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GEOMETRIC INEQUALITIES FOR PLANE HEDGEHOGS

Abstract. We define and study an algebraic area for plane hedgehogs (envelopes parametrized by their Gauss map). In particular, we study the extension to hedgehogs of the Minkowski inequality for plane convex bodies.

I. Introduction and statement of results

Let C be a convex curve of class C^1 in the Euclidean plane $\mathbb{E}^2$. We may consider C as the envelope of the family of support lines given by

$$x\cos\theta + y\sin\theta = p(\theta), \tag{1}$$

where the support function $p(\theta) = h(\cos\theta, \sin\theta)$ is defined as the signed distance of the support line to C with exterior normal vector $u(\theta) = (\cos\theta, \sin\theta)$ from the origin.

Given any $h \in C^2(\mathbb{S}^1; \mathbb{R})$, we may always consider the envelope $\mathcal{H}_h$ of the family of lines given by (1). Partial differentiation of (1) yields

$$-x\sin\theta + y\cos\theta = p'(\theta), \tag{2}$$

and from (1) and (2), the parametric representation of $\mathcal{H}_h$ is

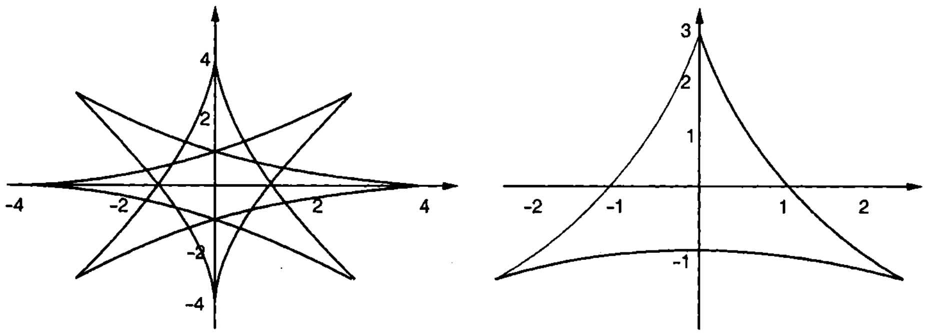
$$\begin{cases} x = p(\theta)\cos\theta - p'(\theta)\sin\theta \\ y = p(\theta)\sin\theta + p'(\theta)\cos\theta. \end{cases} \tag{3}$$

We say that $\mathcal{H}_h$ is the hedgehog defined by the support function h. In general, $\mathcal{H}_h$ is not a convex curve of class C^1. Since (3) implies

$$\frac{dx}{d\theta} = -(p+p'')(\theta)\sin\theta \ \text{ and } \ \frac{dy}{d\theta} = (p+p'')(\theta)\cos\theta,$$

we see that the condition that $\mathcal{H}_h$ is a convex curve of class C^1 is simply that $p + p''$ (which may be identified with the radius of curvature) has no zero. However, if $\mathcal{H}_h$ is not too singular (i.e. if $\mathcal{H}_h$ has a well-defined tangent line at every point), then $\mathcal{H}_h$ is a curve that has exactly one oriented tangent line in each direction (see Fig. 1.a).

For a study of hedgehogs (envelopes parametrized by their Gauss map) in $\mathbb{E}^{n+1}$, we refer to the paper of R. Langevin, G. Levitt and H. Rosenberg [2].



1.a 1.b

Fig. 1

When $h \in C^2(\mathbb{S}^1;\mathbb{R})$ is the support function of a convex curve C, it is well-known (see for example [5]) that

$$l(h) = \int_0^{2\pi} p(\theta)\, d\theta = \int_0^{2\pi} \frac{p(\theta) + p(\theta + \pi)}{2}\, d\theta$$

and

$$a(h) = \frac{1}{2} \int_0^{2\pi} p(\theta)(p(\theta) + p''(\theta))\, d\theta = \frac{1}{2} \int_0^{2\pi} (p(\theta)^2 - p'(\theta)^2)\, d\theta$$

are respectively the perimeter and the area of the convex body with boundary C. But for any $h \in C^2(\mathbb{S}^1;\mathbb{R})$, $a(h)$ (resp. $l(h)$) may be interpreted as an algebraic area (resp. length) of the hedgehog $\mathcal{H}_h$. More precisely, $a(h)$ can be viewed as the integral over $\mathbb{E}^2 - \mathcal{H}_h$ of the index $i_h(p)$ defined as algebraic intersection number of an oriented half-line with origin p with $\mathcal{H}_h$ equipped with its transverse orientation (this number is independent of the oriented half-line for an open dense set of directions). For example, if $p(\theta) = \sin 3\theta$ (see Fig. 1.b) the algebraic area of the hedgehog $\mathcal{H}_h$ is equal to -4π, i.e. $-2area(D)$, where D is the domain delimited by $\mathcal{H}_h$. The minus sign comes from the fact that D is concave at the regular points of its boundary and the factor 2 from the fact that the parametrization describes the curve twice.

In this paper, we prove the following results.

PROPOSITION 1. *Let V be the real vector space of plane hedgehogs defined up to a translation. The map $a : V \longrightarrow \mathbb{R}$, $h \longmapsto a(h)$ is a nondegenerate quadratic form.*

REMARK 1. The symmetric bilinear form $a(h,k)$ obtained by polarizing a may be interpreted as an algebraic mixed area of $\mathcal{H}_h$ and $\mathcal{H}_k$:

$$a(h,k) = \frac{1}{2}\int_0^{2\pi} p(\theta)(q(\theta)+q''(\theta))\,d\theta = \frac{1}{2}\int_0^{2\pi} (p(\theta)q(\theta) - p'(\theta)q'(\theta))\,d\theta,$$

where $p(\theta) = h(\cos\theta, \sin\theta)$ and $q(\theta) = k(\cos\theta, \sin\theta)$.

The following lemma can be viewed as a geometrical interpretation of Wirtinger's lemma.

LEMMA 1. *Let F be the subspace of V defined by the condition $l(h) = 0$. Then, we have:*

(i) $\forall h \in F$, $a(h) \leq 0$

(ii) $\forall h \in F$, $a(h) = 0 \Longleftrightarrow h = 0_V$.

REMARK 2. We define the (signed) width $w_h(\theta)$ of $\mathcal{H}_h$ in the direction $u(\theta) = (\cos\theta, \sin\theta)$ by

$$w_h(\theta) = p(\theta) + p(\theta+\pi), \text{ where } p(\theta) = h(\cos\theta, \sin\theta),$$

which is the (signed) distance between the two support lines of $\mathcal{H}_h$ orthogonal to $u(\theta)$.

Thus, elements of F may be interpreted as plane hedgehogs of zero mean width. Of course, plane hedgehogs of constant width zero are in F: such hedgehogs are said to be projective for, if they are not too singular, they have exactly one nonoriented tangent line in each direction (see Fig.1.b). Note that a plane hedgehog C of constant width $2r$ can be seen as the sum of a circle with radius r with a projective hedgehog (which may be interpreted as the locus of the middles of C's diameters). For a study of projective hedgehogs in $\mathbb{E}^{n+1}$, see [3]. A survey of convex bodies of constant width is given by Chakerian and Groemer [1].

As a corollary, we have the following result.

PROPOSITION 2. *The map $\sqrt{-a} : F \to \mathbb{R}_+$, $h \longmapsto \sqrt{-a(h)}$ is a norm associated with a scalar product. In particular, for any $(h,k) \in F^2$, we have the following inequalities*

$$\sqrt{-a(h+k)} \leq \sqrt{-a(h)} + \sqrt{-a(k)}, \tag{4}$$

$$a(h,k)^2 \leq a(h)\,a(k), \tag{5}$$

with equalities if and only if $\mathcal{H}_h$ and $\mathcal{H}_k$ are homothetic.

REMARK 3. For convenience, in this paper "$\mathcal{H}_h$ and $\mathcal{H}_k$ are homothetic" means "there exists $(\lambda,\mu) \in \mathbb{R}^2 - \{(0,0)\}$ such that $\lambda h + \mu k = 0_V$".

REMARK 4. Inequality (4) (resp. (5)) has to be compared with the Brunn-Minkowski inequality (resp. Minkowski inequality) in the plane $\mathbb{E}^2$ (see for example [4]): for any pair (H, K) of plane convex bodies, we have

$$\sqrt{a(H+K)} \geq \sqrt{a(H)} + \sqrt{a(K)} \tag{6}$$

and

$$a(H,K)^2 \geq a(H)\, a(K), \tag{7}$$

where $a(L)$ (resp. $a(H,K)$) is the area (resp. the mixed area) of L (resp. (H,K)).

The following result extends the Minkowski inequality to a pair of plane hedgehogs provided that one of them has a positive area.

THEOREM 1. *Let $(h,k) \in V^2$ be such that $a(h) > 0$ or $a(k) > 0$. Then, we have*

$$a(h,k)^2 \geq a(h)\, a(k), \tag{8}$$

with equality if and only if $\mathcal{H}_h$ and $\mathcal{H}_k$ are homothetic.

The following corollary extends the classical isoperimetric inequality to hedgehogs.

COROLLARY 1. *For any hedgehog $h \in V$, we have $a(h) \leq \frac{1}{4\pi} l(h)^2$, with equality if and only if $\mathcal{H}_h$ is a circle or a point.*

For a convex curve whose support function is of class C^3, we can sharpen this result as follows.

PROPOSITION 3. *Let C be a convex curve in $\mathbb{E}^2$. If its support function $p(\theta) = h(\cos\theta, \sin\theta)$ is of class C^3, then we have*

$$0 \leq l(h)^2 - 4\pi a(h) \leq -4\pi a(h'), \tag{9}$$

where $h'(\cos\theta, \sin\theta) = p'(\theta)$, and $a(h')$ is the area of the evolute curve of C.

The following corollary generalizes Lemma 1 (and therefore Wirtinger's lemma).

COROLLARY 2. *Let $g \in V$ be such that $a(g) > 0$ and let F_g be the subspace orthogonal (relatively to the nondegenerate quadratic form a) to the subspace spanned by g : $F_g = \{h \in V \mid a(g,h) = 0\}$. Then, we have*

(i) $\forall h \in F_g$, $a(h) \leq 0$
(ii) $\forall h \in F_g$, $a(h) = 0 \Longleftrightarrow h = 0_V$.

In particular, any hedgehog orthogonal to a convex one has a nonpositive area.

Thus, we have the following generalization of Proposition 2.

THEOREM 2. *Let $g \in V$ be a hedgehog such that $a(g) > 0$. The map $\sqrt{-a}$: $F_g \longrightarrow \mathbb{R}_+, h \longmapsto \sqrt{-a(h)}$ is a norm associated with a scalar product. In* *particular, inequalities (4) and (5) hold for any $(h,k) \in (F_g)^2$, with equalities if and only if $\mathcal{H}_h$ and $\mathcal{H}_k$ are homothetic.*

The following proposition relates plane hedgehogs of zero relative mean width to subspaces F_g. Consider an arbitrary norm $\|\cdot\|$ on $\mathbb{E}^2$ and denote by $q(\theta) = k(\cos\theta, \sin\theta)$ the support function of $K = \{v \in \mathbb{E}^2 \mid \|v\| = 1\}$. We define the (signed) width of $\mathcal{H}_h$ relative to K, or simply the K-width of $\mathcal{H}_h$, in direction of $u(\theta) = (\cos\theta, \sin\theta)$ by

$$w_h(K,\theta) = 2\frac{w_h(\theta)}{w_k(\theta)} = \frac{w_h(\theta)}{q(\theta)},$$

which is the (signed) distance (relatively to $\|\cdot\|$) between the two support lines of $\mathcal{H}_h$ orthogonal to $u(\theta)$.

PROPOSITION 4. *There exists a convex hedgehog g such that F_g is the subspace of V constituted by plane hedgehogs of zero mean K-width, which is defined by the condition*

$$\int_0^{2\pi} w_h(K,\theta)\, d\theta = 0 \text{ i.e. } \int_0^{2\pi} \frac{p(\theta)}{q(\theta)}\, d\theta = 0,$$

where $p(\theta) = h(\cos\theta, \sin\theta)$.

II. Further remarks and proof of results

Proof of Proposition 1. Obviously, the map a is a quadratic form on V. Though the nondegenaracy of a comes from Lemma 1, we give a direct proof of it. For any $h \in V - \{0_V\}$, consider the Fourier expansion of $p(\theta) = h(\cos\theta, \sin\theta)$, namely

$$p(\theta) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta),$$

and let $k(\cos\theta, \sin\theta) = p(\theta) - a_0$. We get easily

$$a(h,k) = -\frac{\pi}{2} \sum_{n=2}^{+\infty} (n^2 - 1)(a_n^2 + b_n^2),$$

so that $a(h,k) < 0$ unless $p(\theta) = a_0 + (a_1 \cos\theta + b_1 \sin\theta)$ with $a_0 \neq 0$, but in this case $\mathcal{H}_h$ is a circle so that $a(h) = a(h,h) > 0$. This completes the proof.□

REMARK 5. Note that any hedgehog $\mathcal{H}_g$ has a unique representation of the form

$$\mathcal{H}_g = \mathcal{H}_h + \mathcal{H}_k,$$

where $\mathcal{H}_h$ is projective and $\mathcal{H}_k$ centred (i.e. centrally symmetric with center at the origin), given by

$$g = h + k, \text{ where } h(u) = \frac{1}{2}(g(u) - g(-u)) \text{ and } k(u) = \frac{1}{2}(g(u) + g(-u)).$$

Using Proposition 2, we see easily that the nondegenerate quadratic form a defines a notion of orthogonality for which $P^{\perp} = Q$, where P (rep. Q) denotes the subspace of projective (resp. centred) hedgehogs. So $V = P \oplus Q = P \oplus P^{\perp}$.

Proof of Theorem 1. Suppose for example that $a(k) > 0$ and let $A : \mathbb{R} \to \mathbb{R}$ be the quadratic function defined by

$$A(t) = a(h + tk) = a(h) + 2ta(h,k) + t^2 a(k).$$

Since $a(k) > 0$, we have $A(t) > 0$ for large t. Moreover, from Lemma 1 $l(k) \neq 0$ and we can consider $t = -\frac{l(h)}{l(k)}$ for which $h + tk \in F$ so that $A(t) < 0$ unless $h + tk = 0_V$. Therefore, the discriminant of $A(t)$, namely

$$\triangle = 4(a(h,k)^2 - a(h)a(k)),$$

must satisfy $\triangle > 0$ unless $h + tk = 0_V$. To conclude, note that we have obviously $\triangle = 0$ when $\mathcal{H}_h$ and $\mathcal{H}_k$ are homothetic.□

Proof of Corollary 1. Take $k = 1$ in inequality (8).□

Proof of Proposition 3. By the Cauchy-Schwarz inequality, we have

$$l(h)^2 \leq 2\pi \int_0^{2\pi} (p + p'')(\theta)^2 \, d\theta = 4\pi(a(h) - a(h')),$$

and thus (9) considering Corollary 1. Moreover, it is easy to check that the support function of the evolute curve of C is given by $q(\theta) = p'(\theta - \frac{\pi}{2})$. This completes the proof.□

Remark 6. It follows immediately from Lemma 1 that evolute curves (i.e. the locus of centers of curvature) of hedgehogs are hedgehogs with nonpositive area.

Corollary 2 is an immediate consequence of Theorem 1.

Proof of Proposition 4. By the fact that $q(\theta) = k(\cos\theta, \sin\theta)$ is the support function of a centred hedgehog (see Remark 5), the general solution of the differential equation $y + y'' = \frac{1}{q}$, namely

$$y(\theta) = \Big(\int \frac{\cos\theta}{q(\theta)} \, d\theta\Big) \sin\theta - \Big(\int \frac{\sin\theta}{q(\theta)} \, d\theta\Big) \cos\theta,$$

is a 2π-periodic function, and thus defines a hedgehog g (up to a translation) which is convex since $y + y'' = \frac{1}{q} > 0$. The hedgehog g has the desired property.□

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