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Mursaleen, Mushir A. Khan, Qamaruddin

DIFFERENCE SEQUENCE SPACES DEFINED BY ORLICZ FUNCTIONS

Abstract. The idea of difference sequences was introduced by H. Kizmaz [1]. In this paper we define some difference sequence spaces by Orlicz Function and establish some inclusion relations.

1. Introduction

Let ℓ_∞, c, c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. Recently Kizmaz [1] has defined the following sequence spaces

$$\ell_\infty(\Delta) = \{x = (x_k) : \Delta x \in \ell_\infty\},$$
$$c(\Delta) = \{x = (x_k) : \Delta x \in c\},$$
$$c_0(\Delta) = \{x = (x_k) : \Delta x \in c_0\},$$

where $\Delta x = (\Delta x_k)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that

$$M(2u) \leq KM(u)(u \geq 0).$$

The Δ_2-condition is equivalent to $M(\ell u) \leq K \cdot \ell M(u)$, for all values of u and for $\ell > 1$.

An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x q(t)dt$, where q known as the kernal of M, is right differentiable for $t \geq 0$, $q(0) = 0$, $q(t) > 0$ for $t > 0$, q is non-decreasing and $q(t) \to \infty$ as $t \to \infty$.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz functions to construct Orlicz sequence space,

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\varrho}\right) < \infty, \text{ for some } \varrho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \varrho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\varrho}\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space.

In this paper, we define the following sequence spaces

$$c_0(\Delta, M) = \left\{ x = (x_k) : \lim_{k\to\infty} M\left(\frac{|\Delta x_k|}{\varrho}\right) = 0, \text{for some} \varrho > 0 \right\};$$

$$c(\Delta, \ell) = \left\{ x = (x_k) : \lim_{k\to\infty} M\left(\frac{|\Delta x_k - \ell|}{\varrho}\right) = 0, \text{ for some } \varrho > 0, \ \ell \in \mathbb{C} \right\};$$

$$\ell_\infty(\Delta, M) = \left\{ x = (x_k) : \sup_{k\ge 0} M\left(\frac{|\Delta x_k|}{\varrho}\right) < \infty, \text{ for some } \varrho > 0 \right\};$$

for an Orlicz function M.

It is easy to show that these sequence spaces are normed spaces with the norm

$$\|x\|_\Delta = \inf \left\{ \varrho > 0 : \sup_{k\ge 0} M\left(\frac{|\Delta x_k|}{\varrho}\right) \le 1 \right\}.$$

2. Main results

THEOREM 2.1. *$\ell_\infty(\Delta, M)$ is a Banach space with the norm*

$$\|x\|_\Delta = \inf \left\{ \varrho > 0 : \sup_{k\ge 0} M\left(\frac{|\Delta x_k|}{\varrho}\right) \le 1 \right\}.$$

Proof. Let (x^i) be any Cauchy sequence in $\ell_\infty(\Delta, M)$, where $x^i = (x^i_k) = (x^i_1, x^i_2, \ldots) \in \ell_\infty(\Delta, M)$ for each $i \in \mathbb{N}$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$\|x^i - x^j\|_\Delta < \frac{\epsilon}{rx_0}, \quad \text{for all} i, j \ge N.$$

Using the definition of norm, we get

$$\sup_{k\ge 0} M\left(\frac{|\Delta x^i_k - \Delta x^j_k|}{\|x^i - x^j\|_\Delta}\right) \le 1, \quad \text{for all } i, j \ge N,$$

$$M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{\|x^i - x^j\|_\Delta}\right) \le 1, \quad \text{for each } k \ge 0$$

and for all $i, j \ge N$.

Hence we can find $r > 0$ with $M(\frac{rx_0}{2}) \ge 1$, such that

$$M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{\|x^i - x^j\|_\Delta}\right) \le M\left(\frac{rx_0}{2}\right).$$

This implies that

$$|\Delta x_k^i - \Delta x_k^j| < \frac{x_0 r}{2} \cdot \frac{\epsilon}{rx_0} = \frac{\epsilon}{2}.$$

Hence (Δx_k^i) is a Cauchy sequence in $\mathbb{R}$. Therefore for each $\epsilon(0 < \varepsilon < 1)$, there exists a positive integer N such that $|\Delta x^i - \Delta x^j| < \epsilon$, for all $i, j \ge N$.

Using the continuity of M, we can find that

$$\sup_{k \ge N} M\left(\frac{|\Delta x_k^i - \lim_{j\to\infty} \Delta x_k^j|}{\varrho}\right) \le 1.$$

Thus

$$\sup_{k \ge N} M\left(\frac{|\Delta x_k^i - \Delta x_k|}{\varrho}\right) \le 1.$$

Taking infimum of such ϱ's we get

$$\inf\left\{\varrho > 0 : \sup_{k \ge N} M\left(\frac{|\Delta x_k^i - \Delta x|}{\varrho}\right) \le 1\right\} < \epsilon,$$

for all $i \ge N$ and $j \to \infty$. Since $(x^i) \in \ell_\infty(\Delta, M)$ and M is an Orlicz function (hence continuous), it follows that $x \in \ell_\infty(\Delta, M)$.

This completes the proof of the theorem.

THEOREM 2.2. *Let M be an Orlicz function which satisfies Δ_2-condition. Then*

(a) $c_0(\Delta) \subset c_0(\Delta, M)$;
(b) $c(\Delta) \subset c(\Delta, M)$;
(c) $\ell_\infty(\Delta) \subset \ell_\infty(\Delta, M)$.

Proof. (c) Let $x \in \ell_\infty(\Delta)$, this implies that $|\Delta x_k| \le N$ for all k. So that

$$M\left(\frac{|\Delta x_k|}{\varrho}\right) \le M\left(\frac{N}{\varrho}\right) \le K \cdot \ell M(N), \quad \text{by } \Delta_2\text{-condition.}$$

Hence $\sup_{k\ge0} M(\frac{|\Delta x_k|}{\varrho}) < \infty$. This shows that $\ell_\infty(\Delta) \subset \ell_\infty(\Delta, M)$. Proofs of (a) and (b) follow similarly.

3. Paranormed sequence spaces

Let $p = (p_k)$ be any sequence of positive real numbers. Then in the same way, we can also define the following sequence spaces for an Orlicz function M as c_0, c and ℓ_∞ were extended to $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ (cf. [4], [5]). For some $\varrho > 0$,

$$c_0(\Delta, M, p) = \left\{x = (x_k) : \lim_{k\to\infty} \left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k} = 0\right\},$$

$$c(\Delta, M, p) = \left\{x = (x_k) : \lim_{k\to\infty} \left(M\left(\frac{|\Delta x_k - \ell|}{\varrho}\right)\right)^{p_k} = 0\right\},$$

$$\ell_\infty(\Delta, M, p) = \left\{x = (x_k) : \sup_{k\geq 0} \left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k} < \infty\right\},$$

when p_k = constant, for all k, then $c_0(\Delta, M, p) = c_0(\Delta, M)$, $c(\Delta, M, p) = c(\Delta, M)$ and $\ell_\infty(\Delta, M, p) = \ell_\infty(\Delta, M)$.

It is easy to show that these sequence spaces are paranormed spaces with

$$G(x) = \inf\left\{\varrho^{p_n/H} : \left\{\sup_{k\geq 0}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k}\right\}^{1/H} \leq 1\right\},$$

where $H = \max(1, \sup_{k\geq 0} p_k)$.

THEOREM 3.1. *$\ell_\infty(\Delta, M, p)$ is a complete paranormed space with*

$$G(x) = \inf\left\{\varrho^{p_n/H} : \left\{\sup_{k\geq 0}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k}\right\}^{1/H} \leq 1\right\}.$$

Proof. Let (x^i) be any Cauchy sequence in $\ell_\infty(\Delta, M, p)$. Let $r, x_0 > 0$ be fixed. Then for each $\frac{\epsilon}{rx_0} > 0$, there exists a positive integer N such that

$$G(x^i - x^j) < \frac{\epsilon}{rx_0}, \text{for all} i, j \geq N.$$

Using the definition of paranorm, we get

$$\left\{\sup_{k\geq 0}\left(M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{G(x^i - x^j)}\right)\right)^{p_k}\right\}^{1/H} \leq 1, \quad \text{for all } i, j \geq N.$$

Thus

$$\sup_{k\geq 0}\left(M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{G(x^i - x^j)}\right)\right)^{p_k} \leq 1, \quad \text{for all} i, j \geq N.$$

It follows that $M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{G(x^i - x^j)}\right) \leq 1$, for each $k \geq 0$ and for all $i, j \geq N$. For $r > 0$ with $M\left(\frac{rx_0}{2}\right) \geq 1$, we have

$$M\left(\frac{|\Delta x_k^i - \Delta x_k^j|}{G(x^i - x^j)}\right) \leq M\left(\frac{rx_0}{2}\right).$$

This implies that

$$|\Delta x_k^i - \Delta x_k^j| \le \frac{rx_0}{2} \cdot \frac{\epsilon}{rx_0} = \frac{\epsilon}{2}.$$

Hence (Δx^i) is a Cauchy sequence in $\mathbb{R}$. Therefore for each $\epsilon (0 < \epsilon < 1)$, there exists a positive integer N such that

$$|\Delta x^i - \Delta x| < \epsilon, \quad \text{for all } i, j \ge N.$$

Using the continuity of M, we find that

$$\left\{ \sup_{k \ge N} \left(M\left(\frac{|\Delta x_k^i - \lim_{j\to\infty} \Delta x_k^j|}{\varrho} \right) \right)^{p_k} \right\}^{1/H} \le 1.$$

Thus

$$\left\{ \sup_{k \ge N} \left(M\left(\frac{|\Delta x_k^i - \Delta x|}{\varrho} \right) \right)^{p_k} \right\}^{1/H} \le 1.$$

Taking infimum of such ϱ's we get

$$\inf \left\{ \varrho^{p_n/H} : \left\{ \sup_{k \ge N} \left(M\left(\frac{|\Delta x_k^i - \Delta x|}{\varrho} \right) \right)^{p_k} \right\}^{1/H} \le 1 \right\} < \epsilon,$$

for all $i, j \ge N$ and $j \to \infty$. Since $(x^i) \in \ell_\infty(\Delta, M, p)$ and M is continuous. It follows that $x \in \ell_\infty(\Delta, M, p)$.

This completes the proof of the theorem.

THEOREM 3.2. *Let $0 < p_k \le q_k < \infty$ for each k. Then*

$$c_0(\Delta, M, p) \subseteq c_0(\Delta, M, q).$$

Proof. Let $x \in c_0(\Delta, M, p)$. Then there exists some $\varrho > 0$ such that

$$\lim_{k\to\infty} \left(M\left(\frac{|\Delta x_k|}{\varrho} \right) \right)^{p_k} = 0.$$

This implies that $M(\frac{|\Delta x_k|}{\varrho}) \le 1$ for sufficiently large k, since M is non-decreasing. Hence we get

$$\lim_{k\to\infty} \left(M\left(\frac{|\Delta x_k|}{\varrho} \right) \right)^{q_k} \le \lim_{k\to\infty} \left(M\left(\frac{|\Delta x_k|}{\varrho} \right) \right)^{p_k} = 0,$$

i.e. $x \in c_0(\Delta, M, q)$.

This completes the proof of the theorem.

THEOREM 3.3. (a) *Let $0 < \inf p_k \le p_k \le 1$. Then $c_0(\Delta, M, p) \subseteq c_0(\Delta, M)$,*
(*b*) *Let $1 \le p_k \le \sup p_k < \infty$. Then $c_0(\Delta, M) \subseteq c_0(\Delta, M, p)$.*

Proof. (a). Let $x \in c_0(\Delta, M, p)$, that is $\lim_{k\to\infty} (M(\frac{|\Delta x_k|}{\varrho}))^{p_k} = 0$. Since $0 < \inf p_k \le p_k \le 1$,

$$\lim_{k\to\infty}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right) \le \lim_{k\to\infty}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k} = 0,$$

and hence $x \in c_0(\Delta, M)$.

(b). Let $p_k \ge 1$ for each k and $\sup_k p_k < \infty$. Let $x \in c_0(\Delta, M)$, then for each ϵ $(0 < \epsilon < 1)$ there exists a positive integer N such that

$$\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right) \le \epsilon, \quad \forall k \ge N.$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\lim_{k\to\infty}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right)^{p_k} \le \lim_{k\to\infty}\left(M\left(\frac{|\Delta x_k|}{\varrho}\right)\right) \le \varepsilon < 1.$$

Therefore $x \in c_0(\Delta, M, p)$.

This completes the proof of the theorem.

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Mursaleen, Mushir A. Khan
DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002 INDIA;

Qamaruddin
DEPARTMENT OF APPLIED MATHEMATICS
Z.H. COLLEGE OF ENGG. AND TECHNOLOGY
ALIGARH MUSLIM UNIVERSITY
ALIGARH-202002 INDIA

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