DEMONSTRATIO MATHEMATICA
Vol. XXXII No 1 1999

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IRREDUCIBLE *-REPRESENTATIONS ON BANACH *-ALGEBRAS

Dedicated to Professor Mursaleen

Abstract. In this paper, using the notion of g-bounded linear functionals on a Banach *- algebra A, a characterization of representable functionals on A in terms of the pure states of A is given. Further, it is also shown that the extreme points of the set of all g-bounded positive linear functionals on A are simply the pure states of A.

1. Introduction

Let A be a complex Banach *- algebra. We assume neither the existence of a unit element nor that the involution is continuous.

A B^*-semi-norm on A is a function $\eta : A \to R^+$ such that for all $a, b \in A$ and $\alpha \in C$,

1) $\eta(a+b) \leq \eta(a) + \eta(b)$,
2) $\eta(\alpha a) = |\alpha|\eta(a)$,
3) $\eta(ab) \leq \eta(a)\eta(b)$,
4) $\eta(a^*a) = (\eta(a))^2$.

Let $P(A)$ denote the set of all B^*-semi-norms on A and define

$$g(a) = \sup\{\eta(a) : \eta \in P(A)\}.$$

A linear functional f on A is g-bounded if there exists a constant $M_f > 0$ (depends on f) such that for all $a \in A$, $|f(a)| \leq Mg(a)$. If A' is the dual space of A then the norm of any g-bounded functional f is defined as follows

$$|f|_g = \sup\{f(a) : g(a) \leq 1\}.$$

1991 *Mathematics Subject Classification*: 46J99, 46J15.
Key words and phrases: Banach *-algebra, cyclic representation, pure states.

We denote by $D(g)$ the set of all g-bounded functionals f on A with $|f|_g \leq 1$. A g-bounded linear functional f on A will be called a state of A if $|f|_g = 1$.

An A^*-representation of A is a mapping π from A into $B(H)$, where $B(H)$ denotes the algebra of all bounded linear operators on a Hilbert space H, such that for all a in A and $\alpha \in C$,

1) $\pi(a+b) = \pi(a) + \pi(b)$
2) $\pi(\alpha a) = \alpha\pi(a)$,
3) $\pi(ab) = \pi(a)\pi(b)$,
4) $\pi(a^*) = \pi(a))^*$.

We define an involution and norm on A^+, the unitization of A, as follows:

i) $(a, \alpha)^* = (a^*, \overline{\alpha})$, for all $a \in A$,
ii) $\|(a, \alpha)\| = \|a\| + |\alpha|$ for all $a \in A$.

Then the Banach *-algebra A is isometrically and *-isomorphically embedded in the unital Banach *-algebra A^+.

DEFINITION 1.1. Let T be a self-adjoint operator on H and f be the positive linear functional represented by (π, x) defined by $|f|_T(a) = (\pi(a)Tx, Tx)$ for all $a \in A$.

The following results are known in the case of B^*-algebras (in which case g coincides with the original norm) and proofs can be found in [5]. They are included for the sake of completeness.

THEOREM 1.1. *Let u and v be positive g-bounded linear functionals on A. Then for all $a \in A$ we have the following:*

i) $|u(a)|^2 \leq |u|_g u(a^*a)$,
ii) $|u+v|_g = |u|_g + |v|_g$,
iii) $|u|_g = \sup\{u(a^*a) : g(a) \leq 1\}$,
iv) *u and v are hermitian functionals.*

THEOREM 1.2. *If f is a g-bounded positive linear functional on A then f can be represented by (π, x) where x is a cyclic vector for the *-representation π and moreover $|f|_g = \|x_f\|_f^2$.*

REMARK 1.1. In a nonunital complex Banach *-algebra A the g-bounded functionals are characterized by the representable functionals, (see Theorem 3.1 of [5]). On the other hand, if A has a unit element then every positive linear functional is g-bounded (see Remark 2.2 of [5]). If A has an approximate identity and isometric involution, then each functional is g-bounded if and only if it is continuous. This follows by Theorem 2.2 of [5].

In the light of Remark 1.1, one can find stronger results on representable functionals on any complex Banach *-algebras.

Proposition 1.1. *If (π, x) is a cyclic representation of a positive linear functional f, then $\pi \cong \pi_f$ and $\|x\|^2 = |f|_g$.*

Proof. If x and $x - f$ are the cyclic vectors for π and π_f respectively then $(\pi(a)x, x) = f(a) = (\pi_f(a)x_f, x_f)$ for all $a \in A$. Thus, by a result in [2], (2.4.1), it follows that $\pi \cong \pi_f$ and $|f|_g = \|x\|^2 = \|x_f\|_f^2$.

The following lemma is a generalization of Theorem 2.5.1 of Dixmier [2].

Lemma 1.1. *Let π be an *-representation of A on H, and let $x \in H$. Let f be the positive linear functional represented by (π, x). Then*

1) *for a self-adjoint operator T on H such that $T\pi(A) = \pi(A)T$ and $0 \leq T \leq I_H$, f_T is g-bounded positive linear functional on A with $f_T \leq f$.*

2) *if the representation of f by (π, x) is cyclic, the mapping $T \to f_T$ is one-to-one.*

3) *any positive g-bounded linear functional f_1, on A, dominated by f, is of type f_T.*

Proof. 1) By the construction of f_T and the hypothesis, f_T is a positive linear g-bounded functional. Also for all $a \in A$

$$f_T(a^*a) = \|\pi(a)Tx\|^2 = \|T\pi(a)x\|^2 \leq \|\pi(a)x\|^2 = f(a^*a).$$

Thus $f_T \leq f$.

2) Let $f_{T_1} - f_{T_2} = 0$. Then for all $a \in A$, $(\pi(a)x, T_1^2 x) = (\pi(a)x, T_2^2 x)$. Since x is a cyclic vector for π, it follows that $(T_2^1 - T_2^2)x = 0$. Moreover $(T_1^2 - T_2^2)\pi(a)x = 0$ for all $a \in A$. Thus $T_1^2 = T_2^2$ and since T_1 and T_2 are both positive, we have $T_1 = T_2$.

3) Let f_1 be a g-bounded positive linear functional on A with $f_T \leq f$. Then for all $a \in A$ we have

$$|f_1(b^*a)|^2 \leq f_1(a^*a)f_1(b^*b) \leq f(a^*a)f(b^*b) = \|\pi(a)x\|^2\|\pi(b)x\|^2.$$

On the subspace $\pi(A)x$ of H we define for all $a, b \in A$, $\varphi(\pi(a)x, \pi(b)x) = f_1(b^*a)$. Then φ is bounded, symmetric, and a positive sesquilinear functional on $\pi(A)x$ and so can be uniquely extended to the Hilbert space $K = \overline{\pi}(A)x$. Hence, by Theorem 21.1, [1] there exists a bounded self-adjoint linear operator S on K with $0 \leq S \leq I_K$ such that for all $a, b \in A$

$$f_1(b^*a) = (\pi(a)x, S\pi(b)x)$$

since $(\pi(a)x, S\pi(c)\pi(b)x) = (\pi(a)x, \pi(c)S\pi(b)x)$ for all $a, b, c \in A$, therefore for all $a \in A$

$$\pi(a)S = S\pi(a) \text{ on } K.$$

As S is invariant under $\pi(A)$ the orthogonal projection i_K on K commutes with $\pi(A)$, since Si_K is a bounded linear operator on H with $0 \leq$

$Si_K \leq I_K$. Let T be the positive square root of Si_K. Then $0 \leq T \leq I_H$ and $T\pi(A) = \pi(A)T$. Hence

$$\begin{aligned} f_1(b^*a) &= (\pi(a)x, Si_K(\pi(b)x) \\ &= (\pi(a)x, T^2\pi(b)x) \\ &= (\pi(a)Tx, \pi(b)Tx) \\ &= (\pi(b^*a)Tx, Tx) \\ &= f_T(b^*a). \end{aligned}$$

That is $f_1(b^*a) = f_T(b^*a)$ for all $a, b \in A$. Now f_1 and f_T are both g-bounded positive linear functionals and hence by Theorem 1.1 we can consider the associated positive linear functionals $\overline{f}_1$ and $\overline{f}_T$ on the quotient *-algebra A/J_g where $J_g = \{a \in A : g(a) = 0\}$, and F_1 and F_T on A_g is the completion of A/J_g. Hence $\overline{f}_1(x_b^* x_a) = \overline{f}_T(x_b^* x_a)$, $x_b, x_a \in A/J_g$ and therefore $F_1(b^*a) = F_T(b^*a)$, for all $a, b \in A_g$. Since A_g is a B^*-algebra, it has an approximate identity e_λ where λ is in the directed set I_λ. In this case $F_1(b^*) = F_1(b^*e_\lambda) = F_T(b^*e_\lambda) = F_T(b^*)$ for all $b \in A_g$. Hence,

$$\overline{f}_1(x_a) = \overline{f}_T(x_a), x_a \in A/J_g$$

so that for all $a \in A$, $f_1(a) = f_T(a)$.

2. Pure states and irreducible *-representations

Ultimately in this section we prove a characterization of those representable functionals which can be represented by a topologically irreducible representation.

DEFINITION 2.1. A positive linear functional f is a pure state of A if it is non-zero and g-bounded and if any g-bounded positive linear functional dominated by f is of the form βf with β in the closed interval $[0, 1]$. $D_p(A)$ denotes the set of all pure states of A.

THEOREM 2.1. *Let (π, x) be a cyclic representation of a positive linear functional f. Then π is topologically irreducible and non-zero if and only if f is a pure state of A.*

Proof. Suppose that π is topologically irreducible and non-zero. Since x is cyclic vector of π, it follows that for all a in A

$$\begin{aligned} \eta_f(a) &= \sup\{[f(y^*a^*ay)]^{1/2} : f(y^*y) \leq 1\} \\ &= \sup\{\|\pi(a)\pi(y)x\| : \|\pi(y)x\| \leq 1\} = |\pi(a)|. \end{aligned}$$

Thus if π is non-zero then f is also non-zero.

Let f_1 be a positive g-bounded functional on A dominated by f. Then by Part 3) of Lemma 1.1 there exists a self-adjoint operator T on H with $0 \leq T \leq I_H$ and $T\pi(A) = \pi(A)T$, such that for all a in $Af_1(a) = (\pi(a)Tx, Tx)$.

Using the fact that π is irreducible and $T\pi(A) = \pi(A)T$ ([2], 2.3.1), we have for all scalars λ in $C, T = \lambda I_H$ since $0 \leq T \leq I_H$ and it follows that λ is real and $0 \leq \lambda \leq 1$. Thus for all $a \in A$

$$f_1(a) = (\pi(a)\lambda x, \lambda x) = \lambda^2 f(a).$$

Hence f is pure.

Conversely, suppose that f is a pure state. Let M be a subspace of H invariant under $\pi(A)$. Let i_H denote the orthogonal projection of H onto M. Then i_H is a self-adjoint with $0 \leq i_H \leq i_H$ and $i_H\pi(A) = \pi(A)i_H$. Hence by Lemma 1.1 (1) the functional f_{i_H} defined by

$$f_{i_H}(a) = (\pi(a)i_H x, i_H x), \quad \text{for all } a \in A$$

is positive g-bounded functional dominated by f. Since f is pure, $f_{i_H} = \lambda f$ with $0 \leq \lambda \leq 1$ so that for all a in A

$$(\pi(a)i_H x, i_H x) = (\pi(a)\lambda^{1/2}x, \lambda^{1/2}x).$$

Now from Lemma 1.1 (2) we have $i_H = \lambda^{1/2}I_H$ since (π, x) is a cyclic representation of f. Note that $\lambda = 0$ or $\lambda = 1$ since $i_H^2 = i_h$. Thus $i_H = 0$ or $i_H = I_H$ and consequently $M = \{0\}$ or $M = H$. Hence, π is topologically irreducible and $\pi \neq 0$.

COROLLARY 2.1. *If f is a pure state of A then there exists a non-zero topologically irreducible *-representation of A such that for all $a \in A \eta_f(a) = |\pi(a)|$.*

Proof. Let f be a pure state of A. By Theorem 1.2. f has a cyclic representation (π, x), which by Proposition 1.1 is unique up to equivalence, such that $\|x\|^2 = |f|g = 1$. Thus by Theorem 2.1, π is irreducible and non-zero and $\eta_f(a) = |\pi(a)|$ for all $a \in A$.

COROLLARY 2.2. *If π is a non-zero topologically irreducible *-representation of A on H, then there exists a pure state f of A such that for all $a \in A$*

$$\eta_f(a) = |\pi(a)|.$$

Proof. Suppose π is a non-zero topologically irreducible *-representation of A on H. Let f be the positive linear functional represented by the pair (π, x) for $x \in H$. By Theorem 2.1, f is pure and by Proposition 1.1 $\|x\|^2 = |f|g = 1$. Thus, f is a pure state of A and it is obvious to see that for all $a \in A$, $\eta_f(a) = |\pi(a)|$.

REMARK 2.1. The above Corollaries 2.1 and 2.2 give a mapping from $D_p(A)$ onto A, the set of equivalence classes of non-zero topologically irreducible *-representations of A.

3. The extreme points of $D(g)$

The following theorem shows that if A is a complex Banach *-algebra, the extreme points of $D(g)$ are just the pure states of A.

THEOREM 3.1. *The extreme points of $D(g)$ are the pure states of A.*

Proof. Let $f_1 + f_2 = 0$ and for all $a \in A$. But $|f_1(a)|^2 \leq f_1(a^*a) = 0$ so $f_1(a) = 0$ for all $a \in A$. Thus $f_1 = f_2 = 0$ and 0 is an extreme point of $D(g)$.

Next, let f be a pure state of A, and suppose for any $f_1, f_2 \in D(g)$, $f = \frac{1}{2}(f_1 + f_2)$. Then $0 \leq \frac{1}{2}f_1 \leq f$ and $\frac{1}{2}f_1 = \lambda f$ with $0 \leq \lambda \leq 1$. But $1 = |f|_g = \frac{1}{2}(|f_1|g + |f_2|g)$, where $|f_1|g \leq 1$ and $|f_2|g \leq 1$. Hence $|f_1|_g = |f_2|_g = 1$ and $\lambda = \frac{1}{2}$. Thus $f = f_1 = f_2$ so that f is an extreme point of $D(g)$.

Now suppose that f is an extreme point of $D(g)$ such that $f \neq 0$. In this case $|f|_g = 1$. Suppose $f = f_1 + f_2$. Then $|f|_g = |f_1|_g + |f_2|_g = 1$. If $|f_1|_g \neq 0$ and $|f_2|_g \neq 0$ then

$$f = |f_1|_g\left(\frac{f_1}{|f_1|_g}\right) + |f_2|_g\left(\frac{f_2}{|f_2|_g}\right).$$

Since $\frac{f_1}{|f_1|_g}$ and $\frac{f_2}{|f_2|_g}$ belong to $D(g)$ and since f is an extreme point of $D(g)$ it follows that $\frac{f_1}{|f_1|_g} = \frac{f_2}{|f_2|_g} = f$. Thus, $0 \leq f_1 \leq f$ implies that $f_1 = |f_1|_g f$ with $0 \leq |f_1|_g \leq 1$ which proves that f is a pure state.

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Received June 9, 1998.