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INVERSIVE CLOSURE OF METRIC AFFINE SPACE AND ITS AUTOMORPHISMS

1. Basic notions and constructions

Let $\mathfrak{F}$ be a commutative field and $\mathbb{W}$ be a vector space over $\mathfrak{F}$. Then the projective space $\mathfrak{P} = \mathbf{P}(\mathbb{W})$ over $\mathbb{W}$ is the structure $\langle X, \mathcal{L}\rangle$ with points (X) being 1-dimensional subspaces of $\mathbb{W}$ and lines $(\mathcal{L})$ being 2-dimensional subspaces (cf [1]). Let $\mathcal{P}$ be the set of planes in $\mathfrak{P}$.

Let μ be a nondegenerate symmetric bilinear form on $\mathbb{W}$. Then μ determines a quadric

$$Q = Q(\mu) = \{\langle u\rangle : \mu(u,u) = 0, u \neq \theta_{\mathbb{W}}\},$$

where $\theta_{\mathbb{W}}$ is the zero-vector of $\mathbb{W}$. Let $p \in Q(\mu)$; The structure

$$\mathfrak{P}_p = \langle\{k \in \mathcal{L} : p \in k\}, \{S \in \mathcal{P} : p \in S\}, \subset\rangle$$

is a projective space, canonically isomorphic to $\mathbf{P}(\mathbb{V}_{/p})$. The set

$$T_p(Q) = \{q \in X : \mu(p,q) = 0\}$$

is the tangent hyperplane of $Q(\mu)$ in p. Then the structure $\mathbf{A}_p(Q)$ obtained by removing from $\mathfrak{P}_p$ elements contained in $T_p(Q)$ is an affine space (cf. [2]). Its horizon, i.e. the projective space $\mathfrak{T}_p(Q)$, defined on directions, can be identified with the restriction of $\mathfrak{P}_p$ to elements contained in $T_p(Q)$.

Note that every point x of $Q(\mu) \setminus T_p(Q)$ can be identified with the line $L_{\mathfrak{P}}(p,x) = \overline{p,x}$ of $\mathfrak{P}$ joining p and x. Thus there is a natural one-to-one correspondence given by the map

$$\pi : Q \setminus T_p \ni x \mapsto \overline{p,x}$$

between points of $Q \setminus T_p$ and points of $\mathbf{A}_p(Q)$.

Moreover, we can consider a correlation ω_μ given by the condition

$$\langle w_1\rangle \omega_\mu \langle w_2\rangle :\Leftrightarrow \mu(w_1, w_2) = 0$$

for $\langle w_1\rangle, \langle w_2\rangle \in X$. For l being a line of $\mathbf{A}_p(Q)$, l is a plane in $\mathfrak{P}$ through p; we denote $\tilde{l} = l \cap T_p(Q)$. Note that $\tilde{l}$ can be considered as the direction of l.

The correlation ω_μ defines a (restricted) correlation in $\mathfrak{T}_p$, which enables us to define orthogonality of lines of $\mathbf{A}_p(Q)$ as follows. Let l_1, l_2 be lines of $\mathbf{A}_p(Q)$, we define

$$l_1 \perp_{Q,p} l_2 :\Leftrightarrow \tilde{l}_1 \omega_\mu \tilde{l}_2.$$

So in fact $\mathbf{A}_p(Q)$ should be considered as a metric affine space,

$$\mathbf{M}_p(Q) = (\mathbf{A}_p(Q), \perp_{Q,p});$$

we call it a *derived space of a quadric* in the point p.

Quite frequently the derived affine space is defined as a coimage of $\mathbf{A}_p(Q)$ under π (cf. [3]). Then lines of this space are obtained by deleting from the set of all generators and all conics on Q through p all the elements contained in $[p]_Q = Q \cap T_p$, while $[p]_Q$ can be defined as the union of all generators of Q passing through p. An explicit definition of the orthogonality on $\pi^{-1}(\mathbf{A}_p(Q))$ formulated entirely in terms of the geometry on the quadric Q is much more complicated and will not be cited here.

2. Inversive closure and "sterographical co-projection"

The affine space $\mathbf{A}(\mathbb{V})$ over a vector space $\mathbb{V} = \langle V, +, \theta_{\mathbb{V}}, \cdot\rangle$ over $\mathfrak{F}$ is defined in a natural way. Then every nondegenerate bilinear symmetric form ξ on $\mathbb{V}$ determines the orthogonality $\perp_\xi$ of lines of $\mathbf{A}(\mathbb{V})$ (cf. [4]). Let $\mathfrak{M} = (\mathbf{A}(\mathbb{V}), \perp_\xi)$ be a metric affine space defined as above. We will construct a quadric Q contained in a projective space so as $\mathfrak{M} \cong \mathbf{M}_p(Q)$ for some point $p \in Q$.

Let $\mathbb{W} := \mathfrak{F} \oplus \mathbb{V} \oplus \mathfrak{F}$. $\mathbb{W}$ is a vector space over $\mathfrak{F}$. A subspace $\langle[\alpha, u, \beta]\rangle$ spanned by a vector $[\alpha, u, \beta]$ of $\mathbb{W}$ will be denoted by $\langle\alpha, u, \beta\rangle$. In the projective space $\mathbf{P}(\mathbb{W})$ we consider a form μ given by the formula

$$\mu([\alpha_1, u_1, \beta_1], [\alpha_2, u_2, \beta_2]) = \alpha_1\alpha_2 + \xi(u_1, u_2) - \beta_1\beta_2.$$

Let $Q = Q(\mu)$, $p = \langle 1, \theta, 1\rangle$; of course $p \in Q$. We easily calculate

$$T_p = T_p(Q) = \{\langle\alpha, u, \beta\rangle : \alpha = \beta\}.$$

First we consider the map $\psi : V \to \mathbf{P}(\mathbb{W})$ defined by

$$\psi(u) = \langle 1, u, 0\rangle \quad \text{for } u \in V.$$

Set $S^\psi = \{[\alpha, u, 0] : \alpha \in \mathfrak{F}, u \in \mathbb{V}\}$. Then S^ψ is a hyperplane of $\mathbf{P}(\mathbb{W})$ and $S^\psi \setminus Im\psi = T_p$. Then the map

$$\psi^*: V \ni u \mapsto \overline{(p, \psi(u)},$$

where $\overline{(p, \psi(u)}$ is the line joining in $\mathbf{P}(\mathbb{W})$ points $p, \psi(u)$, maps points of $\mathbf{A}(\mathbb{V})$ onto points of $\mathbf{A}_p(Q)$. It is seen that ψ^* also maps lines onto lines in the sense explained below. We find a point $\hat{\psi}(u) = \pi^{-1}(\psi(u))$ of intersection of the line through $\psi(u)$ and p with the quadric Q.Then we obtain a transformation $\hat{\psi} : V \to Q$ given by the formula

$$\hat{\psi}(u) = \langle 1 - \xi(u,u), 2u, -1 - \xi(u,u) \rangle.$$

Then, clearly, if l is a line of $\mathfrak{M}$, then $\hat{\psi}(l)$ corresponds to the intersection of Q with the plane $\psi^*(l) = \overline{\psi(l), p}$ of $\mathbf{P}(\mathbb{W})$ joining p and $\psi(l)$, being a line of $\mathbf{A}_p(Q)$. Formally

$$\hat{\psi(l)} = (\psi^*(l) \cap Q) \setminus T_p(Q).$$

If l is a line of $\mathfrak{M}$ we put $l_\infty := \psi(l) \cap T_p$. It is seen that $\widetilde{\psi^*(l)} = \overline{p, l_\infty}$. One easily calculates that for $l = u + \langle v \rangle$ being a line of $\mathfrak{M}$, $l_\infty = \langle 0, v, 0 \rangle$.

PROPOSITION 2.1. *If l is an isotropic line of $\mathfrak{M}$ then $\hat{\psi}(l)$ is a generator of Q.*

Proof. Let $l = u + \langle v \rangle$ and $v \perp_\xi v$, then $l_\infty = [0, v, o]$; hence $p\omega_\mu l_\infty$ and $l_\infty \in Q$. Therefore $\overline{p, l_\infty}$ is contained in $\psi^*(l) \cap Q$ and thus $\psi^*(l) \cap Q$ is a sum of two lines intersecting in p. This yields $\hat{\psi}(l)$ is a line contained in Q. ∎

PROPOSITION 2.2. *If $m, n \in \mathfrak{M}$ then $m \perp_\xi n \Leftrightarrow \psi^*(m) \perp_{Q,p} \psi^*(n)$.*

Proof. It is seen that $m \perp_\xi n$ iff $m_\infty \omega_\mu n_\infty$. On the other hand $m_\infty, n_\infty \in Q$, so we have

$$m \perp_\xi n \Leftrightarrow L(p, m_\infty)\omega_\xi L(p, n_\infty) \Leftrightarrow \widetilde{\psi^*(m)}\omega_\mu \widetilde{\psi^*(n)} \Leftrightarrow \psi^*(m) \perp_{Q,p} \psi^*(n). \ \blacksquare$$

As a corollary we get

THEOREM 2.3. *The map ψ^* $(\hat{\psi})$ is an isomorphism between $\mathfrak{M}$ and derived space of Q in p.*

3. Extending of automorphisms

We know that every automorphism of $\mathfrak{M}$ is a superposition of a translation and a semilinear bijection of $\mathbb{V}$. We will prove that every automorphism of $\mathfrak{M}$ can be extended to an automorphism of quadric Q "closing" this space, constructed in the previous section.

LEMMA 3.1. *Let σ be an automorphism of $\mathfrak{F}$ and let φ be a σ-semilinear map of $\mathbb{V}$. If $\varphi \in Aut(\mathfrak{M})$ then there exists a σ-semilinear map ϕ of $\mathbb{W}$ preserving Q such that*

$$\varphi^{\hat{\psi}} = \phi_{|Q \setminus T_p}.$$

Proof. Suppose φ preserves the relation $\perp_\xi$, i.e. $u \perp_\xi v \Leftrightarrow \varphi(u) \perp_\xi \varphi(v)$ for all $u, v \in V$. It is known that then there exists a constant c such that $\xi(\varphi(u), \varphi(v)) = c\sigma(\xi(u,v))$. We are going to find $\varphi^{\hat{\psi}} = \hat{\psi} \circ \varphi \circ \hat{\psi}^{-1}$.

First we find $\hat{\psi}^{-1}(\langle\alpha,u,\beta\rangle) = \frac{u}{\alpha-\beta}$. Then one may calculate that $\varphi^{\hat{\psi}}$: $(Q \setminus T_p) \to (Q \setminus T_p)$ has the form

$$\varphi^{\hat{\psi}}(\langle\alpha,u,\beta\rangle)$$
$$= \langle\sigma^2(\alpha-\beta) - \xi(\varphi(u),\varphi(u)), 2\sigma(\alpha-\beta)\varphi(u), -\sigma^2(\alpha-\beta) - \xi(\varphi(u),\varphi(u))\rangle.$$

Clearly, the map ϕ defined on $\mathbb{W}$ by

$$\phi([\alpha,u,\beta]) = [\sigma(\alpha-\beta) + c\sigma(\alpha+\beta), 2\varphi(u), -\sigma(\alpha-\beta) + c\sigma(\alpha+\beta)]$$

is σ-semilinear. For $\langle\alpha,u,\beta\rangle \in Q$ it holds

$$\xi(u,u) = \beta^2 - \alpha^2.$$

Moreover, if $\langle\alpha,u,\beta\rangle \notin T_p$, then $\alpha \neq \beta$. This together with the condition

$$\xi(\varphi(u),\varphi(v)) = c\sigma(\xi(u,v))$$

yields for $\langle\alpha,u,\beta\rangle \in Q \setminus T_p$:

$$\varphi^{\hat{\psi}}(\langle\alpha,u,\beta\rangle) = \phi(\langle\alpha,u,\beta\rangle). \ \blacksquare$$

The map ϕ constructed above will be denoted by $\tilde{\varphi}$. Note that φ is linear iff $\tilde{\varphi}$ is so.

LEMMA 3.2. *If φ is a translation on $\mathbb{V}$, then there is a linear map $\phi = \tilde{\varphi}$ of $\mathbb{W}$, preserving Q, such that*

$$\varphi^{\hat{\psi}} = \phi_{|Q\setminus T_p}.$$

Proof. By the assumption, there is a vector a such that $\varphi(u) = a + u$ for all $u \in V$. Then, calculating as in the proof of 3.1 we get for $\langle\alpha,u,\beta\rangle \in Q \setminus T_p$

$$\varphi^{\hat{\psi}}(\langle\alpha,u,\beta\rangle)$$
$$= \langle(\alpha-\beta)(2\alpha - (\alpha-\beta)\xi(a,a)) + 2\xi(a,u), 2(\alpha-\beta)(a(\alpha-\beta)+u),$$
$$(\alpha-\beta)(2\beta - (\alpha-\beta)\xi(a,a)) - 2\xi(a,u)\rangle.$$

Consider the map ϕ defined by

$$\phi([\alpha,u,\beta]) = [(2\alpha - (\alpha-\beta)\xi(a,a)) + 2\xi(a,u), 2(a(\alpha-\beta)+u),$$
$$(2\beta - (\alpha-\beta)\xi(a,a)) - 2\xi(a,u)].$$

The map ϕ is linear and $\phi(\langle\alpha,u,\beta\rangle) = \varphi^{\hat{\psi}}(\langle\alpha,u,\beta\rangle)$ for $\langle\alpha,u,\beta\rangle \in Q \setminus T_p$. $\blacksquare$

Gathering together 3.1 and 3.2 we obtain

COROLLARY 3.3. *Let Q be a quadric in a projective space $\mathfrak{P}$ closing $\mathfrak{M}$, as constructed above, let $p \in Q$. (Note that $\mathfrak{M} \cong \mathbf{M}_p(Q)$). For every $f \in$*

Aut($\mathfrak{M}$) *there exists an automorphism* $g = \tilde{(f)}$ *of* $\mathfrak{P}$ *such that*

$$f^{\hat{\psi}} = g_{|Q \setminus T_p}.$$

The map f *is affine iff* $\tilde{f}$ *is projective.*

Taking into account the fact that $\mathfrak{M} = \mathbf{M}_p(Q)$ is definable in $(\mathbf{P}(\mathbb{W}), Q)$ we can state that the map

$$\mathrm{Aut}(\mathfrak{M}) \ni f \mapsto \Psi(f) \in \mathrm{Aut}(\mathbf{P}(\mathbb{W}))$$

defined by the condition: $\Psi(f)$ is an extension of $f^{\hat{\psi}}$, is a (well defined) isomorphism between $\mathrm{Aut}(\mathfrak{M})$ and $\mathrm{Aut}(\mathbf{P}(\mathbb{W}), Q)_{\langle p \rangle}$.

References

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