extrinsically locally symmetric ([12]). Further if $\bar{\nabla} h$ is recurrent (see Section 3) then the immersion is called a recurrent immersion ([10]). Recurrent immersions are special semi-parallel immersions. We recall that the immersion f is a semi-parallel immersion if the tensor field $\bar{R}\cdot h$ vanishes on M. Semi-parallel

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immersions have been studied intensively by J. Deprez, F. Dillen, Ü. Lumiste, V. Mirzoyan and K. Riives. For more information see [9] and [14] and the literature cited there.

2. Planar normal sections

Let $f : (M,g) \longrightarrow (N,\tilde{g})$ be an isometric immersion from an m-dimensional Riemannian manifold (M,g) into $(m+d)$-dimensional Riemannian manifold $(N,\tilde{g})$, $m \geq 2, d \geq 1$. Let ∇, $\bar{\nabla}$ and D denote the covariant derivatives in $T(M)$, $T^{\perp}(M)$ and N, respectively. Thus D_X is just the directional derivative in the direction X in N. For tangent vector fields X, Y and Z and the normal vector field ξ over M we have $D_X Y = \nabla_X Y + h(X,Y)$ and $D_X \xi = -A_\xi X + \bar{\nabla}_X \xi$, where h is the *second fundamental form* and A_ξ is the *shape operator* of M. We have also $h(X,Y) = h(Y,X)$ and $g(h(X,Y),\xi) = g(A_\xi X, Y)$. A submanifold M is called *totally geodesic* if its second fundamental tensor h vanishes on M ([6]). We define $\bar{\nabla} h$ and $\bar{\nabla}\bar{\nabla} h$ as usual by

$$\begin{aligned}(\bar{\nabla}_X h)(Y,Z) &= D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),\\ (\bar{\nabla}_W \bar{\nabla}_X h)(Y,Z) &= D_W((\bar{\nabla}_X h)(Y,Z)) - (\bar{\nabla}_X h)(\nabla_W Y, Z)\\ &\quad -(\bar{\nabla}_X h)(Y, \nabla_W Z) - (\bar{\nabla}_Y h)(\nabla_W X, Z),\end{aligned}$$

respectively, where $X, Y, Z \in T_x(M)$. The equations of Gauss, Codazzi and Ricci of M in N are the following

$$\begin{aligned}g(R(X,Y)Z,W) &= \tilde{g}(\tilde{R}(X,Y)Z,W) + g(h(X,Y),h(Z,W))\\ &\quad -g(h(X,W),h(Y,Z)),\\ \tilde{g}(R(X,Y)Z,\nu) &= g((\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),\nu),\\ \tilde{g}(R(X,Y)\xi,\nu) &= g(R^D(X,Y)\xi,\eta) - g([A_\xi, A_\eta]X,Y),\end{aligned}$$

where $X,Y,Z,W \in T_x(M)$ and $\xi,\eta,\nu \in N_x(M)$.

From now on we assume that $f : M \longrightarrow \mathbb{R}^{m+d}$ is an isometric immersion from an m-dimensional Riemannian manifold M into $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$, i.e. M is a submanifold of $\mathbb{R}^{m+d}$. For $x \in M$ and a non-zero vector X in $T_x(M)$, we define the $(d+1)$-dimensional affine subspace $E(x,X)$ of $\mathbb{R}^{m+d}$ by $E(x,X) = x + \mathrm{span}\{X, N_x(M)\}$. In a neighbourhood of x, the intersection $M \cap E(x,X)$ is a regular curve $\gamma : (-\varepsilon,\varepsilon) \longrightarrow M$. We suppose that the parameter $t \in (-\varepsilon,\varepsilon)$ is a multiple of the arc-length, such that $\gamma(0) = x$ and $\gamma'(0) = X$. Each choice of $X \in T(M)$ yields a curve which is called the *normal section* of M at x in the direction od X, where $X \in T_x(M)$ ([7]).

The immersion f (or the submanifold M) is said to have *pointwise k-planar normal sections* (Pk-PNS) if for each normal section γ the higher order derivatives $\{\gamma'(0), \gamma''(0), \ldots, \gamma^{(k+1)}(0)\}$ are linearly dependent as vectors in $\mathbb{R}^{m+d}$ ([1] - [3], [8]).

The immersion f (or the submanifold M) is said to have *proper pointwise k-planar normal sections* (PPk-PNS) if it has Pk-PNS property and if it does not have *pointwise* $(k-1)$*-planar normal sections* (P(k-1)-PNS), where $k > 1$.

PROPOSITION 2.1 ([1]). *M has P2-PNS if and only if for each $x \in M$ and each $X \in T_xM$ the vectors $h(X,X)$ and $(\bar{\nabla}_X h)(X,X)$ are linearly dependent vectors in $N_x(M)$.*

PROPOSITION 2.2 ([2]). *M has P3-PNS if and only if for each $x \in M$ and each $X \in T_xM$ the vectors $h(X,X)$, $(\bar{\nabla}_X h)(X,X)$ and $(\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X, X)$ are linearly dependent vectors in $N_x(M)$.*

DEFINITION 2.3. A submanifold M of $\mathbb{R}^{m+d}$ is said to be *isotropic* if for each point x of M and each unit vector $X \in T_xM$, $\|h(X,X)\|$ depends only on x and not on X at x. M is isotropic if and only if $\langle h(X,X), h(X,Y)\rangle = 0$ for any $X, Y \in T_xM$ with $\langle X, Y\rangle = 0$ ([15]).

THEOREM 2.4 ([13]). *Let M be an isotropic submanifold in $\mathbb{R}^{m+d}$. Then M has P3-PNS if and only if for each $x \in M$ and each $X \in T_xM$ the vectors $h(X,X)$, $(\bar{\nabla}_X h)(X,X)$ and $(\bar{\nabla}_X \bar{\nabla}_X h)(X,X)$ are linearly dependent vectors in $N_x(M)$.*

THEOREM 2.5 ([1]). *Let M be an m-dimensional submanifold of $\mathbb{R}^{m+d}$. Then M has P2-PNS if and only if*

$$\|h(X,X)\|^2(\bar{\nabla}_X h)(X,X) = \langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle h(X,X).$$

DEFINITION 2.6. A submanifold M is called *spherical* if M lies in a hypersphere of $\mathbb{R}^{m+d}$ ([7]).

THEOREM 2.7 ([7]). *Let M be an m-dimensional spherical submanifold of $\mathbb{R}^{m+d}$. Then M has P2-PNS if and only if M has parallel second fundamental form, i.e. $\bar{\nabla} h = 0$.*

THEOREM 2.8 ([1]). *Let M be an m-dimensional submanifold of $\mathbb{R}^{m+d}$ with P2-PNS property. If M does not have parallel second fundamental form (i.e. $\bar{\nabla} h \neq 0$) then M must be hypersurface.*

THEOREM 2.9 ([8]). *Let M be a surface of $\mathbb{R}^{2+d}$. Then M has P2-PNS property if and only if M is one of the following surfaces:*

(i) *a surface which lies locally in an affine 3-space $\mathbb{R}^3$ of $\mathbb{R}^{2+d}$, $d > 1$, or*

(ii) *an open portion of the product of two plane circles, i.e. $S^1(a) \times S^1(b) \subset \mathbb{R}^4$ (flat torus) or*

(iii) *an open portion of Veronese surface* V^2 *in an affine* 5*-space* $\mathbb{R}^5$ *of* $\mathbb{R}^{2+d}$, $d > 3$.

COROLLARY 2.10 ([2]). *Let* M *be a surface of* $\mathbb{R}^{2+d}$. *Then* M *has P2-PNS property if and only if* M *is one of the following surfaces*:

(i) *a surface which lies locally in an affine* 3*-space* $\mathbb{R}^3$ *of* $\mathbb{R}^{2+d}$, $d > 1$,
  (a) $\bar{\nabla} h \neq 0$ (*quadrics, etc.*),
  (b) $\bar{\nabla} h = 0$ (*sphere or cylinder*) *or*
(ii) $S^1(a) \times S^1(b) \subset \mathbb{R}^4$ (*flat torus*) *or*
(iii) $V^2 \subset \mathbb{R}^5$ (*Veronese surface*).

THEOREM 2.11 ([2]). *Let* M *be a* m*-dimensional submanifold of* $\mathbb{R}^{m+d}$. *Then* M *has P3-PNS property if and only if*

$$\begin{aligned}
&\{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle^2\} \\
&\{(\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X,X)\} \\
= &\{\langle h(X,X), (\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X,X)\rangle \|(\bar{\nabla}_X h)(X,X)\|^2 \\
- &\langle (\bar{\nabla}_X h)(X,X), (\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X,X)\rangle \\
&\langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle\} h(X,X) \\
+ &\{\|h(X,X)\|^2 \langle (\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X,X), (\bar{\nabla}_X h)(X,X)\rangle \\
- &\langle h(X,X), (\bar{\nabla}_X \bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X,X)\rangle \\
&\langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle\} (\bar{\nabla}_X h)(X,X).
\end{aligned}$$

REMARK 2.12. Every hypersurface has Pk-PNS property for arbitrary $k > 1$ and every surface of codimension r has Pk-PNS property for arbitrary $k > r$.

3. Immersions of recurrent type

We denote by $\nabla^p T$ the covariant differential of the pth order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇. According to [16], the tensor T is said to be 1*-recurrent*, resp. 2*-recurrent*, if the following condition holds on M

$$(1) \qquad \begin{aligned} &(\nabla T)(X_1,\ldots,X_k;X)T(Y_1,\ldots,Y_k) \\ &\quad = (\nabla T)(Y_1,\ldots,Y_k;X)T(X_1,\ldots,X_k), \end{aligned}$$

resp.

$$\begin{aligned} &(\nabla^2 T)(X_1,\ldots,X_k;X,Y)T(Y_1,\ldots,Y_k) \\ &\quad = (\nabla^2 T)(Y_1,\ldots,Y_k;X,Y)T(X_1,\ldots,X_k), \end{aligned}$$

where $X, Y, X_1, Y_1, \ldots, X_k, Y_k \in \Xi(M)$, $\Xi(M)$ being the Lie algebra of the vector fields on M. From (1) it follows that at a point $x \in M$ if the tensor

T is non-zero then there exists a unique 1-form ϕ, resp. a $(0,2)$-tensor ψ, defined on a neighbourhood U of x, such that

$$(2) \qquad \nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|),$$

resp.

$$\nabla^2 T = T \otimes \psi,$$

holds on U, where $\|T\|$ denotes the norm of T, $\|T\|^2 = g(T,T)$. The tensor T is said to be *generalized* 2-*recurrent* if

$$\begin{aligned} &((\nabla^2 T)(X_1,\dots,X_k;X,Y) - (\nabla T \otimes \phi)(X_1,\dots,X_k;X,Y))T(Y_1,\dots,Y_k) \\ &= ((\nabla^2 T)(Y_1,\dots,Y_k;X,Y) - (\nabla T \otimes \phi)(Y_1,\dots,Y_k;X,Y))T(X_1,\dots,X_k) \end{aligned}$$

holds on M, where ϕ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor T is non-zero then there exists unique a $(0,2)$-tensor ψ, defined on a some neighbourdood U of x such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi,$$

holds on U. As an immediate consequence of (2) we have the following relation

$$(3) \qquad R \cdot T = 0,$$

i.e. T is a semi-symmetric tensor. We recall that the $(0,k+2)$-tensor $R \cdot T$ is defined by

$$\begin{aligned} (R \cdot T)(X_1,\dots,X_k;X,Y) &= (\mathcal{R}(X,Y) \cdot T)(X_1,\dots,X_k) \\ = -T(\mathcal{R}(X,Y)X_1, X_2,\dots,X_k) &- \dots - T(X_1,X_2,\dots,X_{k-1},\mathcal{R}(X,Y)X_k), \end{aligned}$$

where the curvature operator $\mathcal{R}(X,Y)$ is defined by

$$\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

and $[X,Y]$ is the Lie bracket of X and Y. If the tensor ∇T vanishes on M then T is called parallel. Evidently, every parallel tensor T fulfils (3).

We adopt the above definitions to define weak recurrent immersions.

Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion from an m-dimensional Riemannian manifold M into $(m+d)$-dimensional Euclidean space $\mathbb{R}^{m+d}$. The immersion f (or the submanifold M) is called *weak* 1-*recurrent* if there exists a 1-form η on M such that at every point $x \in M$ and for every vector $X \in T_x(M)$ we have

$$(4) \qquad (\bar{\nabla}_X h)(X,X) = h(X,X)\eta(X),$$

whenever $h(X,X) \neq 0$ holds at x.

REMARK. Every recurrent immersion is weak 1-recurrent. The converse statement is not true.

LEMMA 3.1 ([6]). *Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be a recurrent immersion then f is semiparallel.*

DEFINITION 3.2. Let M be an m-dimensional submanifold of $\mathbb{R}^{m+d}$. For each point $x \in M$, the first normal space ${N_x}^1(M)$ at x is defined by (see [7]): ${N_x}^1(M) = \text{span}\{h(X,Y) : X, Y \in T_x(M)\} \subset N_x(M)$.

LEMMA 3.3 ([6]). *Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be a weak 1-recurrent immersion and let $x \in M$. Then $(\bar{\nabla}_X h)(X,X) = 0$ or $\dim({N_x}^1(M)) \leq 1$ holds at x.*

THEOREM 3.4. *Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. Then M has PP2-PNS property if and only if M (or f) is weak 1-recurrent.*

Proof. Let X be a vector at a point x at which $h(X,X) \neq 0$. Suppose that M has PP2-PNS property. Then by Theorem 2.5 we have at x

$$(\bar{\nabla}_X h)(X,X) = \frac{\langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle}{\|h(X,X)\|^2} h(X,X).$$

We put

$$\phi(X) = \frac{\langle h(X,X), (\bar{\nabla}_X h)(X,X)\rangle}{\|h(X,X)\|^2}.$$

If we consider ϕ as a 1-form then the immersion f must be weak 1-recurrent. Conversely, suppose that if M (or f) is weak 1-recurrent then by Theorem 2.5 we can deduce that M has PP2-PNS property. This completes the proof of the theorem.

DEFINITION 3.5. Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. The immersion f (or the submanifold M) is called AW(3) type if for each $X \in T_x(M)$ the following equation holds ([3]):

$$\begin{aligned} (5) \qquad & \|h(X,X)\|^2\{(\bar{\nabla}_X\bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X, X)\} \\ &= \langle h(X,X), (\bar{\nabla}_X\bar{\nabla}_X h)(X,X) + 3h(A_{h(X,X)}X, X)\rangle h(X,X). \end{aligned}$$

EXAMPLE 3.6 ([5]). The helical cylinder H^2 embedded in $\mathbb{R}^4$ by

$$x(\theta,\phi) = \{(\theta, a\cos\phi, a\sin\phi, b\phi) : \ \theta, \phi \in \mathbb{R}, \theta \neq 0\}$$

is of type AW(3).

DEFINITION 3.7. Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. The immersion f (or the submanifold M) is called *weak 2-recurrent* (or *weak* *birecurrent*) if there exists a 2-form ϕ on M such that at every point $x \in M$ and for every vector $X \in T_x(M)$ we have

$$(6) \qquad (\bar{\nabla}_X \bar{\nabla}_X h)(X, X) = h(X, X)\psi(X, X),$$

whenever $h(X, X) \neq 0$ holds at x.

THEOREM 3.8. *Let M be an isotropic submanifold in $\mathbb{R}^{m+d}$. Then the following statements are equivalent*:

(i) *M is of AW(3) type and*
(ii) *M is weak 2-recurrent.*

Proof. Let X be a vector at a point x at which $h(X, X) \neq 0$. Suppose that M is isotropic submanifold of AW(3) type. Then by Definition 3.5 $(\bar{\nabla}_X \bar{\nabla}_X h)(X, X) + 3h(A_{h(X,X)}X, X)$ and $h(X, X)$ are linearly dependent. Since M is isotropic $(\bar{\nabla}_X \bar{\nabla}_X h)(X, X)$ and $h(X, X)$ are linearly dependent too. This means that

$$(\bar{\nabla}_X \bar{\nabla}_X h)(X, X) = \frac{\langle h(X, X), (\bar{\nabla}_X \bar{\nabla}_X h)(X, X)\rangle}{\|h(X, X)\|^2} h(X, X).$$

We put

$$\psi(X, X) = \frac{\langle h(X, X), (\bar{\nabla}_X \bar{\nabla}_X h)(X, X)\rangle}{\|h(X, X)\|^2}.$$

If we consider ψ as a 2-form then, by Definition 3.7, f is weak 2-recurrent. Conversely, if M is weak 2-recurrent then, by (5) and (6), M is of AW(3) type. This completes the proof of the theorem.

DEFINITION 3.9. Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. The immersion f (or the submanifold M) is called *generalized weak 2-recurrent* (or *generalized weak birecurrent*) if there exist a 2-form ψ and 1-form ϕ on M such that at every point $x \in M$ and for every vector $X \in T_x(M)$ we have

$$(7) \qquad (\bar{\nabla}_X \bar{\nabla}_X h)(X, X) = h(X, X)\psi(X, X) + (\bar{\nabla}_X h)(X, X)\phi(X),$$

whenever $h(X, X) \neq 0$ and $(\bar{\nabla}_X h)(X, X) \neq 0$ hold at x.

THEOREM 3.10. *Let $f : M \longrightarrow \mathbb{R}^{m+d}$ be an isometric immersion. Then the following statements are equivalent*:

(i) *M is isotropic submanifold with PP3-PNS property,*
(ii) *M (or the immersion f) is generalized weak 2-recurrent.*

Proof. Suppose that M is isotropic submanifold with PP3-PNS property. Then by Theorem 2.4 the vectors $h(X, X)$, $(\bar{\nabla}_X h)(X, X)$ and

$(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)$ are linearly dependent in $N_x(M)$. So combining Proposition 2.2 and Theorem 2.11 with Theorem 2.4 we have

$$\begin{aligned}&\{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2\}(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\\ &\quad = \{\langle h(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\|(\bar{\nabla}_X h)(X,X)\|^2\\ &\quad - \langle(\bar{\nabla}_X h)(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle\}h(X,X)\\ &\quad + \{\|h(X,X)\|^2\langle(\bar{\nabla}_X\bar{\nabla}_X h)(X,X),(\bar{\nabla}_X h)(X,X)\rangle\\ &\quad - \langle h(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle\}(\bar{\nabla}_X h)(X,X).\end{aligned}$$

Since M has PP3-PNS property then by definition it does not have P2-PNS. Thus

$$\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 \neq \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2$$

holds at every point $x \in M$. Hence taking

$$\begin{aligned}\psi(X,X) &= \frac{\langle h(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\|(\bar{\nabla}_X h)(X,X)\|^2}{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2}\\ &- \frac{\langle(\bar{\nabla}_X h)(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle}{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2}\end{aligned}$$

and

$$\begin{aligned}\phi(X) &= \frac{\langle(\bar{\nabla}_X h)(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\|h(X,X)\|^2}{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2}\\ &- \frac{\langle h(X,X),(\bar{\nabla}_X\bar{\nabla}_X h)(X,X)\rangle\langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle}{\|h(X,X)\|^2\|(\bar{\nabla}_X h)(X,X)\|^2 - \langle h(X,X),(\bar{\nabla}_X h)(X,X)\rangle^2}\end{aligned}$$

we obtain (7), for each $X \in T_x(M)$. Conversely, if M is generalized weak 2-recurrent submanifold then by Definition 3.9 and Theorem 2.4, M must be isotropic with P3-PNS property.

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