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ONE-STEP DIFFERENCE METHODS FOR MIXED TYPE DIFFERENTIAL-FUNCTIONAL EQUATIONS

1. Introduction

Suppose that $E = (0,a) \times (-b,b)$ where $a > 0$, $b = (b_1,\ldots,b_n) \in R^n$, $b_i > 0$, $i = 1,..,n$ and $B = [-\tau_0, 0] \times [-\tau, \tau]$, $\tau_0 \in R_+$, $\tau = (\tau_1,\ldots,\tau_n) \in R^n_+$. For $c = b + \tau$ let $E_0 = [-\tau_0, 0] \times [-c,c]$ and $\partial_0 E = (0,a) \times ([-c,c] \setminus (-b,b))$. We use vectorial inequalities if the same inequalities hold between their corresponding components.

For a function $z : [-\tau_0, a) \times [-c,c] \to R$ and for a point $(x,y) \in E$ we define the function $z_{(x,y)} : B \to R$ by the formula

$$z_{(x,y)}(t,s) = z(x+t, y+s),\ (t,s) \in B.$$

Then the function $z_{(x,y)}$ is the restriction of z to the set $[x-\tau_0, x] \times [y-\tau, y+\tau]$ and this restriction is shifted to the set B.

Let $C(X,Y)$ denote the class of all continuous mappings from X into Y where X, Y are metric spaces.

Suppose that $f : E \times C(B,R) \to R$ and $\varphi : E_0 \cup \partial_0 E \to R$ are given functions. Consider the initial - boundary value problem

$$(1)\qquad \begin{aligned} &D_x z(x,y) = f(x,y,z_{(x,y)}),\\ &z(x,y) = \varphi(x,y),\ (x,y) \in E_0 \cup \partial_0 E, \end{aligned}$$

where $D_x z$ denotes the derivative of z with respect to x.

Our formulation of the differential - functional problem is motivated by a general model of the ordinary differential - funtional equations (see [2], [7]).

We consider classical solutions of the problem (1). Let $E^* = E_0 \cup \partial_0 E \cup E$. A function $v : E^* \to R$ is solution of (1) if it is continuous on E^*, it has derivative with respect to x on E and it satisfies the problem (1).

The existence of solutions of the problem (1) was discussed in [3], where also is a review of papers dedicated to applications of problems of the type (1).

The problem (1) is a particular case of the problem of the type

$$\begin{aligned} D_x z(x,y) &= g(x,y,z_{(x,y)}, D_y z(x,y)), \\ z(x,y) &= \varphi(x,y), \ (x,y) \in E_0 \cup \partial_0 E \end{aligned} \tag{2}$$

where $g : E \times C(B,R) \times R^n \to R$. Difference schemes of the Euler type for the problem (2) were considered in [1], [4], [5], [6].

In our paper we consider the Euler difference — functional method for the problem (1) and the one — step methods more effective than this one. We give a constructive way to obtain the above mentioned methods.

2. Euler method for initial — boundary value problems

We will denote by Z the set of integers. For $y, \bar{y} \in R^n_+$, $y = (y_1, \ldots, y_n)$, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n)$, we put $y * \bar{y} = (y_1\bar{y}_1, \ldots, y_n\bar{y}_n)$.

We define the mesh on the sets B and E^*. Let $h = (h_0, h')$, $h' = (h_1, \ldots, h_n)$, $h_i > 0$ for $i = 0, 1, \ldots, n$, be the step of a mesh. Suppose that there are $N_0 \in Z$, $N = (N_1, \ldots, N_n) \in Z^n$ such that $N_0 h_0 = \tau_0$, $N * h' = \tau$. Let Δ be the set of all h having the above property. We assume that $\Delta \neq \emptyset$ and that there is a sequence $\{h^{(k)}\}$, $h^{(k)} \in \Delta$, such that $\lim_{k\to\infty} h^{(k)} = 0$.

For any $h \in \Delta$ we put

$$x^{(i)} = ih_0, \qquad y^{(m)} = m * h'$$

where $m = (m_1, \ldots, m_n) \in Z^n$ and $y^{(m)} = (y_1^{(m_1)}, \ldots, y_n^{(m_n)})$.

Write $R_h = \{x^{(i)} : i \in Z\}$ and $R_h^{1+n} = \{(x^{(i)}, y^{(m)}) : (i,m) \in Z^{1+n}\}$ and

$$B_h = B \cap R_h^{1+n}, \ E_{0.h} = E_0 \cap R_h^{1+n}, \ E_h = E \cap R_h^{1+n}, \ \partial_0 E_h = \partial_0 E \cap R_h^{1+n},$$
$$E_h^* = E^* \cap R_h^{1+n}.$$

For a function $z : E_h^* \to R$ and a point $(x^{(i)}, y^{(m)})$ we write $z^{(i,m)} = z(x^{(i)}, y^{(m)})$. For the above z and $0 \le i \le N_0$, $-N \le m \le N$ we define the function $z_{(i,m)} : B_h \to R$ by

$$z_{(i,m)}(t,s) = z(x^{(i)} + t, y^{(m)} + s), \qquad (t,s) \in B_h.$$

The function $z_{(i,m)}$ is the restriction of z to the set $([x^{(i)} - \tau_0, x^{(i)}] \times [y^{(m)} - \tau, y^{(m)} + \tau]) \cap R_h^{1+n}$ and this restriction is shifted to the set B_h.

Let δ be the difference operator

$$\delta z^{(i,m)} = \frac{1}{h_0}(z^{(i+1,m)} - z^{(i,m)}).$$

Let $I_{0.h} = [-\tau_0, 0] \cap R_h$, $I_h = (0,a) \cap R_h$, $I_h^* = [-\tau_0, a) \cap R_h$.
For a function $\eta : I_h^* \to R$ we write $\eta^{(i)} = \eta(x^{(i)})$, $x^{(i)} \in I_h^*$. For the above η and $i \in I_h$ we define the function $\eta_{(i)} : I_{0.h} \to R$ by

$$\eta_{(i)}(t) = \eta(x^{(i)} + t), \ t \in I_{0.h}.$$

We will use the symbol $F(X, Y)$ to denote the class of all functions from X into Y, where X and Y are sets.

In the sequell we will need the following operators $T_h : F(B_h, R) \to C(B, R)$ and $T_{0.h} : F(I_{0.h}, R) \to C([-\tau_0, 0], R)$.

Let $z \in F(B_h, R)$. For every $(x, y) \in B$ there is $(x^{(i)}, y^{(m)}) \in B_h$ such that

$$x^{(i)} \le x \le x^{(i+1)}, \; y^{(m)} \le y \le y^{(m+1)}$$

where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$. Then we put

$$(T_h z)(x, y) = \frac{x - x^{(i)}}{h_0} \sum_{\zeta \in S_+} z^{(i+1, m+\zeta)} \Big(\frac{y - y^{(m)}}{h'}\Big)^{\zeta} \Big(1 - \frac{y - y^{(m)}}{h'}\Big)^{1-\zeta} +$$
$$+ \Big(1 - \frac{x - x^{(i)}}{h_0}\Big) \sum_{\zeta \in S_+} z^{(i, m+\zeta)} \Big(\frac{y - y^{(m)}}{h'}\Big)^{\zeta} \Big(1 - \frac{y - y^{(m)}}{h'}\Big)^{1-\zeta}$$

where

$$S_+ = \{\zeta = (\zeta_1, \ldots, \zeta_n) : \; \zeta_k \in \{0, 1\}, \; k = 1, \ldots, n\},$$
$$\Big(\frac{y - y^{(m)}}{h'}\Big)^{\zeta} = \prod_{k=1}^{n} \Big(\frac{y_k - y_k^{(m_k)}}{h_k}\Big)^{\zeta_k},$$
$$\Big(1 - \frac{y - y^{(m)}}{h'}\Big)^{1-\zeta} = \prod_{k=1}^{n} \Big(1 - \frac{y_k - y_k^{(m_k)}}{h_k}\Big)^{1-\zeta_k}.$$

We adopt the convection that $0^0 = 1$.

Let $\eta \in F(I_{0.h}, R)$. For every $x \in [-\tau_0, 0]$ there is $x^{(i)} \in I_{0.h}$ such that $x^{(i)} \le x \le x^{(i+1)}$. Then we put

$$(T_{0.h}\eta)(x) = \frac{x - x^{(i)}}{h_0} \eta^{(i+1)} + \Big(1 - \frac{x - x^{(i)}}{h_0}\Big) \eta^{(i)}.$$

It is easy to see that $T_{0.h}$ is a particular case of T_h.

Let $h \in \Delta$. Suppose that $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ is a given function. We consider the difference - functional equation

$$\delta z^{(i,m)} = f(x^{(i)}, y^{(m)}, T_h z_{(i,m)}) \tag{3}$$

with the initial - boundary condition

$$z^{(i,m)} = \varphi_h^{(i,m)} \text{ on } E_{0.h} \cup \partial_0 E_h. \tag{4}$$

We say that the method (3)–(4) is convergent if for every solution v of the problem (1) there is solution z_h of method (3)–(4) and the function $\tilde{\alpha} : \Delta \to R_+$ such that $\lim_{h \to 0} \tilde{\alpha}(h) = 0$ and

$$|v^{(i,m)} - z_h^{(i,m)}| \le \tilde{\alpha}(h) \;\text{ on }\; E_h^*.$$

For $w : D \to R$, $D \subset R^{1+n}$ let the symbol w^h denotes the restriction of w to the set $D \cap R_h^{1+n}$.

In the sequel we will need the following lemmas.

LEMMA 1. *Suppose that $w : B \to R$ is of class C^2. Then there exists $c \in R_+$ such that*

$$\| T_h(w^h) - w \|_0 \le c|h|^2$$

where $|h| = \max\{h_i : i = 0, 1, \ldots, n\}$ and $\| \cdot \|_0$ is the maximum norm in $C(B, R)$.

Proof. The operator T_h has the following properties

$$(5) \qquad \sum_{\zeta \in S_+} \left(\frac{y - y^{(m)}}{h}\right)^{\zeta} \left(1 - \frac{y - y^{(m)}}{h}\right)^{1-\zeta} = 1,$$

$$(6) \sum_{\zeta \in S_+} \left(\frac{y - y^{(m)}}{h}\right)^{\zeta} \left(1 - \frac{y - y^{(m)}}{h}\right)^{1-\zeta} h_k \zeta_k = y_k - y_k^{(m_k)}, \quad 1 \le k \le n,$$

where $y^{(m)} \le y \le y^{(m+1)}$. The relations (5), (6) can be proved by the mathematical induction with respect to n. We obtain the estimate

$$|(T_h w^h)(x, y) - w(x, y)| \le$$
$$\le \sup \left\{ \left| \frac{\partial^2 w}{\partial r \partial s}(x, y) \right| : (x, y) \in B,\ r, s \in \{x, y_1, \ldots, y_n\} \right\} \sum_{i,j=0}^{n} h_i h_j,$$

what shows the Lemma 1.

For a function $\omega : [-\tau_0, a_0) \to R$, where $a_0 > a$, and for a point $x \in [0, a)$ we define the function $\omega_{(x)} : [-\tau_0, 0] \to R$ by

$$\omega_{(x)}(t) = \omega(x + t),\ t \in [-\tau_0, 0].$$

The following assumption will be needed throughout the paper.

ASSUMPTION H_0. Suppose that $\sigma : [0, a_0) \times C([-\tau_0, 0], R_+) \to R_+$ where $a_0 > a$, is a function of variables (x, p) such that

1) σ is continuous, $\sigma(x, 0) = 0$ for $x \in [0, a_0)$ and $\omega(x) = 0,\ x \in [-\tau_0, a_0)$, is a unique solution of the problem

$$\omega'(x) = \sigma(x, \omega_{(x)}),\quad \omega(x) = 0 \text{ for } x \in [-\tau_0, 0];$$

2) σ is nondecreasing with respect to x and p.

LEMMA 2. *Suppose that $\sigma : [0, a_0) \times C([-\tau_0, 0], R_+) \to R_+$ satisfies the Assumption H_0 and let A, B are the constants such that the problem*

$$(7) \qquad \omega'(x) = \sigma(x, T_{0.h}(\omega_{(x)})^h) + A,\quad \omega(x) = B \ \text{ for } \ x \in [-\tau_0, 0]$$

has a solution on $[-\tau_0, a]$. Suppose that the function $\eta : I_h^* \to R$ satisfies

$$\eta^{(i+1)} \le \eta^{(i)} + h_0 \sigma(x^{(i)}, T_{0.h} \eta_{(i)}) + h_0 A,\quad \eta^{(i)} \le B \ \text{ on } \ I_{0.h}.$$

Under these assumptions we have $\eta^{(i)} \le \bar{\omega}(x^{(i)}), x^{(i)} \in I_h^*$ where $\bar{\omega}$ is a solution of the problem (7).

Proof. Let $\bar{\omega}$ be a solution of the problem (7). For each $x^{(i)} \in I_h$ there exists $\theta_i \in (0,1)$ such that $\bar{\omega}(x^{(i+1)}) = \bar{\omega}(x^{(i)}) + h_0\bar{\omega}'(x^{(i)} + \theta_i h_0)$.

From the Assumption H_0 it follows that $\bar{\omega}'$ is nondecreasing, so we have $\bar{\omega}(x^{(i+1)}) \geq \bar{\omega}(x^{(i)}) + h_0\bar{\omega}'(x^{(i)})$ and the difference - functional inequality

$$\bar{\omega}^{(i+1)} \geq \bar{\omega}^{(i)} + h_0\sigma(x^{(i)}, T_{0.h}(\bar{\omega}_{(x^{(i)})})^h) + h_0 A$$

is satisfied. Thus $\eta^{(i)} \leq \bar{\omega}(x^{(i)}),\ x^{(i)} \in I_h^*$.

Let the operator $V : C(B,R) \to C([-\tau_0, 0], R)$ be defined by

$$(Vw)(x) = \max\{|w(x,y)| : y \in [-\tau, \tau]\},\ w \in C(B,R),\ x \in [-\tau_0, 0].$$

Now we are ready to formulate the main theorem of this section.

THEOREM 1. *Suppose that*

1) *the Assumption H_0 is satisfied, the function $f : E \times C(B,R) \to R$ is uniformly continuous and the problem* (1) *has an unique solution of the class C^2 on E^*;*

2) *the function f satisfies inequality*

$$|f(x,y,q) - f(x,y,\bar{q})| \leq \sigma(x, V(q - \bar{q})),$$

where $(x,y,q), (x,y,\bar{q}) \in E \times C(B,R)$;

3) *there is $\alpha_0 : \Delta \to R_+$ such that* $\lim_{h\to 0}\alpha_0(h) = 0$ *and*

$$|\varphi^{(i,m)} - \varphi_h^{(i,m)}| \leq \alpha_0(h) \text{ on } E_{0.h} \cup \partial_0 E_h.$$

Then the method (3)–(4) *is convergent.*

Proof. Let $\bar{u}$ be the solution of (1) and let $\bar{z}_h$ be the solution of (3). We define

$$e^{(i,m)} = \bar{z}_h^{(i,m)} - \bar{u}^{(i,m)} \text{ on } E_h^*$$

and

$$\varepsilon^{(i)} = \max\{|e^{(i,m)}| : (x^{(i)}, y^{(m)}) \in E_h^*\} \text{ on } I_h^*.$$

Since there is $\theta = \theta^{(i,m)} \in (0,1)$ such that

$$\bar{u}^{(i+1,m)} = \bar{u}^{(i,m)} + h_0 f(x^{(i)} + \theta h_0, y^{(m)}, \bar{u}_{(x^{(i)}+\theta h_0, y^{(m)})}), \tag{8}$$

and there is $\beta : \Delta \to R_+$ such that $\lim_{h\to 0}\beta(h) = 0$ and

$$|f(x^{(i)} + \theta h_0, y^{(m)}, \bar{u}_{(x^{(i)}+\theta h_0, y^{(m)})}) - f(x^{(i)}, y^{(m)}, \bar{u}_{(x^{(i)}, y^{(m)})})| \leq \beta(h) \tag{9}$$

we have

$$\begin{aligned}|e^{(i+1,m)}| &= |e^{(i,m)} + h_0(f(x^{(i)}, y^{(m)}, T_h(\bar{z}_h)_{(i,m)}) \\ &\qquad - f(x^{(i)} + \theta h_0, y^{(m)}, \bar{u}_{(x^{(i)}+\theta h_0, y^{(m)})}))| \\ &\leq \varepsilon^{(i)} + h_0(\sigma(x^{(i)}, V(T_h(\bar{z}_h)_{(i,m)} - \bar{u}_{(i,m)}^h)) \\ &\qquad + \sigma(x^{(i)}, V(T_h u_{(i,m)}^h - u_{(x^{(i)}, y^{(m)})})) + \beta(h))\end{aligned}$$

$$\leq \varepsilon^{(i)} + h_0(\sigma(x^{(i)}, T_{0.h}\varepsilon_{(i)}) + \sigma(a, c|h|^2) + \beta(h)).$$

Thus $\varepsilon^{(i+1)} \leq \varepsilon^{(i)} + h_0(\sigma(x^{(i)}, T_{0.h}\varepsilon_{(i)}) + \gamma(h))$, where $\gamma(h) = \sigma(a, c|h|^2) + \beta(h)$. From the Lemma 2 it follows that

$$\varepsilon^{(i)} \leq \omega(x^{(i)}; h),$$

where $\omega(\cdot; h)$ is the solution of the problem

$$\omega'(x) = \sigma(x, T_{0.h}(\omega_{(x)})^h) + \gamma(h), \quad \omega(x) = \alpha_0(h) \quad \text{for } x \in [-\tau_0, 0].$$

From the assumptions it follows that $\lim_{h \to 0} \omega(x^{(i)}; h) = 0$ and this ends the proof of theorem.

The estimation of the discretization error is established by our next theorem.

THEOREM 2. *Suppose that assumptions* 1)–2) *of Theorem* 1 *are satisfied and*

1) *the function v is a solution of the problem* (1) *on E^* and there are the functions $\gamma, \gamma_0 : \Delta \to R_+$ such that*

$$|\delta(v^h)^{(i,m)} - f(x^{(i)}, y^{(m)}, T_h(v^h)_{(i,m)})| \leq \gamma(h) \quad \text{on } E_h,$$

$$|v^{(i,m)} - \varphi_h^{(i,m)}| \leq \gamma_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h;$$

2) *$z_h : E_h^* \to R$ is a solution of the method* (3)–(4).

Then

$$|v^{(i,m)} - z_h^{(i,m)}| \leq \omega(x^{(i)}; h),$$

where $\omega(\cdot; h)$ is a solution of the problem

$$\omega'(x) = \sigma(x, T_{0.h}(\omega_{(x)})^h) + \gamma(h), \quad \omega(x) = \gamma_0(h), \ x \in [-\tau_0, 0]. \tag{10}$$

Proof. We only give the main idea of the proof. Set

$$e^{(i,m)} = v^{(i,m)} - z_h^{(i,m)} \quad \text{on } E_h^*$$

and

$$\varepsilon^{(i)} = \max\{|e^{(i,m)}| : (x^{(i)}, y^{(m)}) \in E_h^*\} \quad \text{on } I_h^*.$$

An analysis similar to that in the proof of Theorem 1 shows that

$$\varepsilon^{(i+1)} \leq \varepsilon^{(i)} + h_0\sigma(x^{(i)}, T_{0.h}\varepsilon_{(i)}) + h_0\gamma(h)$$

and our assertion follows from Lemma 2.

3. One-step difference methods

Let $h \in \Delta$. Suppose that $\Phi_h : E_h \times F(E_h^*, R) \to R$, $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ are given functions.

Suppose that the function Φ_h satisfies the Volterra condition, i.e. if $(x, y) \in E_h$, $z, \bar{z} \in F(E_h^*, R)$ and $z(\bar{x}, \bar{y}) = \bar{z}(\bar{x}, \bar{y})$ for $(\bar{x}, \bar{y}) \in E_h^*$, $\bar{x} \leq x$ then $\Phi_h(x, y, z) = \Phi_h(x, y, \bar{z})$.

We consider the one-step difference method

$$\text{(11)}\qquad \begin{aligned} \delta z^{(i,m)} &= \Phi_h(x^{(i)}, y^{(m)}, z), \\ z^{(i,m)} &= \varphi_h^{(i,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \end{aligned}$$

We say that the method (11) is convergent if for every solution v of the problem (1) there is solution z_h of method (11) and the function $\tilde{\alpha} : \Delta \to R_+$ such that $\lim_{h\to 0} \tilde{\alpha}(h) = 0$ and

$$|v^{(i,m)} - z_h^{(i,m)}| \leq \tilde{\alpha}(h) \text{ on } E_h^*.$$

We introduce the following assumption.

ASSUMPTION H_1. Suppose that $\sigma_h : I_h \times F(I_h^*, R_+) \to R_+$ is a function of variables (x,p) and

1) the function σ_h satisfies the following Volterra condition: if $x \in I_h$, $\omega, \bar{\omega} \in F(I_h^*, R_+)$ and $\omega(\bar{x}) = \bar{\omega}(\bar{x})$ for $\bar{x} \leq x$ then $\sigma_h(x,\omega) = \sigma_h(x,\bar{\omega})$;

2) $\sigma_h(x,0) = 0$ for $x \in I_h$ and $\eta^{(i)} = 0$ on I_h^* is an unique solution of the problem

$$\text{(12)}\qquad \eta^{(i+1)} = \eta^{(i)} + h_0\sigma_h(x^{(i)}, \eta), \quad \eta^{(i)} = 0 \quad \text{on } I_{0.h};$$

3) the difference problem (12) is stable in the following sence: if $\eta_h : I_h^* \to R$ is the solution of the problem

$$\eta^{(i+1)} = \eta^{(i)} + h_0\sigma_h(x^{(i)}, \eta) + h_0\gamma(h), \quad \eta^{(i)} = \alpha_0(h) \quad \text{on } I_{0.h}$$

where $\alpha_0, \gamma : \Delta \to R_+$ and $\lim_{h\to 0} \alpha_0(h) = \lim_{h\to 0} \gamma(h) = 0$, then there is a function $\beta : \Delta \to R_+$ such that $\eta_h(x) \leq \beta(h)$ for $x \in I_h^*$ and $\lim_{h\to 0} \beta(h) = 0$;

4) σ_h is nondecreasing with respect to p.

Denote by V_h the operator $V_h : F(E_h^*, R) \to F(I_h^*, R)$ defined by

$$(V_h w)(x) = \max\{|w(x,y)| : (x,y) \in E_h^*\}, \ w \in F(E_h^*, R), \ x \in I_h^*.$$

The following theorem gives the sufficient conditions for convergence of the method (11).

THEOREM 3. *Suppose that*

1) *the function $f : E \times C(B,R) \to R$ of the variables (x,y,q) is uniformly continuous and the problem (1) has an unique solution on E^*;*

2) *the function $\sigma_h : I_h \times F(I_h^*, R_+) \to R_+$ satisfies the Assumption H_1;*

3) *the function $\Phi_h : E_h \times F(E_h^*, R) \to R$ satisfies inequality*

$$|\Phi_h(x,y,w) - \Phi_h(x,y,\bar{w})| \leq \sigma_h(x, V_h(w - \bar{w})) \text{ on } E_h \times F(E_h^*, R);$$

4) *the following compatibility conditions are satisfied*

$$|\Phi_h(x,y,(\bar{u}^h)) - f(x,y,\bar{u}_{(x,y)})| \leq \alpha(h) \quad \text{on } E_h,$$

$$|\varphi_h^{(i,m)} - \varphi^{(i,m)}| \leq \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h,$$

where $\alpha, \alpha_0 : \Delta \to R_+$, $\lim_{h\to 0}\alpha(h) = \lim_{h\to 0}\alpha_0(h) = 0$ *and* $\bar{u}$ *is the solution of the problem* (1).

Then the method (11) *is convergent.*

Proof. Let $\bar{u}$ be the solution of (1) and let $\bar{z}$ be the solution of (11). We define

$$e^{(i,m)} = \bar{z}^{(i,m)} - \bar{u}^{(i,m)} \quad \text{on } E_h^*$$

and

$$\varepsilon^{(i)} = \max\{|e^{(i,m)}| : (x^{(i)}, y^{(m)}) \in E_h^*\} \quad \text{on } I_h^*.$$

Using (8) and (9) from the proof of the Theorem 1 we have

$$\begin{aligned}
&|e^{(i+1,m)}| \\
&= |e^{(i,m)} + h_0(\Phi_h(x^{(i)}, y^{(m)}, \bar{z}) - f(x^{(i)} + \theta h_0, y^{(m)}, \bar{u}_{(x^{(i)}+\theta h_0, y^{(m)})}))| \\
&\le \varepsilon^{(i)} + h_0(\sigma_h(x^{(i)}, \varepsilon) + \alpha(h) + \beta(h)).
\end{aligned}$$

Thus $\varepsilon^{(i)} \le \eta_h^{(i)}$ where η_h is the solution of the problem

$$\begin{cases} \eta^{(i+1)} = \eta^{(i)} + h_0\sigma_h(x^{(i)}, \eta) + h_0(\alpha(h) + \beta(h)) \\ \eta^{(i)} = \alpha_0(h) \quad \text{on } I_{0.h}. \end{cases}$$

From our assumptions it follows that $\lim_{h\to 0}\eta_h^{(i)} = 0$ and this completes the proof of Theorem 3.

Now we formulate the theorem on estimation of the discretization error.

THEOREM 4. *Suppose that*

1) *the assumptions* 1)–3) *of Theorem* 3 *are satisfied;*

2) *the function* v *is solution of the problem* (1) *on* E^* *and there are the functions* $\gamma, \gamma_0 : \Delta \to R_+$ *such that*

$$|\delta(v^h)^{(i,m)} - \Phi_h(x^{(i)}, y^{(m)}, v^h)| \le \gamma(h) \quad \text{on } E_h,$$
$$|v^{(i,m)} - \varphi_h^{(i,m)}| \le \gamma_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h;$$

3) $z_h : E_h^* \to R$ *is solution of the method* (11).

Then

$$|v^{(i,m)} - z_h^{(i,m)}| \le \eta_h^{(i)} \quad \text{on } E_h^*, \tag{13}$$

where η_h *is solution of the problem*

$$\begin{cases} \eta^{(i+1)} = \eta^{(i)} + h_0\sigma_h(x^{(i)}, \eta) + h_0\gamma(h) \\ \eta^{(i)} = \gamma_0(h) \text{ on } I_{0.h}. \end{cases} \tag{14}$$

Proof. This follows by the same method as in Theorem 2.

The expression $\delta(v^h)^{(i,m)} - \Phi_h(x^{(i)}, y^{(m)}, v^h)$, where v is the solution of the problem (1), is called the local discretization error of the method (11). It has an influence on the estimation of the discretization error. For $\sigma_h(x^{(i)},p) = L\max\{p^{(j)} : x^{(j)} \in I_h^*,\ j \leq i\},\ L \geq 0$, the estimation (13) has the form

$$|v^{(i,m)} - z_h^{(i,m)}| \leq \gamma(h)\frac{e^{Lx^{(i)}}-1}{L} + \gamma_0(h)e^{Lx^{(i)}} \leq (\frac{\gamma(h)}{L} + \gamma_0(h))e^{La}.$$

We say that the method (11) is of order k if the local discretization error, with the solution v of suitable class, is $O(|h|^k)$.

4. Examples

Now we give a way to construct the methods of different orders.

Let $h \in \Delta$. Let $k \geq 1$ be integer and $(x,y) \in E_h$. We define the sets $B_k^{(x,y)} = ([x-k\tau_0, x]\times[y-k\tau, y+k\tau])\cap E^*$ and $B_{h.k}^{(x,y)} = B_k^{(x,y)} \cap E_h^*$. For $w : E_h^* \to R$ let $w_{(x,y).k}$ denote the restriction of the function w to the set $B_{h.k}^{(x,y)}$, i.e. $w_{(x,y).k} = w_{|B_{h.k}^{(x,y)}}$. Let the operator $T_{h.k}^{(x,y)} : F(B_{h.k}^{(x,y)}, R) \to C(B_k^{(x,y)}, R)$ have the following property

$$\|T_{h.k}^{(x,y)} z^h - z\|_0 \leq c|h|^k$$

for $z : B_k^{(x,y)} \to R$ of suitable class and for some constant $c \in R_+$. For $k=1$ and $k=2$ let the operator $T_{h.k}^{(x,y)}$ be defined analogously to this one in Section 2. In these cases, for simplicity of notation, we write T_h instead of $T_{h.k}^{(x,y)}$.

For the problem (1) we can define $\Phi_h : E_h \times F(E_h^*, R) \to R$ in the following way

(15) $\quad \Phi_h(x,y,w) = f(x,y,T_h w_{(x,y).1}),\ (x,y,w) \in E_h \times F(E_h^*, R).$

Difference method (11) with such Φ_h is the Euler method for the differential - functional problem (1). This method is of order 1. The comparison function $\sigma_h : I_h \times F(I_h^*, R_+) \to R_+$ for Φ_h given by (15) can be constructed in the following way. If we assume that

$$|f(x,y,q) - f(x,y,\bar{q})| \leq \sigma(x, V(q-\bar{q})) \quad \text{on } E \times C(B,R),$$

where σ satisfies the Assumption H_0, then we put

$$\sigma_h(x^{(i)},p) = \sigma(x^{(i)}, T_{0.h}p_{(i)}), \qquad x^{(i)} \in I_h, p \in F(I_h^*, R_+).$$

If the function u is a solution of class C^3 of the problem (1) on E^* then the first and second order derivatives with respect to x of the solution u are completely determinated by the functions f, φ and their derivatives.

We have

$$D_x u(x,y) = f(x,y,u_{(x,y)}),$$

$$D_x^2 u(x,y) = D_x f(x,y,u_{(x,y)}) + D_q f(x,y,u_{(x,y)})(D_x u)_{(x,y)}$$
$$= D_x f(x,y,u_{(x,y)}) + D_q f(x,y,u_{(x,y)}) f^{\varphi}[u]_{(x,y)},$$

where $f^{\varphi}[u](x,y) = D_x\varphi(x,y)$ for $(x,y) \in E_0 \cup \partial_0 E$ and $f^{\varphi}[u](x,y) = f(x,y,u_{(x,y)})$ for $(x,y) \in E$.

Since

$$\text{(16)} \quad \frac{u(x+h_0,y)-u(x,y)}{h_0} = f(x,y,u_{(x,y)}) + \frac{h_0}{2}(D_x f(x,y,u_{(x,y)}) + D_q f(x,y,u_{(x,y)}) f^{\varphi}[u]_{(x,y)}) + O(h_0^2)$$

it seems natural to choose the function $\Phi_h : E_h \times F(E_h^*, R) \to R$ as follows

$$\Phi_h(x,y,w) = f(x,y,T_h w_{(x,y).1}) + \frac{h_0}{2} D_x f(x,y,T_h w_{(x,y).1}) + \frac{h_0}{2} D_q f(x,y,T_h w_{(x,y).1}) f^{\varphi}[T_h w_{(x,y).2}]_{(x,y)}.$$

The difference method (11) with the above Φ_h is the method of order 2. It is called the Taylor method of order 2.

The another method of order 2 uses the function $\Phi_h : E_h \times F(E_h^*, R) \to R$ defined by

$$\Phi_h(x,y,w) = q_1 f(x,y,T_h w_{(x,y).1}) + q_2 f(x+\alpha h_0, y, T_h w_{(x,y).1} + \beta h_0 f^{\varphi}[T_h w_{(x,y).2}]_{(x,y)}).$$

We choose the parameters q_1, q_2, α, β in the following way. For the function Φ_h we write

$$\Phi_h(x,y,w) = q_1 f(x,y,T_h w_{(x,y).1}) + q_2 (f(x,y,T_h w_{(x,y).1}) + \alpha h_0 D_x f(x,y,T_h w_{(x,y).1}) + \beta h_0 D_q f(x,y,T_h w_{(x,y).1}) f^{\varphi}[T_h w_{(x,y).2}]_{(x,y)}) + O(h_0^2).$$

Using Taylor's expansion (16) for the solution u of the problem (1) we put

$$\text{(17)} \qquad q_1 + q_2 = 1, \quad q_2\alpha = \frac{1}{2}, \quad q_2\beta = \frac{1}{2}$$

For every solution of the system (17) we obtain the method of order 2. These methods are called the Runge-Kutta methods of order 2.

In the similar way we can construct the methods of the highter orders. We must only use the suitable Taylor's expansion instead of (16).

5. Numerical examples

Consider the differential–integral problem

$$\text{(18)} \qquad \begin{aligned} D_x u(x,y) &= F(x,y,\textstyle\int_B u(x+t,y+s)\,dt\,ds), \\ u(x,y) &= \varphi(x,y), \ (x,y) \in E_0 \cup \partial_0 E, \end{aligned}$$

where $F : E \times R \to R$ is function of variables (x, y, r).

For the function $w : E_h^* \to R$ and for the point $(x^{(i)}, y^{(m)}) \in E_h^*$ we define

$$J(w; x^{(i)}, y^{(m)}) = H \cdot \sum_{(i',\zeta_0)\in I_0\times S_0} \sum_{(m',\zeta)\in I\times S} w^{(i+i'+\zeta_0, m+m'+\zeta)},$$

where $H = \prod_{j=0}^{n}(\frac{h_j}{2})^{\text{sign}\tau_j}$, $I = I_1 \times \ldots \times I_n$, $S = S_1 \times \ldots \times S_n$ and

$I_j = \{0\}$ for $\tau_j = 0$, $I_j = \{-M_j, \ldots, M_j - 1\}$ for $\tau_j > 0$, $j = 1, \ldots, n$,

$S_j = \{0\}$ for $\tau_j = 0$, $S_j = \{0, 1\}$ for $\tau_j > 0$, $j = 0, 1, \ldots, n$,

$I_0 = \{0\}$ for $\tau_0 = 0$, $I_0 = \{-M_0, \ldots, -1\}$ for $\tau_0 > 0$.

The Euler method for the problem (18) uses $\Phi_h^E : E_h \times F(E_h^*, R) \to R$ given by the following formula

$$\Phi_h^E(x, y, w) = F(x, y, J(w; x, y)).$$

For the Taylor method of order 2 we define $\Phi_h^T : E_h \times F(E_h^*, R) \to R$ such that

$$\Phi_h^T(x, y, w) = F(x, y, J(w; x, y)) + \frac{h_0}{2} D_x F(x, y, J(w; x, y)) + \frac{h_0}{2} D_r F(x, y, J(w; x, y)) \cdot J(v; x, y),$$

where

$$v(x, y) = D_x\varphi(x, y) \quad \text{for } (x, y) \in E_{0.h} \cup \partial_0 E_h$$

and

$$v(x, y) = F(x, y, J(w; x, y)) \quad \text{for } (x, y) \in E_h.$$

The Runge–Kutta method of order 2 for the problem (18) uses the function $\Phi_h^R : E_h \times F(E_h^*, R) \to R$ such that

$$\Phi_h^R(x, y, w) = q_1 F(x, y, J(w; x, y)) + q_2 F(x + \alpha h_0, y, J(w; x, y) + \beta h_0 J(v; x, y))$$

with the parameters q_1, q_2, α, β satisfying the system (17).

In each above methods we use $\varphi_h : E_{0.h} \cup \partial_0 E_h \to R$ such that $\varphi_h(x, y) = \varphi(x, y)$, $(x, y) \in E_{0.h} \cup \partial_0 E_h$.

The numerical results we obtain for the problem (18) with $E = (0, 1) \times (-1, 1)$, $E_0 \cup \partial_0 E = ([0, 1) \times [-1.5, 1.5]) \setminus E$, $B = \{0\} \times [-1.5, 1.5]$ and $F(x, y, r) = r - x^2(y^2 + \frac{1}{12}) + 2xy^2$, $\varphi(x, y) = x^2y^2$ and $h_0 = 0.02$, $h_1 = 0.02$. The table shows some experimental values obtained for the Euler method, the Taylor method and the Runge–Kutta method. In the table the symbols z_h^E, z_h^T, z_h^R denote the solutions of the respective difference-functional methods and $\delta_h^E, \delta_h^t, \delta_h^R$ denote the point distances of the solutions of the difference-functional methods to the solution u of the differential-functional problem, i.e. $\delta_h^X(x,y) = |u(x,y) - z_h^X(x,y)|$, $(x,y) \in E_h$ where $X \in \{E,T,R\}$.

The above values are as follows:

(x,y)	$z_h^E(x,y)$	$\delta_h^E(x,y)$	$z_h^T(x,y)$	$\delta_h^T(x,y)$	$z_h^R(x,y)$	$\delta_h^R(x,y)$
$(0.4,-0.4)$	0.0239	$1.69 \cdot 10^{-3}$	0.0256	$1.57 \cdot 10^{-6}$	0.0256	$2.35 \cdot 10^{-5}$
$(0.4,0.8)$	0.0097	$5.64 \cdot 10^{-3}$	0.1024	$1.52 \cdot 10^{-6}$	0.1023	$6.25 \cdot 10^{-5}$
$(0.8,-0.4)$	0.0979	$4.48 \cdot 10^{-3}$	0.1024	$1.40 \cdot 10^{-5}$	0.1023	$5.12 \cdot 10^{-5}$
$(0.8,0.8)$	0.3970	$1.26 \cdot 10^{-2}$	0.4096	$1.32 \cdot 10^{-5}$	0.4095	$1.31 \cdot 10^{-4}$

The computations were carried out on an IBM AT.

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