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POSITIVE SOLUTIONS OF A NONLINEAR INTEGRAL EQUATION FROM BIOMATHEMATICS

Abstract. The aim of the present paper is to investigate the existence of positive continuous solutions of the nonlinear integral equation $x(t) = \int_{t-\tau}^{t} f(s, x(s))ds$, arising in infectious diseases. We give sufficient conditions ensuring the existence of positive periodic continuous solutions of this equation, provided that f is a continuous function periodic in the first argument. We also study the existence of positive continuous solutions of the initial value problem for the considered equation.

1. Introduction

In the present paper we deal with the delay nonlinear integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \tag{1.1}$$

which can be interpreted as a model for the spread of certain infectious diseases with a contact rate that varies seasonally. In this equation $x(t)$ is the proportion of infectives in a population at time t, τ is the length of time an individual remains infectious, and $f(t, x(t))$ is the proportion of new infectives per unit time.

This model was formulated and investigated for the first time by Cooke and Kaplan in [1]. They considered continuous functions $f : \mathbb{R} \times [0, \infty[\to \mathbb{R}$, $f(t, 0) = 0$ for all $t \in \mathbb{R}$, which generalize $f(t, x) = \alpha(t)x(1 - x)$, where $\alpha(t)$ is the effective contact rate at time t. Obviously $x(t) \equiv 0$ is a trivial solution of (1.1). To find conditions ensuring the existence of at least one positive continuous solution of Eq. (1.1) was the main purpose of numerous investigations.

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Thus, in the papers [1], [2] and [7] there are given sufficient conditions for the existence of nontrivial periodic continuous solutions of (1.1), provided that f is periodic in the first argument, i. e. there exists a positive real number ω such that

$$f(t+\omega,x)=f(t,x) \quad \text{for all } (t,x)\in\mathbb{R}\times[0,\infty[.$$

The papers [4] and [5] deal with the initial value problem for Eq. (1.1). There are given conditions ensuring the existence of positive continuous solutions of (1.1) when we know the proportion $\phi(t)$ of infectives in the population for $t\in[-\tau,0]$,

$$x(t)=\phi(t) \quad \text{for all } t\in[-\tau,0]. \tag{1.2}$$

Clearly, in this case we must suppose that the function ϕ satisfies the condition

$$\phi(0)=\int_{-\tau}^{0} f(s,\phi(s))\,ds. \tag{1.3}$$

In the next two sections we shall establish two very general existence theorems from which several already known results can be derived. The first theorem gives conditions ensuring the existence of positive periodic continuous solutions of Eq. (1.1) while the second one guarantees the existence of positive continuous solutions of (1.1) satisfying the condition (1.2).

2. Positive periodic solutions of Eq. (1.1)

In this section we establish sufficient conditions ensuring the existence of nontrivial positive periodic continuous solutions of (1.1). In order to simplify the formulation of the results, we shall use the following assumptions:

(A_1) $f:\mathbb{R}\times[0,\infty[\,\to\mathbb{R}$ is a continuous function satisfying $f(t,0)=0$ for all $t\in\mathbb{R}$;

(A_2) there exists $c\in\,]0,\infty]$ such that $f(t,x)>0$ for all $(t,x)\in\mathbb{R}\times\,]0,c[$;

(A_3) there exists $\omega>0$ such that

$$f(t+\omega,x)=f(t,x) \quad \text{for all } (t,x)\in\mathbb{R}\times[0,\infty[;$$

(A_4) there exist a positive real number $a<c$ and an integrable function $b:[-\tau,\omega]\to\mathbb{R}$ such that

$$f(t,x)\ge b(t) \quad \text{for all } (t,x)\in[-\tau,\omega]\times[a,c] \tag{2.1}$$

and

$$\int_{t-\tau}^{t} b(s)\,ds\ge a \quad \text{for all } t\in[0,\omega]; \tag{2.2}$$

(A_5) there exist a real number $x_0 \in [a,c]$ and an integrable function $F : [-\tau, \omega] \times [a, x_0] \to \mathbb{R}$ such that $F(t, \cdot)$ is nondecreasing for all $t \in [-\tau, \omega]$,

$$f(t,x) \le F(t,x) \quad \text{for all } (t,x) \in [-\tau,\omega] \times [a, x_0], \tag{2.3}$$

and

$$\int_{t-\tau}^{t} F(s, x_0)\, ds \le x_0 \quad \text{for all } t \in [0,\omega]. \tag{2.4}$$

We are now in position to state the main result in this section.

THEOREM 2.1. *If the assumptions* (A_1)–(A_5) *are fulfilled, then Eq. (1.1) has at least one positive continuous solution $x : \mathbb{R} \to \mathbb{R}$ with period ω and satisfying $x(t) \ge a$ for all $t \in \mathbb{R}$.*

Proof. Let E be the Banach space consisting of all continuous ω-periodic functions $x : \mathbb{R} \to \mathbb{R}$ endowed with the norm

$$\|x\| = \sup\{|x(t)| \mid t \in \mathbb{R}\} = \sup\{|x(t)| \mid t \in [0,\omega]\},$$

and let S be the subset of E defined by

$$S := \{x \in E \mid \forall t \in [0,\omega] : a \le x(t) \le x_0\}.$$

Obviously S is a bounded closed convex subset of E.

Next we define the mapping $A : S \to E$ by

$$Ax(t) := \int_{t-\tau}^{t} f(s, x(s))\, ds.$$

It is immediately seen that A is correctly defined because the assumptions (A_1) and (A_3) guarantee that Ax is a continuous ω-periodic function for every $x \in S$. Moreover, by using the well-known Arzelá-Ascoli theorem, it is easy to prove that A is completely continuous.

We claim that $A(S) \subseteq S$. To see this, let $x \in S$ be arbitrarily chosen. Taking account of assumption (A_4) we have

$$Ax(t) \ge \int_{t-\tau}^{t} b(s)\, ds \ge a \quad \text{for all } t \in [0,\omega]. \tag{2.5}$$

On the other hand, (A_5) ensures that

$$Ax(t) \le \int_{t-\tau}^{t} F(s, x(s))\, ds \le \int_{t-\tau}^{t} F(s, x_0)\, ds \le x_0 \quad \text{for all } t \in [0,\omega]. \tag{2.6}$$

From (2.5) and (2.6) it follows that $Ax \in S$. Since x was arbitrarily chosen in S, we conclude that the inclusion $A(S) \subseteq S$ holds. Now the conclusion of the theorem is a consequence of Schauder fixed point theorem. ∎

Next we illustrate the applicability of Theorem 2.1 by deriving some already known results as its consequences. Let us list three more assumptions for convenience:

(A_6) $\limsup_{x\to+0}(f(t,x)/x) = a_0(t)$ uniformly with respect to $t \in [0,\omega]$ and $\sup\{a_0(t) \mid t \in [0,\omega]\} < 1/\tau$;

(A_7) $\limsup_{x\to\infty}(f(t,x)/x) = a_\infty(t)$ uniformly with respect to $t \in [0,\omega]$ and $\sup\{a_\infty(t) \mid t \in [0,\omega]\} < 1/\tau$;

(A_8) there exists $R > 0$ such that $f(t,x) \le R/\tau$ for all $(t,x) \in [-\tau,\omega] \times [0,R]$.

COROLLARY 2.2 ([2, Theorem 1]). *Suppose that the assumptions* (A_1)–(A_4), (A_6) *and* (A_7) *are satisfied with* $c = \infty$ *in* (A_2). *Then Eq.* (1.1) *has at least one positive continuous solution* $x : \mathbb{R} \to \mathbb{R}$ *with period* ω *and satisfying* $x(t) \ge a$ *for all* $t \in \mathbb{R}$.

Proof. The assumptions (A_6) and (A_7) ensure (see [2]) the existence of real numbers $\mu > \tau$ and $\beta \ge 0$ such that

$$f(t,x) \le \frac{1}{\mu}x + \beta \quad \text{for all } (t,x) \in \mathbb{R} \times [0,\infty[.$$

Thus the assumption (A_5) is also fulfilled if we define $F : \mathbb{R} \times [0,\infty[\to \mathbb{R}$ by $F(t,x) := \frac{1}{\mu}x + \beta$, and choose $x_0 \in [a,\infty[$ such that

$$\tau\left(\frac{1}{\mu}x_0 + \beta\right) \le x_0.$$

Consequently, Theorem 2.1 can be applied. ∎

COROLLARY 2.3 ([2, Theorem 2]). *Suppose that the assumptions* (A_1)–(A_4) *and* (A_8) *are satisfied with* $c = \infty$ *in* (A_2). *Then Eq.* (1.1) *has at least one positive continuous solution* $x : \mathbb{R} \to \mathbb{R}$ *with period* ω *and satisfying* $a \le x(t) \le R$ *for all* $t \in \mathbb{R}$.

Proof. The assumption (A_8) ensures the validity of (A_5) if we put $F(t,x) := \frac{R}{\tau}$ and $x_0 := R$, and Theorem 2.1 applies. ∎

REMARK. For examples of functions satisfying the assumptions in Corollary 2.2 and Corollary 2.3 the reader is referred to [2].

We end this section with an application of Theorem 2.1 to the function $f(t,x) = \alpha(t)x(1-x)$ from which have started all the investigations concerning Eq. (1.1).

COROLLARY 2.4. *Let ω be a positive real number, let $\alpha : \mathbb{R} \to \mathbb{R}$ be a positive ω-periodic continuous function such that*

$$1 < \int_{t-\tau}^{t} \alpha(s)\, ds < 4 \quad \text{for all } t \in [0, \omega],$$

and let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(t, x) := \alpha(t)x(1 - x)$. Then Eq. (1.1) has at least one ω-periodic continuous solution $x : \mathbb{R} \to \mathbb{R}$ such that $0 < x(t) < 1$ for all $t \in \mathbb{R}$.

Proof. Put

$$a_1 := \min \left\{ 1 - \frac{1}{\int_{t-\tau}^{t} \alpha(s)\, ds} \mid t \in [0, \omega] \right\}$$

$$a_2 := \max \left\{ \frac{1}{4} \int_{t-\tau}^{t} \alpha(s)\, ds \mid t \in [0, \omega] \right\}$$

$$a := \min \{a_1, 1 - a_2\}.$$

It is immediately seen that $0 < a \leq \frac{1}{2}$ and $a = \frac{1}{2}$ iff

$$\int_{t-\tau}^{t} \alpha(s)\, ds = 2 \quad \text{for all } t \in [0, \omega].$$

If the above relation holds, then $x(t) \equiv \frac{1}{2}$ is a positive ω-periodic continuous solution of (1.1).

Suppose next that $0 < a < \frac{1}{2}$. Define the functions $b : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R}^2 \to \mathbb{R}$ by

$$b(t) := a(1 - a)\alpha(t)$$

and

$$F(t, x) := \frac{1}{4}\alpha(t),$$

respectively. Then it is easy to check that all the assumptions (A_1)–(A_5) are satisfied if we set $c := 1 - a$ and $x_0 := 1 - a$. Therefore Theorem 2.1 applies. ∎

REMARK. It should be emphasized that none of the theorems given in [2] or [7] can be applied in the conditions of the above corollary.

3. Positive solutions of the initial value problem (1.1)–(1.2)

Our main purpose in this section is to establish a general existence theorem for the initial value problem (1.1)–(1.2). From this theorem we shall derive the main result from [4]. It should be mentioned that our approach is based on the Schauder fixed point theorem instead of the transversality theorem used in [4].

To simplify the formulation of the results in this section, we shall use the following assumptions:

(B_1) $f : [-\tau, \infty[\times[0, \infty[\to [0, \infty[$ is a continuous function;

(B_2) a is a positive real number;

(B_3) $\phi : [-\tau, 0] \to [a, \infty[$ is a continuous function satisfying the condition (1.3);

(B_4) there exists a locally integrable function $b : [-\tau, \infty[\to \mathbb{R}$ such that

$$f(t, x) \geq b(t) \quad \text{for all } (t, x) \in [-\tau, \infty[\times[a, \infty[\tag{3.1}$$

and

$$\int_{t-\tau}^{t} b(s)\, ds \geq a \quad \text{for all } t \geq 0; \tag{3.2}$$

(B_5) there exists a continuous function $F : [0, \infty[\times[a, \infty[\to \mathbb{R}$ such that $F(t, \cdot)$ is nondecreasing for all $t \in [0, \infty[$ and

$$f(t, x) \leq F(t, x) \quad \text{for all } (t, x) \in [0, \infty[\times [a, \infty[. \tag{3.3}$$

THEOREM 3.1. *Suppose that the assumptions* (B_1)–(B_5) *are satisfied, and that* $[0, t_0[$ *with* $0 < t_0 \leq \infty$ *is the maximal interval on which the Cauchy problem*

$$\begin{cases} y'(t) = F(t, y(t)) \\ y(0) = \phi(0) + 1 \end{cases} \tag{3.4}$$

has a solution. Then Eq. (1.1) has at least one continuous solution $x : [-\tau, t_0[\to \mathbb{R}$ *which satisfies the condition (1.2) and such that* $x(t) \geq a$ *for all* $t \in [-\tau, t_0[$.

Proof. Let $(t_n)_{n\geq 1}$ be an increasing sequence of real numbers lying in $]0, t_0[$ which converges to t_0. We set $T := [-\tau, t_0[$ and $T_n := [-\tau, t_n]$ for each positive integer n. It is well-known that $E := C(T)$ is a complete locally convex space with respect to the compact convergence topology generated by the family $\{p_n \mid n \in \mathbb{N}\}$ of semi-norms $p_n : E \to \mathbb{R}$ defined by

$$p_n(x) := \sup \{|x(t)| \mid t \in T_n\}.$$

Moreover, the family $\mathcal{B} := \{B_{r,n} \mid r > 0, n \in \mathbb{N}\}$ of balls

$$B_{r,n} := \{x \in E \mid p_n(x) \leq r\}$$

is a neighbourhood-base at the origin of E with respect to this topology.

Let $y_0 : [0, t_0[\to \mathbb{R}$ be a solution of the Cauchy problem (3.4). We shall prove that

$$(3.5) \qquad y_0(t) > a \quad \text{for all } t \in [0, t_0[.$$

Indeed, $y_0(0) = \phi(0) + 1 > a$. If we suppose that the relation (3.5) doesn't hold, then there exists $t^* \in]0, t_0[$ such that $y_0(t^*) = a$ and $y_0(t) > a$ for all $t \in [0, t^*[$. Taking into account the assumption (B_5) we have

$$y_0'(t) = F(t, y_0(t)) \geq f(t, y_0(t)) \geq 0 \quad \text{for all } t \in [0, t^*],$$

hence y_0 is nondecreasing on $[0, t^*]$. Consequently, we get $a = y_0(t^*) \geq y_0(0)$, which is a contradiction. This contradiction shows that (3.5) holds as claimed.

Let S be the set of all functions $x \in E$ such that $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$ and $a \leq x(t) \leq y_0(t)$ for all $t \in [0, t_0[$. Obviously S is a nonempty closed convex subset of E.

Next we define the mapping $A : S \to E$ by

$$(3.6) \qquad Ax(t) := \begin{cases} \phi(t) & \text{if } t \in [-\tau, 0] \\ \int_{t-\tau}^{t} f(s, x(s))\, ds & \text{if } t \in [0, t_0[. \end{cases}$$

It is immediately seen that for each $x \in S$ the function Ax is continuous. So the mapping A is correctly defined.

Next we prove that $A(S) \subseteq S$. Indeed, let $x \in S$ be arbitrarily chosen. Taking into account the definition (3.6) of A, we need only to prove that

$$(3.7) \qquad a \leq Ax(t) \leq y_0(t) \quad \text{for all } t \in [0, t_0[.$$

From the assumptions (B_3) and (B_4) it follows that

$$Ax(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds \geq \int_{t-\tau}^{t} b(s)\, ds \geq a$$

for all $t \in [0, t_0[$. On the other hand, taking account of assumption (B_5) we get

$$\begin{aligned}(Ax)'(t) = f(t, x(t)) - f(t-\tau, x(t-\tau)) \leq f(t, x(t)) \leq \\ \leq F(t, x(t)) \leq F(t, y_0(t))\end{aligned}$$

for all $t \in [0, t_0[$. Therefore

$$Ax(t) - Ax(0) \leq \int_0^t F(s, y_0(s))\, ds = y_0(t) - y_0(0) \quad \text{for all } t \in [0, t_0[.$$

Since $Ax(0) = \phi(0)$ and $y_0(0) = \phi(0) + 1$, we get

$$Ax(t) \leq y_0(t) - 1 < y_0(t) \quad \text{for all } t \in [0, t_0[.$$

Therefore the relation (3.7) holds. From this relation we conclude that $A(S) \subseteq S$.

In the sequel we shall prove that the mapping A is continuous at each point $x \in S$. To see this, let $V := Ax + B_{r,n}$ ($r > 0, n \in \mathbb{N}$) be a neighbourhood of Ax in E. Put

$$M_n := \sup \{y_0(t) \mid t \in [0, t_n]\}.$$

Since f is continuous on the compact set $[0, t_n] \times [a, M_n]$, there exists a number $q > 0$ such that

$$|f(s,u) - f(s,v)| < \frac{r}{t_n}$$

for all $s \in [0, t_n]$ and all $u, v \in [a, M_n]$ with $|u - v| < q$. Then $U := (x + B_{q,n}) \cap S$ is a neighbourhood of x such that

$$Az \in V \quad \text{for all } z \in U. \tag{3.8}$$

Indeed, if z is any element in U, then $z - x \in B_{q,n}$, whence

$$\begin{aligned} |Ax(t) - Az(t)| &\le \left| \int_0^t f(s, x(s))\, ds - \int_0^t f(s, z(s))\, ds \right| \\ &\le \int_0^{t_n} |f(s, x(s)) - f(s, z(s))|\, ds \le r, \end{aligned}$$

for all $t \in [0, t_n]$. Moreover $|Ax(t) - Az(t)| = 0$ for all $t \in [-\tau, 0]$. Hence the relation (3.8) holds. Consequently, A is continuous at x as claimed.

Finally, we establish the relative compactness of the image $A(S)$ in E.

The image $A(S)$ is equibounded because if $t \in [-\tau, 0]$, then

$$Ax(t) = \phi(t) \quad \text{for all } x \in S,$$

while if $t \in [0, t_0[$, then

$$a \le Ax(t) \le y_0(t) \quad \text{for all } x \in S.$$

The image $A(S)$ is equicontinuous because for every $x \in S$ and every $t, t' \in [0, \tau]$ with $t < t'$ we have

$$\begin{aligned} |Ax(t') - Ax(t)| &= \left| \int_t^{t'} f(s, x(s))\, ds - \int_{t-\tau}^{t'-\tau} f(s, \phi(s))\, ds \right| \\ &\le \int_t^{t'} F(s, x(s))\, ds + \int_{t-\tau}^{t'-\tau} f(s, \phi(s))\, ds \\ &\le \int_t^{t'} F(s, y_0(s))\, ds + \int_{t-\tau}^{t'-\tau} f(s, \phi(s))\, ds \end{aligned}$$

$$= y_0(t') - y_0(t) + \int_{t-\tau}^{t'-\tau} f(s, \phi(s))\, ds,$$

while for every $x \in S$ and every $t, t' \in \left]\tau, t_0\right[$ with $t < t'$ we have

$$\begin{aligned} |Ax(t') - Ax(t)| &= \left| \int_t^{t'} f(s, x(s))\, ds - \int_{t-\tau}^{t'-\tau} f(s, x(s))\, ds \right| \\ &\leq \int_t^{t'} F(s, x(s))\, ds + \int_{t-\tau}^{t'-\tau} F(s, x(s))\, ds \\ &\leq \int_t^{t'} F(s, y_0(s))\, ds + \int_{t-\tau}^{t'-\tau} F(s, y_0(s))\, ds \\ &= y_0(t') - y_0(t) + y_0(t' - \tau) - y_0(t - \tau). \end{aligned}$$

Now the well-known theorem of Ascoli ensures the relative compactness of $A(S)$ in E.

Since all the conditions of Schauder's fixed point theorem ([3], [6]) are satisfied, we conclude that A has at least one fixed point in S. This completes the proof. ■

COROLLARY 3.2 ([4, Theorem 1]). *Suppose that the assumptions* (B_1)–(B_4) *are fulfilled and that there exist the continuous functions* $g : [0, \infty[\to [0, \infty[$ *and* $h : [a, \infty[\to]0, \infty[$ *satisfying the following conditions*:

(i) h *is nondecreasing*;

(ii) $f(t, x) \leq g(t)h(x)$ *for all* $(t, x) \in [0, \infty[\times [a, \infty[$.

If $t_0 \in]0, \infty]$ *is defined by*

$$t_0 := \sup \left\{ t \in]0, \infty[\;\middle|\; \int_0^t g(u)\, du \leq \int_0^\infty \frac{1}{h(u)}\, du \right\},$$

then Eq. (1.1) *has at least one continuous solution* $x : [-\tau, t_0[\to \mathbb{R}$ *which satisfies the condition (1.2) and such that* $x(t) \geq a$ *for all* $t \in [-\tau, t_0[$.

Proof. It is immediately seen that our assumption (B_5) is satisfied if we define $F : [0, \infty[\times [a, \infty[\to \mathbb{R}$ by $F(t, x) := g(t)h(x)$. Put $a_0 := \phi(0) + 1$. Then the Cauchy problem (3.4) has the solution $y_0 : [0, t_0[\to \mathbb{R}$, $y_0(t) := H^{-1}(G(t))$, where $G : [0, \infty[\to \mathbb{R}$ and $H : [a_0, \infty[\to \mathbb{R}$ are the functions defined by

$$G(t) := \int_0^t g(u)\, du \quad \text{and} \quad H(x) := \int_{a_0}^x \frac{1}{h(u)}\, du,$$

respectively. Now the conclusion follows from Theorem 3.1. ■

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