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ANALYTIC SOLUTIONS OF A POLYNOMIAL-LIKE ITERATIVE FUNCTIONAL EQUATION

Abstract. This paper is concerned with a polynomial-like iterative functional equation $\lambda_1 f(z) + \lambda_2 f^{[2]}(z) + \cdots + \lambda_m f^{[m]}(z) = F(z)$, where $f(z)$ is an unknown function, $F(z)$ is a given function, $f^{[i]}$ denotes the i-th iterate of f, and $\lambda_1, \lambda_2, \cdots, \lambda_m$ are complex constants. By constructing a convergent power series solution $\varphi(z)$ of an auxiliary equation of the form $\lambda_1\varphi(\alpha z) + \lambda_2\varphi(\alpha^2 z) + \cdots + \lambda_m\varphi(\alpha^m z) = F(\varphi(z))$, analytic solutions of the form $\varphi(\alpha\varphi^{-1}(z))$ for the original iterative functional equation are obtained.

Iterative root, seen in [1] and [2], is an important problem in the theory of dynamical systems. As a natural generalization, the polynomial-like iterative functional equations in the following form

$$(*) \qquad \lambda_1 f(x) + \lambda_2 f^{[2]}(x) + \cdots + \lambda_m f^{[m]}(x) = F(x)$$

for $x \in R$, $\lambda_i \in R$, $i = 1, 2, \ldots, m$, or some special cases were discussed recently[3–8]. In this note, we will consider the existence of analytic solutions of equation $(*)$ in the complex field. Namely, we consider the equation

$$(1) \qquad \lambda_1 f(z) + \lambda_2 f^{[2]}(z) + \ldots + \lambda_m f^{[m]}(z) = F(z),$$

where $f(z)$ is an unknown function, $F(z)$ is a given complex-valued function of a complex variable, and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are complex constants (but not all zero).

Assume that $F(z)$ is analytic on a neighborhood of the origin, $F(0) = 0$ and $F'(0) = s \neq 0$. To find analytic solution of (1), we first seek an analytic solution $\varphi(z)$ of the auxiliary equation

$$(2) \qquad \lambda_1\varphi(\alpha z) + \lambda_2\varphi(\alpha^2 z) + \ldots + \lambda_m\varphi(\alpha^m z) = F(\varphi(z)),$$

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where α is a root of the algebraic equation

$$\lambda_1 z + \lambda_2 z^2 + \ldots + \lambda_m z^m - s = 0 \tag{3}$$

and satisfies one of the following conditions:

(H1) $0 < |\alpha| < 1$;
(H2) $|\alpha| > 1$;
(H3) $|\alpha| = 1, \alpha$ is not a root of unity, and

$$\log \frac{1}{|\alpha^n - 1|} \leq K \log n, \qquad n = 2, 3, \ldots$$

for some positive constant K. Then we show that (1) has an analytic solution of the form

$$f(z) = \varphi(\alpha \varphi^{-1}(z)) \tag{4}$$

in a neighborhood of the origin. Finally, we show how to derive an explicit power series solution by means of an example.

We begin with the following preparatory lemmas, the proof of the first of which can be found in [1, Chapter 6].

LEMMA 1. *Assume that* (H3) *holds. Then there is a positive number δ such that $|\alpha^n - 1|^{-1} < (2n)^\delta$ for $n = 1, 2, \ldots$ Furthermore, the sequence $\{d_n\}_{n=1}^{\infty}$ defined by $d_1 = 1$ and*

$$d_n = \frac{1}{|\alpha^{n-1} - 1|} \max_{\substack{n = n_1 + \ldots + n_t, \\ 0 < n_1 \leq \ldots \leq n_t,\, t \geq 2}} \{d_{n_1} \ldots d_{n_t}\}, \qquad n = 2, 3, \ldots$$

satisfies

$$d_n \leq (2^{5\delta+1})^{n-1} n^{-2\delta}, \qquad n = 1, 2, \ldots$$

LEMMA 2. *Assume that one of conditions* (H1) *and* (H2) *is fulfilled. Then for any $\eta \in C$, the equation (2) has an analytic solution $\varphi(z)$ in a neighborhood of the origin such that $\varphi(0) = 0$ and $\varphi'(0) = \eta$.*

Proof. Fix an $\eta \in C$. If $\eta = 0$ then the zero function satisfies the assertion. So assume that $\eta \neq 0$. Let

$$F(z) = \sum_{n=1}^{\infty} c_n z^n, \qquad \text{where } c_1 = s. \tag{5}$$

Since $F(z)$ is analytic on a neighborhood of the origin, there exists a positive β such that

$$|c_n| \leq \beta^{n-1}, \qquad n = 2, 3, \ldots \tag{6}$$

Introducing new functions $\widetilde{\varphi}(z) = \beta\varphi(\beta^{-1}z)$ and $\widetilde{F}(z) = \beta F(\beta^{-1}z)$, we obtain from (2) that

$$\lambda_1\widetilde{\varphi}(\alpha z) + \lambda_2\widetilde{\varphi}(\alpha^2 z) + \ldots + \lambda_m\widetilde{\varphi}(\alpha^m z) = \widetilde{F}(\widetilde{\varphi}(z)),$$

what is an equation of the form (2). From (5) and $\widetilde{F}(z) = \beta F(\beta^{-1}z)$, we have

$$\widetilde{F}(z) = \beta F(\beta^{-1}z) = sz + \sum_{n=2}^{\infty} c_n\beta^{1-n}z^n.$$

By (6) it follows that

$$\left|\frac{c_n}{\beta^{n-1}}\right| \leq 1, \quad n = 2, 3, \ldots$$

Therefore, without loss of generality, we assume that

$$(7) \qquad |c_n| \leq 1, \quad n = 2, 3, \ldots.$$

Let

$$(8) \qquad \varphi(z) = \sum_{n=1}^{\infty} b_n z^n$$

be the expansion of a formal solution $\varphi(z)$ of equation (2). Inserting (5) and (8) into (2) and comparing the coefficients we obtain

$$(\lambda_1\alpha + \lambda_2\alpha^2 + \ldots + \lambda_m\alpha^m - s)b_1 = 0$$

and

$$(9) \quad (\lambda_1\alpha^n + \lambda_2\alpha^{2n} + \ldots + \lambda_m\alpha^{mn} - s)b_n = \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} c_t b_{n_1} b_{n_2} \ldots b_{n_t}, n = 2, 3, \ldots$$

Noting that α is a root of (3), we see that $\lambda_1\alpha + \lambda_2\alpha^2 + \ldots + \lambda_m\alpha^m - s = 0$. Hence we choose $b_1 = \eta$ and by (9) we get

$$(10) \quad (\alpha^n - \alpha)\left(\lambda_1 + \sum_{i=1}^{m-1}\sum_{k=0}^{i} \lambda_{i+1}\alpha^{n(i-k)+k}\right)b_n = \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} c_t b_{n_1} b_{n_2} \ldots b_{n_t}, \qquad n = 2, 3, \ldots$$

Now, we show that the power series (8) converges in a neighborhood of the origin. First of all, since

$$\lim_{n\to\infty} \frac{1}{(\alpha^n-\alpha)\Big(\lambda_1+\sum_{i=1}^{m-1}\sum_{k=0}^{i}\lambda_{i+1}\alpha^{n(i-k)+k}\Big)} = \begin{cases} -\frac{1}{s}, & 0<|\alpha|<1, \\ 0, & |\alpha|>1, \end{cases}$$

there exists a positive number M, such that for $n \geq 2$,

$$\left|\frac{1}{(\alpha^n-\alpha)(\lambda_1+\sum_{i=1}^{m-1}\sum_{k=0}^{i}\lambda_{i+1}\alpha^{n(i-k)+k})}\right| \leq M$$

when either (H1) or (H2) is fulfilled. Thus if we define a sequence $\{B_n\}_{n=1}^{\infty}$ by $B_1 = |\eta|$ and

$$B_n = M \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} B_{n_1}B_{n_2}\ldots B_{n_t}, \qquad n=2,3,\ldots,$$

then in view of (10) and the inequality (7), we have

$$|b_n| \leq B_n, \qquad n=1,2,\ldots$$

Now we define

$$G(z)=\sum_{n=1}^{\infty} B_n z^n.$$

Since $G(0) = 0$, there is a positive number δ_1 such that $|G(z)| < 1$ for $|z| < \delta_1$. Then

$$\begin{aligned} G(z) &= \sum_{n=1}^{\infty} B_n z^n = |\eta| z + \sum_{n=2}^{\infty} B_n z^n \\ &= |\eta| z + M\sum_{n=2}^{\infty}\Big(\sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} B_{n_1}B_{n_2}\ldots B_{n_t}\Big) z^n \\ &= |\eta| z + M\sum_{n=2}^{\infty}[G(z)]^n = |\eta| z + M\frac{[G(z)]^2}{1-G(z)}. \end{aligned}$$

Hence

$$G(z)=\frac{1}{2(1+M)}\left\{1+|\eta|z \pm \sqrt{1-2(1+2M)|\eta|z+|\eta|^2z^2}\right\}.$$

But since $G(0) = 0$, only the negative sign of the square root is possible, so that

$$G(z)=\frac{1}{2(1+M)}\left\{1+|\eta|z - \sqrt{1-2(1+2M)|\eta|z+|\eta|^2z^2}\right\}.$$

It follows that the power series $G(z) = \sum_{n=1}^{\infty} B_n z^n$ converges for $|z| < \delta = \min\{\delta_1; \frac{1}{|\eta|}(1 + 2M - 2\sqrt{M + M^2})\}$, which implies that (8) is also convergent in a neighborhood of the origin. The proof is complete.

Next, we consider the case when (H3) holds.

LEMMA 3. *Suppose* (H3) *holds. Furthermore, suppose* $|\lambda_1| > \sum_{i=1}^{m-1}(i+1)|\lambda_{i+1}|$. *Then the equation* (2) *has an analytic solution* $\varphi(z)$ *of the form* (8) *in a neighborhood of the origin such that* $\varphi(0) = 0$ *and* $\varphi'(0) = 1$.

Proof. As in the proof of Lemma 2, we seek a power series solution of the form (8). Then defining $b_1 = 1$, (10) again holds so that

$$|b_n| = \left| \frac{1}{(\alpha^n - \alpha)(\lambda_1 + \sum_{i=1}^{m-1}\sum_{k=0}^{i} \lambda_{i+1}\alpha^{n(i-k)+k})} \times \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} c_t b_{n_1} b_{n_2} \ldots b_{n_t} \right|$$

$$\leq \frac{1}{|\alpha^{n-1} - 1|(|\lambda_1| - \sum_{i=1}^{m-1}(i+1)|\lambda_{i+1}|)} \times \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} |c_t||b_{n_1}||b_{n_2}|\ldots|b_{n_t}|$$

for $n = 2, 3, \ldots$. For convenience in writing, we put

$$N = \left(|\lambda_1| - \sum_{i=1}^{m-1}(i+1)|\lambda_{i+1}|\right)^{-1}.$$

Therefore, in view of (7), the above inequality becomes

$$|b_n| \leq \frac{N}{|\alpha^{n-1} - 1|} \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} |c_t||b_{n_1}||b_{n_2}|\ldots|b_{n_t}|, \qquad n = 2, 3, \ldots.$$

Let us now consider the function

$$G(z) = \frac{1}{2(1+N)}\left\{1 + z - \sqrt{1 - 2(1+2N)z + z^2}\right\},$$

which in view of the binomial series expansion can be written in the form

$$G(z) = z + \sum_{n=2}^{\infty} C_n z^n$$

for $|z| < 1 + 2N - 2\sqrt{N + N^2}$. Since $G(0) = 0$, there is a positive number

σ_1 such that $|G(z)| < 1$ for $|z| < \sigma_1$, and $G(z)$ satisfies the equation

$$G(z) = z + N\frac{[G(z)]^2}{1 - G(z)}.$$

By the method of undetermined coefficients, one can see that the coefficient sequence $\{C_n\}_{n=1}^{\infty}$ satisfies $C_1 = 1$ and

$$C_n = N \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} C_{n_1}C_{n_2}\ldots C_{n_t}, \qquad n = 2,3,\ldots$$

Hence, we easily see that

$$|b_n| \le C_n d_n, \qquad n = 1,2,\ldots,$$

where the sequence $\{d_n\}_{n=1}^{\infty}$ is defined in Lemma 1. Indeed, $|b_1| = 1 = C_1 d_1$. Assume by induction that the above inequality holds for $n = 1,2,\ldots,l$. Then

$$\begin{aligned}|b_{l+1}| &\le \frac{N}{|\alpha^l - 1|} \sum_{\substack{n_1+\ldots+n_t=l+1;\\ t=2,3,\ldots,n}} |b_{n_1}||b_{n_2}|\ldots|b_{n_t}| \\ &\le \frac{N}{|\alpha^l - 1|} \sum_{\substack{n_1+\ldots+n_t=l+1;\\ t=2,3,\ldots,n}} C_{n_1}d_{n_1}C_{n_2}d_{n_2}\ldots C_{n_t}d_{n_t} \\ &\le \frac{C_{l+1}}{|\alpha^l - 1|} \max_{\substack{n_1+\ldots+n_t=l+1;\\ 0<n_1\le\ldots\le n_t, t\ge 2}} \{d_{n_1}\ldots d_{n_t}\} = C_{l+1}d_{l+1},\end{aligned}$$

as desired.

Since $G(z)$ converges in the open disc $|z| < \sigma = \min\{\sigma_1; 1 + 2N - 2\sqrt{N + N^2}\}$, there is a positive A such that

$$C_n \le A^n$$

for $n = 1,2,\ldots$ In view of this and Lemma 1, we finally see that

$$|b_n| \le A^n(2^{5\delta+1})^{n-1}n^{-2\delta}, \qquad n = 1,2,\ldots,$$

which shows that the series (8) converges for $|z| < (A2^{5\delta+1})^{-1}$. The proof is complete.

We now state and prove our main result in this note.

THEOREM. *Suppose the conditions of Lemma* 2 *or Lemma* 3 *are fulfilled. Then equation* (1) *has an analytic solution of the form* (4) *in a neighborhood of the origin, where* $\varphi(z)$ *is an analytic solution of equation* (2).

Proof. In view of Lemma 2 and Lemma 3, we can find a sequence $\{b_n\}_{n=1}^{\infty}$ such that the function $\varphi(z)$ of the form (8) is an analytic solution of (2) in a neighborhood of the origin. Since $\varphi'(0) = \eta \ne 0$, the function $\varphi^{-1}(z)$ is analytic in a neighborhood of the point $\varphi(0) = 0$. If we now define $f(z)$ by means of (4), then

$$\begin{aligned}&\lambda_1 f(z) + \lambda_2 f^{[2]}(z) + \ldots + \lambda_m f^{[m]}(z)\\ &\quad= \lambda_1\varphi(\alpha\varphi^{-1}(z)) + \lambda_2\varphi(\alpha^2\varphi^{-1}(z)) + \ldots + \lambda_m\varphi(\alpha^m\varphi^{-1}(z))\\ &\quad= F(\varphi(\varphi^{-1}(z))) = F(z)\end{aligned}$$

as required. The proof is complete.

We now show how to explicitly construct an analytic solution of an equation of the form (1) by means of an example. Consider the following equation

$$f(f(z)) - f(z) = F(z), \tag{11}$$

where $F(z) = e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$. Obviously, the equation

$$z^2 - z - 1 = 0$$

has two roots

$$\alpha_1 = \frac{1-\sqrt{5}}{2}, \qquad \alpha_2 = \frac{1+\sqrt{5}}{2}$$

and $0 < |\alpha_1| < 1$, $|\alpha_2| > 1$. For α_1, by means of Lemma 2, the auxiliary equation

$$\varphi(\alpha_1^2 z) - \varphi(\alpha_1 z) = F(\varphi(z)) \tag{12}$$

has an analytic solution $\varphi(z)$ in a neighborhood of the origin such that $\varphi(0) = 0$ and $\varphi'(0) = \eta \neq 0$. Let

$$\varphi(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad b_1 = \eta.$$

Inserting this series into (12) and comparing the coefficients, we get

$$(\alpha_1^n - \alpha_1)(\alpha_1^n + \alpha_1 - 1)b_n = \sum_{\substack{n_1+\ldots+n_t=n;\\ t=2,3,\ldots,n}} \frac{1}{t!} b_{n_1} b_{n_2} \ldots b_{n_t}, \qquad n = 2, 3, \ldots \tag{13}$$

It is not difficult to calculate the coefficients b_n by means of (13). Indeed the first few terms are as follows:

$$b_2 = \frac{\varphi''(0)}{2!} = \frac{\eta^2}{4\alpha_1},$$

$$b_3 = \frac{\varphi'''(0)}{3!} = \frac{(2\alpha_1+3)\eta^3}{3!\cdot 6(3\alpha_1+2)},$$

$$\ldots\ldots$$

Next, recall from the proof of the above theorem that $\varphi^{-1}(z)$ is analytic in a neighborhood of the point $\varphi(0) = 0$. Therefore, its derivatives at $z = 0$

can also be determined by

$$(\varphi^{-1})'(0) = \frac{1}{\varphi'(\varphi^{-1}(0))} = \frac{1}{\varphi'(0)} = \frac{1}{\eta},$$

$$(\varphi^{-1})''(0) = -\frac{\varphi''(\varphi^{-1}(0))(\varphi^{-1})'(0)}{(\varphi'(\varphi^{-1}(0)))^2} = -\frac{\varphi''(0)(\varphi^{-1})'(0)}{(\varphi'(0))^2} = -\frac{1}{2\alpha_1\eta},$$

$$\begin{aligned}
&(\varphi^{-1})'''(0) \\
&= -\frac{\{\varphi'''(\varphi^{-1}(0))[(\varphi^{-1})'(0)]^2 + \varphi''(\varphi^{-1}(0))(\varphi^{-1})''(0)\}[\varphi'(\varphi^{-1}(0))]^2}{[\varphi'(\varphi^{-1}(0))]^4} \\
&\quad + \frac{\varphi''(\varphi^{-1}(0))(\varphi^{-1})'(0)\cdot 2\cdot\varphi'(\varphi^{-1}(0))\varphi''(\varphi^{-1}(0))(\varphi^{-1})'(0)}{[\varphi'(\varphi^{-1}(0))]^4} \\
&= -\frac{[\varphi'''(0)\eta^{-2} - \varphi''(0)\frac{1}{2\alpha_1\eta}]\eta^2 - \varphi''(0)\eta^{-1}\cdot 2\cdot\eta\varphi''(0)\eta^{-1}}{[\varphi'(0)]^4} \\
&= \frac{8\alpha_1+5}{12(5\alpha_1+3)\eta}, \qquad \text{etc.}
\end{aligned}$$

Finally, we determine a solution $f(z)$ of (11) by finding its derivatives at $z = 0$:

$$\begin{aligned}
f(0) &= \varphi(\alpha_1\varphi^{-1}(0)) = \varphi(\alpha_1\cdot 0) = 0, \\
f'(0) &= \varphi'(\alpha_1\varphi^{-1}(0))\cdot\alpha_1(\varphi^{-1})'(0) = \alpha_1\varphi'(0)(\varphi^{-1})(0) \\
&= \alpha_1\eta\cdot\frac{1}{\eta} = \alpha_1, \\
f''(0) &= \alpha_1^2\varphi''(\alpha_1\varphi^{-1}(0))[(\varphi^{-1})'(0)]^2 + \alpha_1\varphi'(\alpha_1\varphi^{-1}(0))(\varphi^{-1})''(0) \\
&= \alpha_1^2\varphi''(0)\eta^{-2} + \alpha_1\varphi'(0)(\varphi^{-1})''(0) = \alpha_1^2\frac{\eta^2}{2\alpha_1}\cdot\eta^{-2} - \alpha_1\eta\cdot\frac{1}{2\alpha_1\eta} \\
&= \frac{\alpha_1-1}{2}, \\
f'''(0) &= \varphi'''(\alpha_1\varphi^{-1}(0))\cdot[\alpha_1(\varphi^{-1})'(0)]^3 \\
&\quad + \varphi''(\alpha_1\varphi^{-1}(0))\cdot 2\alpha_1(\varphi^{-1})'(0)\cdot\alpha_1(\varphi^{-1})''(0) \\
&\quad + \varphi''(\alpha_1\varphi^{-1}(0))\alpha(\varphi^{-1})'(0)\cdot\alpha_1(\varphi^{-1})''(0) \\
&\quad + \varphi'(\alpha_1\varphi^{-1}(0))\cdot\alpha_1(\varphi^{-1})'''(0) \\
&= \frac{5\alpha_1+1}{12(3\alpha_1+2)}, \qquad \text{etc.}
\end{aligned}$$

Thus, the desired solution is

$$f(z) = \alpha_1 z + \frac{\alpha_1-1}{4}z^2 + \frac{5\alpha_1+1}{72(3\alpha_1+2)}z^3 + \dots$$

For α_2, by means of the same method we can obtain

$$f(z) = \alpha_2 z + \frac{\alpha_2 - 1}{4} z^2 + \frac{5\alpha_2 + 1}{72(3\alpha_2 + 2)} z^3 + \dots$$

It is clear from the above procedure that with the help of commercial software which is capable of doing exact differentiations, an arbitrary number of terms of the above series can be obtained.

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