DEMONSTRATIO MATHEMATICA
Vol. XXXII No 1 1999

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A FIXED POINT THEOREM IN BANACH SPACES OVER TOPOLOGICAL SEMIFIELDS AND AN APPLICATION

Abstract: A fixed point theorem for three mappings on a Banach space X over a topological semifield is proved. An application is given for the solvability of certain non-linear functional equations in X.

1. Introduction

The notion of a topological semifield has been introduced by M. Antonovskii, V. Boltyanskii and T. Sarymsakov in [1].

Let E be a topological semifield and K be the set of all its positive elements. Take any two elements x, y in E. If $y-x$ is in $\overline{K}$ (in K), this is denoted by $x \ll y$ ($x < y$). As proved in [1], every topological semifield E contains a subsemifield, called the axis of E, which is isomorphic to the field R of real numbers. Consequently by identifying the axis and R, each topological semifield can be regarded as a topological linear space over the field R.

The ordered triple (X, d, E) is called a metric space over the topological semifield if there exists a mapping $d : X \times X \to E$ satisfying the usual axioms for a metric (see [1],[2] and [4]).

Linear spaces considered in this paper are defined on the field R. Let X be a linear space. The ordered triple $(X, \|.\|, E)$ is called a *feeble normed space over the topological semifield* if there exists a mapping $\|.\| : X \to E$ satisfying the usual axioms for a norm (see [1] and [3]).

2. Main result

We use the following definition:

Definition 1. Let $(X, \|.\|, E)$ be a feeble normed space over a topological semifield E and let $d(x, y) = \|x - y\|$ for all x, y in X. A space $(X, \|.\|, E)$ is

[1] Research partially supported by U.G.C. grant, New Delhi, INDIA.
1991 AMS Mathematics Subject Classification: 54H25, 47H10.
Key words and phrases: Banach space, common fixed points, topological semifield.

said to be a Banach space over the topological semifield E if (X, d, E) is a sequentially complete metric space over the topological semifield E.

Now we prove the following result.

THEOREM 1. *Let X be a Banach space over a topological semifield E and let F, G and H be three continuous self mappings of X satisfying the following conditions:*

$$(1) \qquad (1-t)G(X) + tF(X) \subset G(X), \qquad \forall t \in (0,1),$$

$$(2) \qquad (1-t)G(X) + tH(X) \subset G(X), \qquad \forall t \in (0,1),$$

$$(3) \qquad FG = GF, \qquad HG = GH,$$

$$(4) \qquad p\|Gx - Gy\|^m + \|Gy - Hy\|^m \ll q\|Gx - Fx\|^m,$$

$$(5) \qquad p\|Gx - Gy\|^m + \|Gy - Fy\|^m \ll q\|Gx - Hx\|^m$$

for all x, y in X, where $p, m > 0$, and $0 < q < 1$. Then the sequence $\{Gx_n\}$ defined by

$$(6) \qquad Gx_{2n+1} = (1-t)Gx_{2n} + tFx_{2n},$$

$$(7) \qquad Gx_{2n+2} = (1-t)Gx_{2n+1} + tHx_{2n+1}$$

where x_0 is a point in X, $0 < t < 1$ and $0 \leq q - pt^m < 1$, converges to the common fixed point of F, G and H in X.

Proof. Note that the points x_n in the theorem exist because of conditions (1) and (2). Let x_0 in X be an arbitrary point. From (6) and (7), we obtain

$$(8) \qquad \|Gx_{2n+1} - Gx_{2n}\| = t\|Fx_{2n} - Gx_{2n}\|,$$

$$(9) \qquad \|Gx_{2n+2} - Gx_{2n+1}\| = t\|Hx_{2n+1} - Gx_{2n+1}\|.$$

If we put $x = x_{2n}$ and $y = x_{2n+1}$ in (4), we then have from (8) and (9)

$$p\|Gx_{2n} - Gx_{2n+1}\|^m + t^{-m}\|Gx_{2n+2} - Gx_{2n+1}\|^m \ll qt^{-m}\|Gx_{2n+1} - Gx_{2n}\|^m$$

and hence

$$(10) \qquad \|Gx_{2n+1} - Gx_{2n+2}\| \ll (q - pt^m)^{\frac{1}{m}}\|Gx_{2n+1} - Gx_{2n}\|$$

for all n. Putting $x = x_{2n+1}$ and $y = x_{2n+2}$ in (5) and using (8), we get

$$t^{-m}(pt^m\|Gx_{2n+1} - Gx_{2n+2}\|^m + \|Gx_{2n+3} - Gx_{2n+2}\|^m) \ll qt^{-m}\|Gx_{2n+2} - Gx_{2n+1}\|^m.$$

Hence

$$(11)\qquad \|Gx_{2n+3} - Gx_{2n+2}\| \ll (q - pt^m)^{\frac{1}{m}} \|Gx_{2n+2} - Gx_{2n+1}\|$$

for all n. From (10) and (11) we then obtain

$$\|Gx_n - Gx_{n+1}\| \ll (q - pt^m)^{\frac{1}{m}} \|Gx_{n-1} - Gx_n\|$$

which implies that

$$\|Gx_n - Gx_{n+1}\| \ll (q - pt^m)^{\frac{n}{m}} \|Gx_0 - Gx_1\|.$$

Since $0 \leq q - pt^m < 1$, it follows that $\{Gx_n\}$ is a Cauchy sequence. Since X is complete, it then follows that the sequence $\{Gx_n\}$ converges to a point u in X. Using (6) and (7), we see that $\{Fx_{2n}\}$ and $\{Hx_{2n+1}\}$ also converge to u. Since F, G and H are continuous, we have

$$(12)\qquad F(Gx_{2n}) \to Fu, \quad H(Gx_{2n+1}) \to Hu.$$

Since G commutes with F and H, we have

$$F(Gx_{2n}) = G(Fx_{2n}), \quad H(Gx_{2n+1}) = G(Hx_{2n+1})$$

for $n = 0, 1, 2, \ldots$. Letting n tend to infinity, we have

$$Fu = Gu = Hu$$

and then

$$(13)\qquad \begin{aligned} G(Gu) &= G(Fu) = F(Gu) = F(Fu) = F(Hu) \\ &= G(Hu) = H(Gu) = H(Hu). \end{aligned}$$

Now if $Fu \neq H(Fu)$, then by (4), (12) and (13) we have

$$p\|Gu - G(Hu)\|^m + \|G(Hu) - H(Hu)\|^m \ll q\|Gu - Fu\|^m,$$

$$p\|Fu - H(Fu)\|^m + \|G(Hu) - H(Fu)\|^m \ll q\|Fu - Fu\|^m.$$

Hence

$$(14)\qquad Fu = H(Fu)$$

and by (4), (13) and (14), we have

$$Fu = H(Fu) = G(Fu) = F(Fu)$$

which implies that Fu is a common fixed point of F, G and H.

Now to prove the uniqueness. Suppose that u and v are two common fixed points of F, G and H in X. Then by (4), we have

$$p\|Gu - Gv\|^m + \|Gv - Hv\|^m \ll q\|Gu - Fu\|^m$$

$$p\|u - v\|^m + \|v - v\|^m \ll q\|u - u\|^m$$

and so $\|u-v\|^m \ll 0$. This implies the uniqueness of the common fixed point of F, G and H.

We next investigate the solvability of certain non-linear functional equations in a Banach space over a topological semifield.

THEOREM 2. *Let X be a Banach space over a topological semifield E and let F, G and H be three continuous self-mappings on X satisfying conditions* (2), (3), (4) *and* (5) *of Theorem* 1, *let $\{g_{p'}\}$, $\{f_{p'}\}$ and $\{h_{p'}\}$ be sequences of elements in X and let $w_{p'}$ be the unique solution of the system of equations*

$$u - Gu = g_{p'}, \quad Gu - Fu = f_{p'}, \quad Gu - Hu = h_{p'}. \tag{15}$$

If $\lim_{p'\to\infty} \|g_{p'}\| = \lim_{p'\to\infty} \|f_{p'}\| = \lim_{p'\to\infty} \|h_{p'}\| = 0$, *then the sequence $\{w_{p'}\}$ converges to the solution of the equations*

$$u = Gu = Fu = Hu.$$

Proof. Suppose that $\|w_{p'} - Gw_{p'}\| \neq 0$, $\|Gw_{p'} - Fw_{p'}\| \neq 0$ and $\|Gw_{p'} - Hw_{p'}\| \neq 0$. Then by (4), we have for $p' > q'$,

$$\begin{aligned} \|w_{p'} - w_{q'}\| &\ll \|w_{p'} - Gw_{p'}\| + \|Gw_{p'} - Gw_{q'}\| + \|Gw_{q'} - w_{q'}\| \\ &\ll \|g_{p'}\| + p^{-1}[q\|Gw_{p'} - Fw_{p'}\|^m \\ &\quad -\|Gw_{q'} - Hw_{q'}\|^m]^{1/m} + \|g_{q'}\| \\ &= \|g_{p'}\| + p^{-1}[q\|f_{p'}\|^m - \|h_{q'}\|^m]^{1/m} + \|g_{q'}\|. \end{aligned}$$

Letting p', q' tend to infinity, it follows that $\|w_{p'} - w_{q'}\| \to 0$, which implies that $\{w_{p'}\}$ is a Cauchy sequence in X. Since X is complete, it then follows that the sequence $\{w_{p'}\}$ converges to a point w in X.

Since F, G and H are continuous, it follows from (15) that

$$\|w - Gw\| = \lim_{p'\to\infty} \|w_{p'} - Gw_{p'}\| = \lim_{p'\to\infty} \|g_{p'}\| = 0,$$

$$\|Gw - Fw\| = \lim_{p'\to\infty} \|Gw_{p'} - Fw_{p'}\| = \lim_{p'\to\infty} \|f_{p'}\| = 0,$$

$$\|Gw - Hw\| = \lim_{p'\to\infty} \|Gw_{p'} - Hw_{p'}\| = \lim_{p'\to\infty} \|h_{p'}\| = 0.$$

This implies that $w = Gw = Fw = Hw$, completing the proof of the theorem.

We finally note that Theorem 1 and its proof may be modified for a Banach space over a real or complex field by replacing the symbol '$\ll$' with '$\leq$' throughout the text.

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Received August 12, 1997; revised version March 23, 1998.