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A VARIATIONAL CHARACTERIZATION OF THE BEST APPROXIMATION ELEMENT

Abstract. A variational characterization for the best approximtion element from linear subspaces in normed linear spaces is given.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space and G a nondense linear subspace in X. Suppose $x_0 \in X \setminus Cl(G)$ and $g_0 \in G$.

DEFINITION 1.1. The element g_0 will be called *the best approximation element* of x_0 in G if

$$\|x_0 - g_0\| = \inf_{g \in G} \|x_0 - g\| \tag{1.1}$$

and we shall denote by $\mathcal{P}_G(x_0)$ the set of all elements which satisfy (1.1).

The following classic characterization result due to I. Singer (see for example [19, p. 16]) holds.

THEOREM 1.2. *Let X, G, x_0 and g_0 be as above. Then $g_0 \in \mathcal{P}_G(x_0)$ if and only if there exists a functional $f \in X^*$ so that*

$$\|f\| = 1,\ f(g) = 0 \textit{ for all } g \in G \textit{ and } f(x_0 - g_0) = \|x_0 - g_0\|.$$

For some different consequences as well as for the geometrical interpretation of this classic fact see [19, p. 16–26].

Another main result due to I. Singer is embodied in the next theorem [19, p. 57].

THEOREM 1.3. *Let X, G, x_0 and g_0 be as above. Then $g_0 \in \mathcal{P}_G(x_0)$ iff for every $g \in G$ there exists a functional $f^g \in X^*$ so that*

$$f^g \in \mathcal{E}(B_{X^*}),\ \operatorname{Re} f^g(g_0 - g) \geq 0 \textit{ and } f^g(x_0 - g_0) = \|x_0 - g_0\|$$

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where B_{X^*} is the unit closed ball of dual space X^* and $\mathcal{E}(B_{X^*})$ denotes the set of all extremal points of B_{X^*}.

For some interesting consequences of this theorem see [19, p. 52–63] where more details are given.

Another characterization of the best approximation element in terms of the *tangent functional* τ, i.e.,

$$\tau(x,y) := \lim_{t\to 0+}(\|x+ty\| - \|x\|)/t; \quad x,y\in X,\ x\neq 0$$

is embodied in the following theorem [19, p. 82]:

THEOREM 1.4. *With the above assumptions, the following statements are equivalent*:

(i) $g_0 \in \mathcal{P}_G(x_0)$;

(ii) *One has*

$$\tau(x_0-g_0,g)\geq 0 \quad \textit{for all } g\in G;$$

(iii) *We have the double inequality*

$$-\tau(x_0-g_0,-g)\leq 0\leq \tau(x_0-g_0,g) \quad \textit{for all } g\in G,$$

(iv) *For any $g\in G$, there exists $f^g\in X^*$ so that*

$$\|f^g\|=1, \quad \operatorname{Re} f^g(g)=0 \quad \textit{and} \quad f^g(x_0-g_0)=\|x_0-g_0\|.$$

In 1935, G. Birkhoff (see [19, p. 84]) introduced the following concept of orthogonality in normed spaces

$$x\perp y\ (B) \quad \text{iff} \quad \|x+\alpha y\|\geq\|x\| \quad \text{for all } \alpha\in R,$$

which in the case of real prehilbertian spaces coincides with the usual orthogonality associated with the inner product.

By the use of Birkhoff's orthogonality, the following characterization of best approximation elements holds.

LEMMA 1.5. *Let X be a normed space, G its nondense linear subspace, $x_0\in X\setminus \mathrm{Cl}(G)$ and $g_0\in G$. Then $g_0\in\mathcal{P}_G(x_0)$ iff $x_0-g_0\perp G(B)$, i.e., $x_0-g_0\perp g$ for all $g\in G$.*

For other results in connection with best approximation element see the monograph [19] as well as the recent papers [6], [7], [11] and [12].

2. An interpolating theorem

Let $(X,\|\cdot\|)$ be a normed space. The mapping $f: X\to R$ given by $f(x)=\frac{1}{2}\|x\|^2$ is convex on X and thus there exist the following limits (see also [6] and [12]):

$$(x,y)_{i(s)} := \lim_{t\to 0^{-(+)}} (\|y+tx\|^2 - \|y\|^2)/2t, \quad x,y \in X.$$

The mapping $(\cdot,\cdot)_{i(s)}$ will be called *the inferior (superior) semi-inner product* associated with the norm $\|\cdot\|$.

The semi-inner product $(\cdot,\cdot)_{i(s)}$ has the following properties:

(i) $(x,x)_p = \|x\|^2$ for all $x \in X$;

(ii) $(\alpha x, \beta y)_p = \alpha\beta\ (x,y)_p$ if $\alpha\beta \geq 0$ and $x,y \in X$;

(iii) $(-x,y)_p = -(x,y)_q = (x,-y)_p$ for all $x,y \in X$;

(iv) One has the inequality

$$(\|x+ty\|^2 - \|x\|^2)/2t \geq (y,x)_s \geq (y,x)_i \geq (\|x+zy\|^2 - \|x\|^2)/2z$$

for all $x,y \in X$ and $z < 0 < t$;

(v) The following Schwarz's inequality holds

$$|(x,y)_p| \leq \|x\|\,\|y\| \quad \text{for all } x,y \in X;$$

(vi) The mapping $(\cdot,\cdot)_{s(i)}$ is sub(super)-additive in the first variable;

(vii) $(\alpha x + y, x)_p = \alpha\ \|x\|^2 + (y,x)_p$ for all $\alpha \in R$ and $x,y \in X$;

(viii) One has the inequality

$$|(y+z,x)_p - (z,x)_p| \leq \|y\|\,\|x\| \quad \text{for all } x,y,z \in X;$$

(ix) We have

$$x \perp (\alpha x + y)(B) \ \text{ if and only if } (y,x)_i \leq -\alpha\|x\|^2 \leq (y,x)_s$$

where $\alpha \in R$ and $x,y \in X$ and $x \perp y\ (B)$ iff $(y,x)_i \leq 0 \leq (y,x)_s$;

(x) The norm $\|\cdot\|$ is Gâteaux differentiable in $y \in X \setminus \{0\}$, i.e., the space X is smooth in y iff $(x,y)_i = (x,y)_s$ for all $x \in X$ or, iff the mapping $(\cdot,y)_p$ is linear on X;

where $p,q \in \{s,i\}$ and $p \neq q$.

Below we present certain characterization of best approximation element. This characterization gives us a possibility of "interpolation" (estimation) of the bounded linear functionals on real normed spaces (compare with [6]):

THEOREM 2.1. *Let $(X,\|\cdot\|)$ be a real normed space and G its closed linear subspace with $G \neq X$ and $x_0 \in X \setminus G$, $g_0 \in G$. The following statements are equivalent*:

(i) $g_0 \in \mathcal{P}_G(x_0)$;

(ii) *For every $f \in (G \oplus \mathrm{Sp}(x_0))^*$ with* $\mathrm{Ker}(f) = G$, *we have the estimation*

(2.1) $$\|f\|_{G_{x_0}}(x,\lambda_0(x_0-g_0)/\|x_0-g_0\|)_i \le f(x) \le \|f\|_{G_{x_0}}(x,\lambda_0(x_0-g_0)/\|x_0-g_0\|)_s$$

for all $x \in G_{x_0} := G \oplus \mathrm{Sp}(x_0)$, *where* $\|f\|_{G_{x_0}} := \sup\{|f(x)|/\|x\|,\ x \in G_{x_0}\}$ *and* $\lambda_0 := \operatorname{sgn} f(x_0)$.

Proof. We need the following lemma which is interesting by itself (see also [6]).

LEMMA 2.2. *Let* $(X,\|\cdot\|)$ *be as above* $f \in X^* \setminus \{0\}$, $x_0 \in X \setminus \mathrm{Ker}(f)$ *and* $g_0 \in \mathrm{Ker}(f)$. *The following sentences are equivalent*:

(i) $g_0 \in \mathcal{P}_{Ker(f)}(x_0)$;
(ii) *One has the estimation*:

(2.2) $$\|f\|(x,\lambda_0(x_0-g_0)/\|x_0-g_0\|)_i \le f(x) \le \|f\|(x,\lambda_0(x_0-g_0)/\|x_0-g_0\|)_s$$

for all $x \in X$ *and* $\lambda_0 := \operatorname{sgn} f(x_0)$.

Proof. (i)⇒(ii)". Take $g_0 \in \mathcal{P}_{\mathrm{Ker}(f)}(x_0)$ and denote by $w_0 := x_0 - g_0$. Then $w_0 \neq 0$ and by Lemma 1.5 we deduce that $w_0 \perp Ker(f)(B)$. Using the property (ix) of the s.i.p. $(\cdot,\cdot)_{i(s)}$ we have $(y,w_0)_i \le 0 \le (y,w_0)_s$ for every $y \in \mathrm{Ker}(f)$.

Fix $x \in X$. Then the element $y := f(x)w_0 - f(w_0)x$ belongs to $\mathrm{Ker}(f)$, and by the above inequality, we deduce that

(2.3) $$(f(x)w_0 - f(w_0)x, w_0)_i \le 0 \le (f(x)w_0 - f(w_0)x, w_0)_s$$

for all $x \in X$.

Using the properties of $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ one has

$$(f(x)w_0 - f(w_0)x, w_0)_p = f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_p, \quad x \in X.$$

where $p = s$ or $p = i$.

On the other hand, since $w_0 \perp \mathrm{Ker}(f)(B)$ and $w_0 \neq 0$, hence $f(w_0) \neq 0$. Thus we have to consider two cases: a) $f(w_0) > 0$ and b) $f(w_0) < 0$.

a) If $f(w_0) > 0$, then by (2.3) we have successively that

$$0 \le f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_s = f(x)\|w\|^2 + f(w_0)(-x, w_0)_s$$
$$= f(x)\|w_0\|^2 + (-x, f(w_0)w_0)_s = f(x)\|w_0\|^2 - (x, f(w_0)w_0)_i.$$

Therefore

$$f(x) \ge (x, \frac{f(w_0)w_0}{\|w_0\|^2})_i \quad \text{for all } x \in X.$$

Similarly, by (2.3), we deduce that

$$0 \ge f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_i = f(x)\|w_0\|^2 - (x, f(w_0)w_0)_s$$

which implies

$$(2.4') \qquad f(x) \leq \left(x, \frac{f(w_0)w_0}{\|w_0\|^2}\right)_s \quad \text{for all } x \in X.$$

b) Notice that for every $y, z \in X$ we have

$$-(y,z)_i = (-y,z)_s = (-y,-(-z))_s = (y,-z)_s$$

and thus,

$$\begin{aligned} 0 &\leq f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_s = f(x)\|w_0\|^2 + (-f(w_0))(x, w_0)_s \\ &= f(x)\|w_0\|^2 + (x, (-f(w_0))w_0)_s = f(x)\|w_0\|^2 - (x, f(w_0)w_0)_i \end{aligned}$$

which shows that the inequality (2.4) holds. Similarly, we have that (2.4′) is also valid for $f(w_0) < 0$. Consequently, in both cases we can state

$$(2.5) \qquad \left(x, \frac{f(w_0)w_0}{\|w_0\|^2}\right)_i \leq f(x) \leq \left(x, \frac{f(w_0)w_0}{\|w_0\|^2}\right)_s \quad \text{for all } x \in X.$$

Now, let $u := \frac{f(w_0)w_0}{\|w_0\|^2}$. Then by (2.5) we have

$$f(x) \leq (x,u)_s \leq \|x\|\|u\| \quad \text{for all } x \in X$$

and

$$f(x) \geq (x,u)_i = -(x,-u)_s \geq -\|x\|\|u\| \quad \text{for all } x \in X.$$

The latter implies that

$$-\|u\| \leq \frac{f(x)}{\|x\|} \leq \|u\| \quad \text{for all } x \in X$$

i.e., $\|f\| \leq \|u\|$.

On the other hand, we have

$$\|f\| \geq \frac{f(u)}{\|u\|} \geq \frac{(u,u)_i}{\|u\|} = \|u\|$$

what shows that $\|f\| = \|u\| = \frac{|f(w_0)|}{\|w_0\|}$. But $f(w_0) = f(x_0)$ and then

$$\|f\| = \frac{|f(x_0)|}{\|x_0 - g_0\|} = \frac{f(x_0)\lambda_0}{\|x_0 - g_0\|}, \quad \text{i.e., } f(x_0) = \lambda_0\|f\|\|x_0 - g_0\|$$

which implies, by (2.5), that the estimation (2.2) holds.

(ii)⇒(i) Suppose that (2.2) holds for all $x \in X$. Then we get

$$(x, \lambda_0(x_0 - g_0)/\|x_0 - g_0\|)_i \leq 0 \leq (x, \lambda_0(x_0 - g_0)/\|x_0 - g_0\|)_s$$

for all $x \in \mathrm{Ker}(f)$, what gives, by (ix), that

$$\lambda_0(x_0 - g_0)/\|x_0 - g_0\| \perp \mathrm{Ker}(f)(B).$$

If $\lambda_0 > 0$, then, obviously, by the above relation we get that $(x_0 - g_0) \perp \operatorname{Ker}(f)(B)$, and, by Lemma 1.5, we conclude that $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$.

If $\lambda_0 < 0$, then also $-(x_0 - g_0) \perp \operatorname{Ker}(f)(B)$ or $(x_0 - g_0) \perp (-\operatorname{Ker}(f))(B)$ and since $-\operatorname{Ker}(f) = \operatorname{Ker}(f)$, we obtain that $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$, and the proof of the lemma is completed.

Now, the proof of the theorem follows by the above lemma applied to the normed linear space G_{x_0} where G is a hyperplane.

The following corollary is important as it gives a criterion of representation for the continuous linear functionals in terms of semi-inner products $(\cdot,\cdot)_{i(s)}$.

COROLLARY 2.3. *With the above assumptions and if $x_0 - g_0$ is a point of smoothness of the normed space X, then $g_0 \in \mathcal{P}_G(x_0)$ if and only if for every $f \in G^*_{x_0}$ with* $\operatorname{Ker}(f) = G$, *one has the representation*

$$f(x) = \|f\|_{G_{x_0}}(x, \lambda_0(x_0 - g_0)/\|x_0 - g_0\|)_p \quad \text{for all } x \in G_{x_0},\ p \in \{s, i\}.$$

3. A variational characterization

The following theorem contains a variational characterization of best approximation element.

THEOREM 3.1. *Let $(X, \|\cdot\|)$ be a real normed space and G a closed linear subspace in X with $G \neq X$ and $x_0 \in X \setminus G$, $g_0 \in G$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x_0)$;

(ii) *For every $f \in G^*_{x_0}$ with* $\operatorname{Ker}(f) = G$, *the element*

$$u_0 := f(x_0)(x_0 - g_0)/\|x_0 - g_0\|^2$$

minimizes the quadratic functional $F_f : G_{x_0} \to R$ given by $F_f(x) = \|x\|^2 - 2f(x)$.

To prove this theorem, we need the following lemma which is also interesting by itself.

LEMMA 3.2. *Let $(X, \|\cdot\|)$ be a real normed space, $f \in X^* \setminus \{0\}$ and $w \in X \setminus \{0\}$. The following statements are equivalent:*

(i) *One has the estimation*

$$(x, w)_i \leq f(x) \leq (x, w)_s \quad \text{for all } x \in X; \tag{3.1}$$

(ii) *The element w minimizes the quadratic functional*

$$F_f : X \to R, \qquad F_f(u) := \|u\|^2 - 2f(u).$$

Proof. (i)⇒(ii). If w satisfies the relation (3.1), then for $x = w$ we obtain $f(w) = \|w\|^2$.

Now, let $u \in X$. Then

$$\begin{aligned} F_f(u) - F_f(w) &= \|u\|^2 - 2f(u) - \|w\|^2 + 2f(w) \\ &= \|u\|^2 - 2f(u) + \|w\|^2 \geq \|u\|^2 - 2(u,w)_s + \|w\|^2 \\ &\geq \|u\|^2 - 2\|u\|\|w\| + \|w\|^2 = (\|u\| - \|w\|)^2 \geq 0 \end{aligned}$$

which shows that w minimizes the functional F_f.

(ii)⇒(i). If w minimizes the functional F_f, then for all $u \in X$ and $\lambda \in R$ we have:

$$F_f(w + \lambda u) - F_f(w) \geq 0.$$

On the other hand, a simple calculation yields

$$F_f(w + \lambda u) - F_f(w) = \|w + \lambda u\|^2 - \|w\|^2 - 2\lambda f(u)$$

and thus we obtain

$$2\lambda f(u) \leq \|w + \lambda u\|^2 - \|w\|^2 \tag{3.2}$$

for all $u \in X$ and $\lambda \in R$. Assume that $\lambda > 0$. Then by (3.2) we have:

$$f(u) \leq \frac{\|w + \lambda u\|^2 - \|w\|^2}{2\lambda}, \qquad u \in X.$$

Passing $\lambda \to 0+$ then $f(u) \leq (u,w)_s$ for all $u \in X$.

Putting $(-u)$ instead of u we get $f(u) \geq -(-u,w)_s = (u,w)_i$ for all $u \in X$, and the lemma is thus proved.

The above lemma gives us the following criterion of representation for the continuous linear functionals in normed linear spaces.

COROLLARY 3.2. *Let $(X, \|\cdot\|)$ be a real normed space and $f \in X^* \setminus \{0\}$, $w \in X \setminus \{0\}$. Then w is a point of smoothness of X and it minimizes the functional F_f if and only if one has the representation*

$$f(x) = (x,w)_p \quad \textit{for all } x \in X,$$

where $p = s$ or $p = i$.

Proof of Theorem 3.1. (i)⇒(ii). If $g_0 \in \mathcal{P}_G(x_0)$, then by Theorem 2.1. we deduce that for every $f \in G^*_{x_0}$ with $\mathrm{Ker}(f) = G$ one has the estimation (2.1). If in this relation we put $x = \lambda_0(x_0 - g_0)$, a simple calculation gives us

$$\|f\|_{G_{x_0}} = \frac{|f(x_0)|}{\|x_0 - g_0\|}$$

and then (2.1) becomes

$$\left(x, \frac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_i \le f(x) \le \left(x, \frac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_s \quad \text{for all } x \in G_{x_0}.$$

Now if we apply Lemma 3.2 for $u_0 = f(x_0)(x_0 - g_0)/\|x_0 - g_0\|^2$ on the space G_{x_0} we conclude that u_0 minimizes the quadratic functional F_f on the space G_{x_0}.

(ii)$\Rightarrow$(i). If u_0 given above minimizes the functional F_f on G_{x_0}, then by Lemma 3.2 we derive that the estimation (3.3) holds and, furthermore, the interpolation (2.1) is valid. Thus, by Theorem 2.1, we get that $g_0 \in \mathcal{P}_G$, and the proof is completed.

4. Applications in the space $C(D)$

Let D be a compact Hansdorff set and $C(D)$ the space of continuous real-valued functions defined on D with the norm

$$\|f\|_\infty = \max_{x\in D} |f(x)|.$$

Then for $f, g \in C(D)$, $f \ne 0$, we have (see e.g. [18]):

$$\tau_+(f,g) := \lim_{t\to 0+} \frac{\|f + tg\|_\infty - \|f\|_\infty}{t} = \max_{x\in A}[g(x)\operatorname{sgn}(g(x))]$$

where $A := \{x \in D \mid |f(x)| = \|f\|_\infty\}$.

First of all, let us observe that for all $f, g \in C(D)$ we get

$$\begin{aligned}(f,g)_s &:= \lim_{t\to 0+} \frac{\|g + tf\|_\infty^2 - \|g\|_\infty^2}{2t} \\ &= \|g\|_\infty \lim_{t\to 0+} \frac{\|f + tg\|_\infty - \|f\|_\infty}{t} = \|g\|_\infty \tau_+(g,f) \\ &= \|g\|_\infty \max_{x\in B}[f(x)\operatorname{sgn}(g(x))]\end{aligned}$$

where

$$B = \{x \in D \mid |g(x)| = \|g\|_\infty\}$$

and, obviously

$$\begin{aligned}(f,g)_i &= -(-f,g)_s = -\|g\|_\infty \max_{x\in B}[-f(x)\operatorname{sgn}(g(x))] \\ &= \|g\|_\infty \min_{x\in B}[f(x)\operatorname{sgn}(g(x))]\end{aligned}$$

for all $f, g \in C(D)$.

Now using Theorem 2.1 we can state the following characterization of best approximation element in the space $C(D)$.

PROPOSITION 4.1. *Let Λ be a linear space in $C(D)$, $f_0 \in C(D) \setminus Cl(\Lambda)$ and $g_0 \in \Lambda$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_\Lambda(f_0)$;

(ii) *For every bounded linear functional Φ defined on subspace $\Lambda_{f_0} := \Lambda \oplus \mathrm{Sp}(f_0)$ with $\mathrm{Ker}(\Phi) = \Lambda$ we have the estimation*

$$\frac{\|\Phi\|_{\Lambda_{f_0}}}{\|f_0 - g_0\|_\infty} \min_{x \in C}[f(x)\,\mathrm{sgn}[\lambda_0(f_0 - g_0)]] \le \Phi(f) \le \frac{\|\Phi\|_{\Lambda_{f_0}}}{\|f_0 - g_0\|_\infty} \max_{x \in C}[f(x)\,\mathrm{sgn}[\lambda_0(f_0 - g_0)]]$$

for all $f \in \Lambda_{f_0}$ where

$$\lambda_0 := \mathrm{sgn}[\Phi(f_0)], \quad \|\Phi\|_{\Lambda_{f_0}} := \sup\{\frac{|\Phi(f)|}{\|f\|_\infty}, \quad f \in \Lambda_{f_0} \setminus \{0\}\}$$

and

$$C := \{x \in D \mid |f_0(x) - g_0(x)| = \|f_0 - g_0\|_\infty\}.$$

The following variational characterization of best approximation in $C(D)$ holds (see Theorem 3.1).

PROPOSITION 4.2. *Let Λ be a linear subspace in $C(D)$, $f_0 \in C(D) \setminus Cl(\Lambda)$ and $g_0 \in \Lambda$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_\Lambda(f_0)$;

(ii) *For every Φ a bounded linear functional on Λ_{f_0} with $\mathrm{Ker}(\Phi) = \Lambda$ the element*

$$h_0 := \Phi(f)(f_0 - g_0)/\|f_0 - g_0\|_\infty^2$$

minimizes the quadratic functional $F_\Phi : \Lambda_{f_0} \to R$ given by $F_\Phi(f) := \|f\|_\infty^2 - 2\Phi(f)$.

5. Applications in the space $\mathbf{L}^1(D, \sum, \nu)$

Let D be a set, $\sum$ a σ-field of subsets of D and ν a positive measure on $\sum$. By $L^1(D, \sum, \nu)$ we mean the usual space of real-valued ν-measurable functions f defined on D for which $|f|$ is ν-measurable, and

$$\|f\|_1 = \int_D |f|\,d\nu.$$

For $f \in L^1(D, \sum, \nu)$, let

$$Z(f) := \{x \mid f(x) = 0\}.$$

Then $Z(f)$ is ν-measurable and for all $f, g \in L^1(D, \sum, \nu)$, $f \neq 0$ we have (see e.g. [18]) that

$$\tau_+(f,g) = \int_D g \cdot (\operatorname{sgn} f) d\nu + \int_{Z(f)} |g| d\nu.$$

Now, let us observe that for all $f, g \in L^1(D, \sum, \nu)$ we have

$$\begin{aligned}(f,g)_s &= \|g\|_1 \tau_+(f,g) \\ &= \|g\|_1 \int_D f \cdot (\operatorname{sgn} g) d\nu + \|g\|_1 \int_{Z(g)} |f|\, d\nu\end{aligned}$$

and

$$(f,g)_i = -(-f,g)_s = \|g\|_1 \int_D f \cdot (\operatorname{sgn} g) d\nu - \|g\|_1 \int_{Z(g)} |f|\, d\nu.$$

Using Theorem 2.1. we can state the following characterization of the best approximation element in the space $L^1(D, \sum, \nu)$.

PROPOSITION 5.1. *Let Λ be a linear subspace in $L^1(D, \sum, \nu)$, $f_0 \in L^1(D, \sum, \nu) \setminus Cl(\Lambda)$ and $g_0 \in \Lambda$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_\Lambda(f_0)$;

(ii) *For every Φ a bounded linear functional defined on subspace $\Lambda_{f_0} := \Lambda \oplus \operatorname{Sp}(f_0)$ with $\operatorname{Ker}(\Phi) = \Lambda$ we have the estimation*

$$\frac{\|\Phi\|_{\Lambda_{f_0}}}{\|f_0 - g_0\|_1} \int_D f \cdot \operatorname{sgn}(\lambda_0(f_0 - g_0))]\, d\nu - \|\Phi\|_{\Lambda_{f_0}} \cdot \int_{Z(f_0 - g_0)} |f|\, d\nu \tag{5.1}$$

$$\leq \Phi(f) \leq \frac{\|\Phi\|_{\Lambda_{f_0}}}{\|f_0 - g_0\|_1} \int_D f \cdot [\operatorname{sgn}(\lambda_0(f_0 - g_0))]\, d\nu + \|\Phi\|_{\Lambda_{f_0}} \cdot \int_{Z(f_0 - g_0)} |f|\, d\nu$$

for all $f \in \Lambda_{f_0}$, where

$$\lambda_0 = \operatorname{sgn} \Phi(f_0), \quad \|\Phi\|_{\Lambda_{f_0}} = \sup\{\frac{|\Phi(f)|}{\|f\|_1}, \quad f \in \Lambda_{f_0} \setminus \{0\}\}$$

and

$$Z(f_0 - g_0) = \{x \in D \mid f_0(x) = g_0(x)\}.$$

Finally, the following variational characterization of the best approximation element in $L^1(D, \sum, \nu)$ holds:

PROPOSITION 5.2. *Let Λ be a linear subspace in $L^1(D, \sum, \nu)$, $f_0 \in C(D) \setminus Cl(\Lambda)$ and $g_0 \in \Lambda$. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_\Lambda(f_0)$;

(ii) *For every bounded linear functional* Φ *on* Λ_{f_0} *with* $\mathrm{Ker}(\Phi) = \Lambda$, $h_0 := \Phi(f_0)(f_0 - g_0)/\|f_0 - g_0\|_1^2$ *minimizes the quadratic functional* $F_\Phi : \Lambda_{f_0} \to R$ *given by* $F_\Phi(f) := \|f\|_1 - 2\Phi(f)$.

The proof follows by Theorem 3.1 applied to the space $L^1(D, \sum, \nu)$. We omit the details.

For other characterization of best approximation element in concrete spaces see [1], [3], [14], [16-17] and [19], where further references are given.

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