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SOME FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS SATISFYING AN IMPLICIT RELATION

1. Introduction

Let S and T be two self mappings of a metric space (X,d) Sessa [2] defines S and T to be weakly commuting if $d(STx, TSx) \leq d(Tx, Sx)$ for all x in X Jungck [1] defines S and T to be compatible if $\lim_{n\to\infty}(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$ for some x in X. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but neither implications is reversible [3, Ex. 1] and [1, Ex. 2.2].

LEMMA 1 [4]. *Let f and g be two self mappings of the set $X = \{x, y\}$ with any metric d. If the range of g contains the range of f, then the following statements are equivalent:*

1) *f and g commute,*
2) *f and g weakly commute,*
3) *f and g are compatible.*

By Lemma 1, we suppose that X contains at least three points.

LEMMA 2 [1]. *Let f and g be compatible self mappings on a metric space (X,d). If $f(t) = g(t)$, then $fg(t) = gf(t)$.*

The purpose of this paper is to prove some fixed point theorems for compatible mappings satisfying an implicit relation.

2. Implicit relations

Let $\mathcal{F}$ be the set of all real continuous functions $F(t_1, \ldots, t_6) : R_+^6 \to R$ satisfying the following conditions:

F_1: F is non-increasing in variables t_5 and t_6,
F_2: there exists $h \in (0,1)$ such that for every $u, v \geq 0$ with
(F_a): $F(u, v, v, u, u+v, 0) \leq 0$ or
(F_b): $F(u, v, u, v, 0, u+v) \leq 0$
we have $u \leq h \cdot v$.

F_3: $F(u, u, 0, 0, u, u) > 0, \forall u > 0$.

EXAMPLE 1. $F(t_1, \ldots, t_6) = t_1 - k \cdot \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $k \in (0,1)$. F_1: Obviously.

(F_a): Let be $u > 0$ and $F(u, v, v, u, u+v, 0) = u - k \cdot \max\left\{v, v, u, \frac{1}{2}(u+v)\right\} \leq 0$. If $u \geq v$, then $u \leq k \cdot u < u$, a contradiction. Thus $u < v$ and $u \leq k \cdot v = h \cdot v$, where $h = k \in (0,1)$.

(F_b): Let be $u > 0$ and $F(u, v, u, v, 0, u+v) \leq 0$, then $u \leq h \cdot v$. If $u = 0$, then $u \leq h \cdot v$.

F_3: $F(u, u, 0, 0, u, u) = u - k \cdot u = (1-k)u > 0,\ \forall u > 0$.

EXAMPLE 2. $F(t_1, \ldots, t_6) = t_1^2 - c_1 \cdot \max\{t_2^2, t_3^2, t_4^2\} - c_2 \cdot \max\{t_3 t_5, t_4 t_6\} - c_3 t_5 t_6$, where $c_1 > 0$, c_2, $c_3 \geq 0$, $c_1 + 2c_2 < 1$, and $c_1 + c_3 < 1$.

F_1: Obviously.

(F_a): Let be $u > 0$ and $F(u, v, v, u, u+v, 0) = u^2 - c_1 \cdot \max\{u^2, v^2\} - c_2 v(u+v) \leq 0$. If $u \geq v$, then $u^2(1 - c_1 - 2c_2) \leq 0$ which implies $c_1 + 2c_2 \geq 1$, a contradiction. Thus $u < v$ and $u \leq \sqrt{(c_1 + 2c_2)}v = hv$, where $h = \sqrt{c_1 + 2c_2} < 1$.

(F_b) Let be $u > 0$ and $F(u, v, u, v, 0, u+v) \leq 0$, then $u \leq h \cdot v$. If $u = 0$, then $u \leq hv$.

F_3: $F(u, u, 0, 0, u, u) = u^2(1 - (c_1 + c_3)) > 0, \forall u > 0$.

EXAMPLE 3. $F(t_1, \ldots, t_6) = t_1^2 - t_1(at_2 + bt_3 + ct_4) - dt_5 t_6$, where $a > 0$, $b, c, d \geq 0$, $a + b + c < 1$ and $a + d < 1$.

F_1: Obviously.

(F_a); Let be $u > 0$ and $F(u, v, v, u, u+v, 0) = u^2 - u(av + bv + cu) \leq 0$. Then $u \leq (\frac{a+b}{1-c}) \cdot v = h_1 \cdot v$, where $h_1 = \frac{a+b}{1-c} < 1$.

(F_b): Let be $u > 0$ and $F(u, v, u, v, 0, u+v) = u^2 - u(av + bu + cv) \leq 0$. Then $u \leq (\frac{a+c}{1-b})v = h_2 v$, where $h_2 = \frac{a+c}{1-b} < 1$. Therefore, $u \leq h, v$ where $h = \max\{h_1, h_2\}$.

If $u = 0$ then $u \leq hv$.

F_3: $F(u, u, 0, 0, u, u) = u^2(1 - (a+d)) > 0, \forall u > 0$.

EXAMPLE 4. $F(t_1, \ldots, t_6) = t_1^3 - at_1^2 t_2 - bt_1 t_3 t_4 - ct_5^2 t_6 - dt_5 t_6^2$, where $a > 0$, $b, c, d \geq 0$, $a + b < 1$ and $a + c + d < 1$.

F_1: Obviously.

(F_a): Let be $u > 0$ and $F(u,v,v,u,u+v,0) = u^3 - au^2v - bu^2v \leq 0$, then $u \leq (a+b)v = hv$, where $h = a+b < 1$.

(F_b): Let be $u > 0$ and $F(u,v,u,v,0,u+v) \leq 0$, then $u \leq hv$. If $u = 0$, then $u \leq hv$.

F_3: $F(u,u,0,0,u,u) = u^3(1-(a+d+c)) > 0$, $\forall u > 0$.

EXAMPLE 5. $F(t_1,\dots,t_6) = t_1^3 - c \cdot \frac{t_3^2t_4^2+t_5^2t_6^2}{t_2+t_3+t_4+1}$, where $c \in (0,1)$.

F_1: Obviously.

(F_a): Let be $u > 0$ and $F(u,v,v,u,u+v,0) = u^3 - \frac{cu^2v^2}{u+2v+1} \leq 0$ which implies $u \leq \frac{cv^2}{2v+u+1}$. But $\frac{cv^2}{2v+u+1} \leq cv$ is equivalent to $u+v+1 > 0$ an evident relation. Thus $u \leq cv = hv$, where $h = c < 1$.

(F_b): Let be $u > 0$ and $F(u,v,u,v,0,u+v) \leq 0$, then $u \leq hv$. If $u = 0$ then $u \leq hv$.

F_3: $F(u,u,0,0,u,u)u^3 - \frac{cu^4}{v+1} = u^3 \cdot \frac{(1-c)u+1}{u+1} > 0$, $\forall u > 0$.

REMARK. There exists a function $F \in \mathcal{F}$ which is increasing in variables t_3 or t_4.

EXAMPLE 6. $F(t_1,t_2,\dots,t_6) = t_1^2 - at_2^2 - \frac{bt_5t_6}{t_3^2+t_4^2+1}$, where $a > 0$, $b \geq 0$ and $a+b < 1$.

F_1: Obviously.

(F_a): Let be $u > 0$ and $F(u,v,v,u,u+v,0) = u^2 - av^2 \leq 0$ which implies $u \leq a^{\frac{1}{2}}v = vh$, where $h = a^{\frac{1}{2}} < 1$.

(F_b): Let $u > 0$ be and $F(u,v,u,v,0,u+v) \leq 0$, then $u \leq hv$. If $u = 0$, then $u \leq hv$.

F_3: $F(u,u,0,0,u,u) = u^2(1-a-b) > 0$, $\forall u > 0$.

3. Common fixed point theorems

THEOREM 1. *Let (X,d) be a metric space and $S,T,I,J : (X,d) \to (X,d)$ four mappings satisfying the inequality*

(1) $\quad F(d(Sx,Ty), d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty), d(Jy,Sx)) \leq 0$

for all x,y in X, where F satisfies property (F_3). Then S,T,I,J have at most one common fixed point.

Proof. Suppose that S,T,I,J have two common fixed point z and z' with $z \neq z'$. Then by (1) we have

$$F(d(Sz,Tz'), d(Iz,Jz'), d(Iz,Sz), d(Jz',Tz'), d(Iz,Tz'), d(Jz',Sz)) = F(d(z,z'), d(z,z'), 0, 0, d(z,z'), d(z,z')) \leq 0,$$

a contradiction to (F_3).

THEOREM 2. *Let S, T, I, J be mappings from a complete metric space (X, d) into itself satisfying the conditions:*

(a) $S(X) \subset J(X)$ *and* $T(X) \subset I(X)$,

(b) *one of* S, T, I, J *is continuous*,

(c) S *and* I *as well as* T *and* J *are compatible*,

(d) *the inequality* (1) *holds for all* x, y *in* X, *where* $F \in \mathcal{F}$. *Then* S, T, I, J *have a unique common fixed point.*

Proof. Suppose x_0 an arbitrary point in X. Then, since (a) holds, we can define inductively a sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Sx_{2n}, Tx_{2n+1}, \ldots\} \tag{2}$$

such that $Sx_{2n} = Jx_{2n+1}, Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$. Using inequality (1), we have succesively

$$\begin{aligned}
&F(d(Sx_{2n}, Tx_{2n+1}), d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}),\\
&\qquad d(Jx_{2n+1}, Tx_{2n+1}), d(Ix_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, Sx_{2n})) \leq 0,\\
&F(d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}),\\
&\qquad d(Sx_{2n}, Tx_{2n+1}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), 0) \leq 0.
\end{aligned}$$

By (F_a), we have

$$d(Sx_{2n}, Tx_{2n+1}) \leq h \cdot d(Tx_{2n-1}, Sx_{2n}).$$

Similarly, by (F_b), we have

$$d(Sx_{2n}, Tx_{2n-1}) \leq h \cdot d(Sx_{2n-2}, Tx_{2n-1})$$

and so

$$d(Sx_{2n}, Tx_{2n-1}) \leq (h)^{2n} d(Sx_0, Tx_1) \quad \text{for } n = 0, 1, 2, \ldots.$$

By a routine calculation it follows that (2) is a Cauchy sequence. Since X is complete, the sequence (2) converges to a point z in X. Hence, z is also the limit of the subsequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ of (2).

Let us now suppose that I is continuous, so that the sequence $\{ISx_{2n}\}$ converges to $\{Iz\}$. We have

$$d(SIx_{2n}, Iz) \leq d(SIx_{2n}, ISx_{2n}) + d(ISx_{2n}, Iz).$$

Since I is continuous and S and I are compatible, letting n tend to infinity, we state that the sequence $\{SIx_{2n}\}$ also converges to Iz. Using (1), we have

$$\begin{aligned}
&F(d(SIx_{2n}, Tx_{2n+1}), d(I^2x_{2n}Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}),\\
&\qquad d(Jx_{2n+1}, Tx_{2n+1}), d(I^2x_{2n}, Tx_{2n+1}), d(Jx_{2n+1}, SIx_{2n})) \leq 0.
\end{aligned}$$

Letting n tend to infinity we have, by the continuity of F,

$$F(d(Iz,z),d(Iz,z),0,0,d(Iz,z),d(z,z)) \leq 0,$$

a contradiction to (F_3), if $d(Iz,z) \neq 0$. Thus $Iz = z$. Further, by (1), we have

$$F(d(Sz,Tx_{2n+1}),d(Iz,Jx_{2n+1}),d(Iz,Sz), \\ d(Jx_{2n+1},Tx_{2n+1}),d(Iz,Tx_{2n+1}),d(Jx_{2n+1},Sz)) \leq 0$$

and letting n tend to infinity we get

$$F(d(Sz,z),0,d(z,Sz),0,0,d(z,Sz)) \leq 0$$

which implies, by (F_b), that $z = Sz$. This means that z is in the range of S and, since $S(X) \subset J(X)$, there exists a point u in X such that $Ju = z$. Using (1), we have successively

$$F(d(Sz,Tu),d(Iz,Ju),d(Iz,Sz),d(Ju,Tu),d(Iz,Tu),d(Ju,Sz)) \\ = F(d(z,Tu),0,0,d(z,Tu),d(z,Tu),0) \leq 0$$

which implies by (F_a), that $z = Tu$.

Since $Ju = Tu = z$, by Lemma 2, it follows that $TJu = JTu$ and so $Tz = TJu = JTu = Jz$. Thus, from (1) we have

$$F(d(Sz,Tz),d(Iz,Jz),d(Iz,Sz),d(Jz,Tz),d(Iz,Tz),d(Jz,Sz)) \\ = F(d(z,Tz),d(z,Tz),0,0,d(z,Tz),d(z,Tz)) \leq 0,$$

a contradiction to (F_3), if $z \neq Tz$. Thus $z = Tz = Jz$. We have therefore proved that z is a common fixed point of S,T,I,J. The same result holds, if we assume that J is continuous instead of I.

Now suppose that S is continuous. Then the sequence $\{SIx_{2n}\}$ converges to Sz. We have

$$d(ISx_{2n},Sz) \leq d(ISx_{2n},SIx_{2n}) + d(SIx_{2n},Sz).$$

Since S is continuous and S and T are compatible, letting n tend to infinity, we state that $\{ISx_{2n}\}$ converges to Sz. Using the inequality (1), we have

$$F(d(S^2x_{2n},Tx_{2n+1}),d(ISx_{2n},Jx_{2n+1}),d(ISx_{2n},S^2x_{2n}), \\ d(Jx_{2n+1},Tx_{2n+1}),d(ISx_{2n},Tx_{2n+1}),d(Jx_{2n+1},S^2x_{2n})) \leq 0.$$

Letting n tend to infinity, we have, by continuity of F,

$$F(d(Sz,z),d(Sz,z),0,0,d(Sz,z),d(Sz,z)) \leq 0,$$

a contradiction to (F_3) if $z \neq Sz$. Thus $z = Sz$. This means that z is in the range of S and, since $S(X) \subset J(X)$, there exists a point v in X such that $Jv = z$. Thus, by (1), we have

$$F(d(s^2x_{2n},Tv),d(ISx_{2n},Jv),d(ISx_{2n},S^2x_{2n}),$$
$$d(Jv,Tv),d(ISx_{2n},Tv),d(Jv,S^2x_{2n}))\leq 0.$$

Letting n tend to infinity we get

$$F(d(z,Tv),0,0,d(z,Tv),d(z,Tv),0)\leq 0$$

and, by (F_a), it follows that $z=Tv$. Since $Jv=Tv=z$, by Lemma 2, it follows that $Tz=TJv=JTv=Jz$. Thus, from (1) we have

$$F(d(Sx_{2n},Tz),d(Ix_{2n},Jz),d(Ix_{2n},Sx_{2n}),$$
$$d(Jz,Tz),d(Ix_{2n},Tz),d(Jz,Sx_{2n}))\leq 0.$$

Letting n tend to infinity, we obtain

$$F(d(z,Tz),d(z,Tz),0,0,d(z,Tz),d(z,Tz))\leq 0$$

and, by (F_3), it follows that $z=Tz=Jz$. This means that z is in the range of T and, since $T(X)\subset I(X)$ there exists $w\in X$ such that $Iw=z$. Thus, from (1) we have

$$F(d(Sw,Tz),d(Iw,Jz),d(Iw,Sw),d(Jz,Tz),d(Iw,Tz),d(Jz,Sw))$$
$$=F(d(Sw,z),0,d(z,Sw),0,0,d(z,Sw))\leq 0$$

and, by (F_b), we have $z=Sw=Iw$. Since $Sw=Iw=z$, by Lemma 2, it follows that $z=Sz=SIw=Iz$ and thus $z=Iz$. We have therefore proved that z is a common fixed point of S,T,I and J.

The same result holds, if we assume that T is continuous instead of S. By Theorem 1, z is the unique common fixed point of S,T,I,J.

For a function $f:(X,d)\to(X,d)$ we denote $F_f=\{x\in X: x=f(x)\}$.

THEOREM 3. *Let I,J,S,T be mappings from a metric space (X,d) into itself. If the inequality* (1) *holds for all x,y in X then $(F_I\cap F_J)\cap F_S=(F_i\cap F_J)\cap F_T$.*

Proof. Let $x\in(F_I\cap F_J)\cap F_S$. Then, by (1), we have

$$F(d(Sx,Tx),d(Ix,Jx),d(Ix,Sx),d(Jx,Tx),d(Ix,Tx),d(Jx,Sx))$$
$$=F(d(x,Tx),0,0,d(x,Tx),d(x,Tx),0))\leq 0$$

which implies, by (F_a), that $x=Tx$. Thus $(F_I\cap F_J)\cap F_S\subset(F_I\cap F_J)\cap F_T$. Similarly, we have by (F_b), that $(F_I\cap F_J)\cap F_T\subset(F_I\cap F_J)\cap F_S$.

The Theorems 2 and 3 imply the following one.

THEOREM 4. *Let I,J and $\{T_i\}_{i\in N^*}$ be mappings from a complete metric space into itself such that*

(a) *$T_2(X)\subset I(X)$ and $T_1(X)\subset J(X)$,*
(b) *one of I,J,T_1 and T_2 is continuous,*
(c) *the pairs (T_1,I) and (T_2,J) are compatible,*

(d) *the inequality*

$$F(d(T_ix, T_{i+1}y), d(Ix, Jy), d(Ix, T_ix), d(Jy, T_{i+1}y), d(Ix, T_{i+1}y), d(Jy, T_ix)) \leq 0$$

holds for each x, y *in* X, $\forall i \in N^*$ *and* $F \in \mathcal{F}$. *Then* I, J, *and* $\{T_i\}_{i \in N^*}$ *have a unique common fixed point.*

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