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THEOREMS ON IMPULSIVE PARABOLIC DIFFERENTIAL-FUNCTIONAL INEQUALITIES

Abstract. Theorems on weak parabolic differential-functional inequalities together with initial boundary inequalities and impulsive inequalities, and on uniqueness criteria of solutions of parabolic differential-functional problems in arbitrary parabolic sets are proved.

1. Introduction

The aim of the paper is to prove theorems on weak parabolic differential-functional inequalities together with initial-boundary inequalities and impulsive inequalities. As a consequence of these theorems we obtain uniqueness criteria of classical solutions of nonlinear parabolic differential-functional problems.

The results of the paper, concerning classical parabolic differential-functional inequalities, are related to positions [4]–[6] and [8].

A theorem on weak inequalities with initial-boundary inequalities for unbounded solutions was proved in [5] under an assumption that a stifling divisor exists. In Section 3 of this paper we prove an analogous theorem for bounded solutions of a nonlinear system of inequalities assuming that the solutions satisfy a certain condition at infinity. The idea of the proof of the theorem was drawn out from [6].

To prove the theorems from Section 4 on impulsive parabolic problems some results given in papers [1]–[3] are applied.

Recently, parabolic problems together with impulsive inequalities without the functional argument have been considered in [7].

Key words and phrases: differential-functional inequalities and equations, parabolic problems, impulsive problems, uniqueness criteria.

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2. Preliminaries

For any vectors $z = (z_1, \dots, z_m) \in R^m$, $\bar{z} = (\bar{z}_1, \dots, \bar{z}_m) \in R^m$ we write $z \le \bar{z}$ if $z_i \le \bar{z}_i$ $(i = 1, \dots, m)$.

Let $E \subset R^{n+1}$ be an open set such that the projection of E on the t- axis is the interval $(0, T)$, $T \le \infty$. By D we denote a set of the points $(\bar{t}, \bar{x}) \in \overline{E}$ for which there exists a number $\rho > 0$ such that $\big\{(t, x) : \sum_{j=1}^{n} (x_j - \bar{x}_j)^2 + (t - \bar{t})^2 < \rho, t < \bar{t}\big\} \subset E$.

It is clear that $E \subset D \subset \overline{E}$. Let

$$\sigma := \big(\partial E \cap ((0, T] \times R^n)\big) \backslash D, \quad S_0 := \overline{E} \cap \big(\{0\} \times R^n\big)$$
$$\Sigma := \sigma \cup S_0.$$

For any $t \in (0, T)$ and for any $r > 0$ we define the following sets:

$$S_t := \{x \in R^n : (t, x) \in D\},$$
$$D^t := D \cap ((0, t] \times R^n),$$
$$D^t_r := D^t \cap \{(t, x) \in R^{n+1} : |x| = r\},$$
$$(D^t)_r := D^t \cap \{(t, x) \in R^{n+1} : |x| < r\}$$
$$Z(S_t) := \{z \mid z : S_t \ni x \to z(x) \in R^m\}.$$

ASSUMPTION A. *For each $i \in \{1, \dots, m\}$, we assume that σ^i is a subset (possibly empty) of σ and l^i is a direction such that for every $(t, x) \in \sigma^i$ the direction $l^i = l^i(t, x)$ is orthogonal to the t-axis and the interior of some segment starting at (t, x) in the direction l^i is contained in D. The functions $\alpha^i : \sigma^i \to R_+$, $\beta^i : \sigma^i \to R_+$, where $\beta^i(t, x) > \beta \ge 0$ $(i = 1, \dots, m)$, are given.*

We define two sets: $\tilde{\sigma} := \bigcup_{i=1}^{m} \sigma^i$, $\tilde{D} := D \cup \tilde{\sigma}$.

For each $i \in \{1, \dots, m\}$ a function $u^i : D \cup \sigma^i \to R$ is said to be *σ^i-regular in D* if u^i is continuous in $D \cup \sigma^i$ and the derivatives u^i_t, u^i_x, u^i_{xx} are continuous in D.

We write $u : \tilde{D} \to R^m$ if $u^i : D \cup \sigma^i \to R$ for every $i \in \{1, \dots, m\}$.

A function $u : \tilde{D} \to R^m$ is called *σ-regular in D* if u^i $(i = 1, \dots, m)$ are σ^i-regular in D.

We denote by $C_\sigma(D)$ the set of functions $u : \tilde{D} \to R^m$ which are σ-regular in D and by $C^0_\sigma(D)$ the set of the functions belonging to $C_\sigma(D)$ and bounded in $\tilde{D}$.

For every set $E \subset R^{1+n}$ we denote by *dist*(P, E) the distance between a point P and the set E.

DEFINITION 2.1. We say that a function $u \in C_\sigma(D)$ satisfies a *boundary condition* (BC) with the constant $h \ge 0$ if:

(1) For every $P_\nu = (t_\nu, x_\nu) \in D$ such that $\lim_{\nu\to\infty} \mathrm{dist}(P_\nu, \Sigma \setminus \sigma^i) = 0$ we have

$$\limsup_{\nu\to\infty} u^i(t_\nu, x_\nu) \le h \ (i = 1, \dots, m);$$

(2) For every $(t, x) \in \sigma^i$ $(i = 1, \dots, m)$ we have

$$\beta^i(t,x)u^i(t,x) - \alpha^i(t,x)\bar{D}_{l^i}u^i(t,x) \le h\beta,$$

where β is the constant from Assumption A and

$$\bar{D}_{l^i}u^i(t,x) := \lim sup_{\tau\to 0+} \frac{1}{\tau}[u^i(t, x + \tau vers\ l^i(t,x)) - u^i(t,x)].$$

DEFINITION 2.2. A function $u \in C_\sigma(D)$ satisfies a *limit condition* (LC) if

$$\limsup_{\substack{|x|\to\infty \\ (t,x)\in D\cup\sigma^i}} u^i(t,x) \le 0 \quad (i = 1, \dots, m)$$

almost uniformly with respect to t.

For each $i \in \{1, \dots, m\}$ consider the mapping

$$f^i : D \times R^m \times R^n \times R^{n^2} \times Z(S_t) \ni (t, x, u, q, r, z) \to f^i(t, x, u, q, r, z) \in R,$$

where $q = (q_1, \dots, q_n)$, $r = [r_{jk}]_{n\times n}$, and an operator P^i given by the formula

$$(P^i w)(t,x) := w^i_t(t,x) - f^i(t, x, w(t,x), w^i_x(t,x), w^i_{xx}(t,x), w(t,\cdot)),$$
$$w \in C_\sigma(D), (t,x) \in D,$$

where $w^i_x := grad_x w^i(t,x)$ and $w^i_{xx} := \left[\frac{\partial^2 w^i(t,x)}{\partial x_j \partial x_k}\right]_{n\times n}$ $(i = 1, \dots, m)$.

For each $i \in \{1, \dots, m\}$ the function f^i is said to be *parabolic* in a subset $S \subset D$ with respect to a function $u \in C_\sigma(D)$ if for every real square symmetric matrices $r = [r_{jk}]$, $\tilde{r} = [\tilde{r}_{jk}]$ and $(t,x) \in S$ the following implication holds

$$r \le \tilde{r} \Rightarrow f^i(t, x, u(t,x), u_x(t,x), r, u(t,\cdot))$$
$$\le f^i(t, x, u(t,x), u_x(t,x), \tilde{r}, u(t,\cdot)),$$

where $r \le \tilde{r}$ means that the inequality $\sum_{j,k=1}^n (r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k \le 0$ is satisfied for arbitrary vector $\lambda = (\lambda_1, \dots, \lambda_n) \in R^n$.

3. Differential-functional inequalities with initial-boundary inequalities

We introduce the following assumption:

ASSUMPTION B. *There exists a function $M : D \times R^m \times Z(S_t) \to R^n$ such that for $(t,x) \in D, q \in R^n, r \in R^{n^2}$ and $z, \bar{z} : D \to R^m$ we have*

$$f^i(t,x,z(t,x),q,r,z(t,\cdot)) - f^i(t,x,\bar{z}(t,x),q,r,\bar{z}(t,\cdot)) \leq M^i(t,x,z(t,x)-\bar{z}(t,x),z(t,\cdot)-\bar{z}(t,\cdot)) \tag{3.1}$$

in the set $\{(t,x) \in D : z^i(t,x) > \bar{z}^i(t,x)\}$ $(i = 1,\dots,m)$.

Moreover, there exists a constant $K \in R_+$ such that for arbitrary function $w : D \to R^m$ bounded from above in D the functions M^i $(i = 1,\dots,m)$ satisfy the inequalities

$$M^i(t,x,w(t,x),w(t,\cdot)) \leq K \max_p \sup_{x \in S_t} w^p(t,x) \tag{3.2}$$

in the set $\{(t,x) \in D : w^i(t,x) > 0\}$ $(i = 1,\dots,m)$.

THEOREM 3.1. *Suppose that:*

1. *Assumption B is satisfied;*
2. *$u, v \in C_\sigma(D)$ and $(P^i u)(t,x) \leq (P^i v)(t,x)$ for*
$$(t,x) \in N_i := \{(t,x) \in D : u^i(t,x) > v^i(t,x)\} \quad (i = 1,2,\dots,m);$$
3. *$u - v$ is bounded from above in D and satisfies the condition (LC) and condition (BC) with the constant $h = 0$;*
4. *f^i is parabolic with respect to u or v in N_i $(i = 1,\dots,m)$.*

Then

$$u(t,x) \leq v(t,x) \quad \text{for} \quad (t,x) \in D. \tag{3.3}$$

Proof. We will show that

$$u(t,x) \leq v(t,x) \quad \text{for} \quad (t,x) \in D^{T_0}, \tag{3.4}$$

where $0 < T_0 < \frac{1}{2K}$, arguing by contradiction. Suppose that inequality (3.4) is not true, i.e.,

$$\max_i \sup_{D^{T_0}} \left[u^i(t,x) - v^i(t,x)\right] = H > 0.$$

Obviously, there is an index $j \in \{1,\dots,m\}$ such that

$$\sup_{D^{T_0}} \left[u^j(t,x) - v^j(t,x)\right] = H > 0.$$

By condition (LC), there exists $r_0 > 0$ such that

$$\sup_{\overline{(D^{T_0})_{r_0}}} (u^j - v^j)(t,x) = H.$$

Next, define an auxiliary function w by the formula

$$w(t,x) := u^j(t,x) - v^j(t,x) - \frac{\lambda t}{T_0}, \tag{3.5}$$

where $\frac{H}{2} < \lambda < H, \quad (t,x) \in D^{T_0}$. Function w has the following properties:

(i) $w(t,x) \le H$ for $(t,x) \in D^{T_0}$,
(ii) w satisfies the conditions (BC) with the constant $h = 0$ and (LC),
(iii) if $\omega := \sup_{\overline{(D^{T_0})_{r_0}}} w(t,x)$, then

$$H \ge \omega \ge \sup_{\overline{(D^{T_0})_{r_0}}} \left[(u^j - v^j)(t,x) - \lambda\right] = H - \lambda > 0.$$

Condition (LC) implies that for $\varepsilon = \frac{\omega}{2}$ there exists $r_1 > r_0$ such that for every $r \ge r_1$ we have

$$w(t,x) < \frac{\omega}{2} \qquad \text{for } (t,x) \in D_r^{T_0}. \tag{3.6}$$

Consequently,

$$\sup_{\overline{(D^{T_0})_{r_1}}} w(t,x) = \omega_1 \ge \omega > 0.$$

Observe, from inequality (3.6), that ω_1 cannot be reached on the boundary $D_{r_1}^{T_0}$ and in the exterior of the set $(D^{T_0})_{r_1}$ belonging to D^{T_0}. Put

$$D^{T_0}(\rho) := \{P \in (D^{T_0})_{r_1} : \operatorname{dist}(P, (\Sigma^{T_0})_{r_1}) \ge \rho\},$$
$$D_0(\rho) := (D^{T_0})_{r_1} \setminus D^{T_0}(\rho).$$

Arguing analogously as in Theorem 1 from [5], we obtain that ω_1 cannot be reached close to the boundary $(\Sigma^{T_0})_{r_1}$, i.e. there exists $\rho_0 > 0$ such that

$$\sup_{(t,x)\in\overline{D_0(\rho_0)}} w(t,x) = \omega_2 < \omega_1. \tag{3.7}$$

Fix $\rho_0 > 0$ for which (3.7) holds. Let $\rho_1 := \frac{\rho_0}{2}$ and $D_1 := D^{T_0}(\rho_1)$. Since the function w is continuous in $\overline{D_1}$ and $w(t,x) \le \omega_2 < \omega_1$ for $(t,x) \in \partial D_1$, there is $(\bar{t},\bar{x}) \in intD_1$ such that

$$w(\bar{t},\bar{x}) = \max_{(t,x)\in\overline{D_1}} w(t,x) = \omega_1.$$

Therefore,

$$\begin{aligned} w_t(\bar{t},\bar{x}) &= u_t^j(\bar{t},\bar{x}) - v_t^j(\bar{t},\bar{x}) - \frac{\lambda}{T_0} \ge 0, \\ w_x(\bar{t},\bar{x}) &= u_x^j(\bar{t},\bar{x}) - v_x^j(\bar{t},\bar{x}) = 0, \\ \sum_{l,k=1}^{n} w_{x_l x_k}(\bar{t},\bar{x})\lambda_l\lambda_k &= \sum_{l,k=1}^{n} (u_{x_l x_k}^j(\bar{t},\bar{x}) - v_{x_l x_k}^j(\bar{t},\bar{x}))\lambda_l\lambda_k \le 0. \end{aligned} \tag{3.8}$$

On the other hand, by (3.5), by Assumptions 2,4 of Theorem 1 and by Assumption B, we obtain

$$\begin{aligned}
w_t(\bar t,\bar x)+\frac{\lambda}{T_0} &= u_t^j(\bar t,\bar x)-v_t^j(\bar t,\bar x)\\
&\le f^j(\bar t,\bar x,u,u_x^j,u_{xx}^j,u(\bar t,\cdot))-f^j(\bar t,\bar x,v,v_x^j,v_{xx}^j,v(\bar t,\cdot))\\
&\quad - f^j(\bar t,\bar x,v,v_x^j,u_{xx}^j,v(\bar t,\cdot))+f^j(\bar t,\bar x,v,v_x^j,u_{xx}^j,v(\bar t,\cdot))\\
&\le M^j(\bar t,\bar x,u-v,u(\bar t,\cdot)-v(\bar t,\cdot))\\
&\le K\sup_{S_{\bar t}}\big(u^j(\bar t,x)-v^j(\bar t,x)\big)\le KH.
\end{aligned}$$

Consequently,

$$w_t(\bar t,\bar x)\le KH-\frac{\lambda}{T_0}<H\Big(K-\frac{1}{2T_0}\Big)<0. \tag{3.9}$$

Since (3.9) contradicts to (3.8) then (3.4) holds.

Assuming that for certain integer m inequality $u(t,x)\le v(t,x)$ holds in the set $D^{mT_0}\setminus D^{(m-1)T_0}$ and repeating the above argumentation, we can prove that $u(t,x)\le v(t,x)$ in $(t,x)\in D^{(m+1)T_0}\setminus D^{mT_0}$.

The proof of Theorem 3.1 is complete.

Consider the following problem (called a *mixed parabolic differential-functional problem*):

$$(P_iu)(t,x)=0 \quad \text{for} \quad (t,x)\in D\ (i=1,\dots,m), \tag{3.10}$$

$$u^i(t,x)=\phi_1^i(t,x) \quad \text{for} \quad (t,x)\in\Sigma\backslash\sigma^i\ (i=1,\dots,m), \tag{3.11}$$

$$\beta^i(t,x)u^i(t,x)-\alpha^i(t,x)\frac{du^i}{dl^i}(t,x)=\phi_2^i(t,x) \quad \text{for } (t,x)\in\sigma^i\ (i=1,\dots,m), \tag{3.12}$$

where $\phi_1^i:\Sigma\backslash\sigma^i\to R$ and $\phi_2^i:\sigma^i\to R$ $(i=1,\dots,m)$ are given functions.

A function $u\in C_\sigma(D)$ is said to be a *solution* of problem (3.10) – (3.12) if:

(a) the derivatives $\frac{du^i}{dl^i}$ $(i=1,\dots,m)$ are finite on σ^i $(i=1,\dots,m)$, respectively,

(b) u satisfies formulas (3.10) and (3.12) ,

(c) $\lim_{\substack{(t,x)\to(\bar t,\bar x)\\ (t,x)\in D}} u^i(t,x)=\phi_1^i(\bar t,\bar x)$ $(i=1,\dots,m)$ for all $(\bar t,\bar x)\in\Sigma\setminus\sigma^i$ $(i=1,\dots,m)$.

Let

$$LC^0_\sigma(D) := \Big\{ u \in C^0_\sigma(D) : \lim_{\substack{|x|\to\infty \\ (t,x)\in D\cup\sigma^i}} u^i(t,x) = 0 \ (i = 1,\dots,m),$$
$$\text{almost uniformly with respect to t} \Big\}.$$

As a consequence of Theorem 3.1 we obtain the following theorem about the uniqueness of a classical solution of the parabolic differential-functional problem (3.10)–(3.12).

THEOREM 3.2. *Suppose that Assumption B is satisfied. Then, in the class of all functions u belonging to $LC^0_\sigma(D)$ such that the functions f^i $(i = 1,\dots,m)$ are parabolic with respect to u in D, there exists at most one function u satisfying the mixed parabolic differential-functional problem* (3.10)–(3.12).

4. Differential-functional inequalities with impulsive inequalities

The aim of this section is to prove Theorem 4.1 about weak nonlinear parabolic differential-functional inequalities with nonlinear impulsive inequalities and initial-boundary conditions. The Assumption B of Theorem 4.1 is quite different than assumptions considered in papers [1]–[3].

Let $s \in N$ be an arbitrary fixed number and let $t_1, t_2, \dots, t_s$ be given real numbers such that $0 = t_0 < t_1 < t_2 < \dots < t_s < T$. For any $t \in (0,T)$ we define $\delta_t := \bar{D} \cap (\{t\} \times R^n)$ and introduce the following sets

$$\delta_\star := \bigcup_{j=1}^{s} \delta_{t_j}, \qquad D_\star := \bigcup_{j=1}^{s} D_j,$$

where

$$D_j := D \cap \big[(t_j, t_{j+1}) \times R^n\big] \quad (j = 0, 1, \dots, s-1),$$
$$D_s := D \cap \big[(t_s, T] \times R^n\big].$$

Let $\Sigma_\star := (\sigma \setminus \delta_\star) \cup S_0$.

ASSUMPTION C. *For each $i \in \{1,\dots,m\}$, we assume that $\sigma^i_\star$ is a subset (possibly empty) of $\sigma \setminus \delta_\star$ and l^i is a direction such that for every $(t,x) \in \sigma^i_\star$ the direction $l^i = l^i(t,x)$ is orthogonal to the t-axis and the interior of some segment starting at (t,x) in the direction l^i is contained in D. We assume that the functions $\alpha^i : \sigma^i_\star \to R_+$, $\beta^i : \sigma^i_\star \to R_+$, where $\beta^i(t,x) > \beta \geq 0$ $(i = 1,\dots,m)$, are given.*

We will need two sets

$$\tilde{\sigma}_\star := \bigcup_{i=1}^{m} \sigma^i_\star, \qquad \tilde{D}_\star := D \cup \tilde{\sigma}_\star.$$

For each $i \in \{1,\dots,m\}$, a function $u^i : D \cup \sigma^i_\star \to R$ is said to be *$\sigma^i_\star$-regular in D* if:

(i) u^i is continuous in $\left(D \cup \sigma^i_\star\right) \backslash \delta_\star$,

(ii) for every $(t,x) \in \delta_\star$ and for each sequence $(t_\nu, x_\nu) \in D_\star$ such that $t_\nu > t$ $[t_\nu < t]$ and $\lim_{\nu\to\infty}(t_\nu, x_\nu) = (t,x)$, the limit $\lim_{\nu\to\infty} u^i(t_\nu, x_\nu)$ $(i = 1,\dots,m)$ is finite (obviously this limit does not depend on the choice of the sequence (t_ν, x_ν)), it will be denoted by $u^i(t^+, x)$ $[u^i(t^-, x)]$, respectively and $u^i(t,x) := u^i(t^+, x)$, $(t,x) \in \delta_\star$,

(iii) u^i_t, u^i_x, u^i_{xx} are continuous in $D_\star$.

We write $u : \tilde{D}_\star \to R^m$ if $u^i : D \cup \sigma^i_\star \to R$ for every $i \in \{1,\dots,m\}$.

A function $u : \tilde{D}_\star \to R^m$ is called *$\sigma_\star$-regular in D* if u^i $(i = 1,..,m)$ are $\sigma^i_\star$-regular in D.

We use the notation $PC_\sigma(D) := \left\{u : \tilde{D}_\star \to R^m :\ u \text{ is } \sigma_\star\text{-regular in } D\right\}$.

DEFINITION 4.1. We say that a function $u \in PC_\sigma(D)$ satisfies a *boundary condition* $(BC)_\star$ with the constant $h \geq 0$ if:

(1) For every $P_\nu = (t_\nu, x_\nu) \in D_\star$ such that $\lim_{\nu\to\infty} \operatorname{dist}(P_\nu, \Sigma_\star \setminus \sigma^i_\star) = 0$ we have

$$\limsup_{\nu\to\infty} u^i(t_\nu, x_\nu) \leq h \quad (i = 1,\dots,m);$$

(2) For every $(t,x) \in \sigma^i_\star$ the inequalities

$$\beta^i(t,x)u^i(t,x) - \alpha^i(t,x)\bar{D}_{l^i}u^i(t,x) \leq h\beta, \quad (i = 1,\dots,m),$$

are true, where β is the constant from Assumption C and

$$\bar{D}_{l^i}u^i(t,x) = \lim sup_{\tau\to 0^+} \frac{1}{\tau}[u^i(t, x + \tau vers\ l^i(t,x)) - u^i(t,x)].$$

DEFINITION 4.2. A function $u \in PC_\sigma(D)$ satisfies a *limit condition* $(LC)_\star$ if

$$\limsup_{\substack{|x|\to\infty \\ (t,x)\in D_\star \cup \sigma^i_\star}} u^i(t,x) \leq 0 \quad (i = 1,\dots,m)$$

almost uniformly with respect to t.

For each $i \in \{1,\dots,m\}$ consider the mapping

$$f^i_\star : D_\star \times R^m \times R^n \times R^{n^2} \times Z(S_t) \ni (t,x,u,q,r,z) \to f^i_\star(t,x,u,q,r,z) \in R$$

and an operator $P^i_\star$ given by the formula

$$(P^i_\star w)(t,x) := w^i_t(t,x) - f^i_\star(t,x,w(t,x),w^i_x(t,x),w^i_{xx}(t,x),w(t,\cdot)),$$
$$w \in PC_\sigma(D), (t,x) \in D_\star.$$

For each $i \in \{1,\dots,m\}$, the function $f^i_\star$ is said to be *parabolic* in a subset $S \subset D_\star$ with respect to a function $u \in PC_\sigma(D)$ if for every real square symmetric matrices $r = [r_{jk}]$, $\tilde{r} = [\tilde{r}_{jk}]$ and $(t, x) \in S$ the following implication holds

$$r \leq \tilde{r} \Rightarrow f_\star^i(t, x, u(t, x), u_x(t, x), r, u(t, \cdot)) \leq f_\star^i(t, x, u(t, x), u_x(t, x), \tilde{r}, u(t, \cdot)),$$

where $r \leq \tilde{r}$ means that the inequality $\sum_{j,k=1}^n (r_{jk} - \tilde{r}_{jk})\lambda_j \lambda_k \leq 0$ is satisfied for arbitrary vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$.

For each $i \in \{1, \ldots, m\}$ consider the function

$$h^i : \delta_\star \times R^m \to R.$$

ASSUMPTION D. *The functions h^i $(i = 1, \ldots, m)$ satisfy the following condition*

$$(z, \bar{z} \in R^m, z \leq \bar{z}) \Rightarrow \big(z^i + h^i(t, x, z) \leq \bar{z}^i + h^i(t, x, \bar{z})\big) \quad \text{for } (t, x) \in \delta_\star \ (i = 1, \ldots, m).$$

THEOREM 4.1. *Suppose that:*

1. *Assumption B, where f and D are replaced by $f_\star$ and $D_\star$, respectively, is satisfied;*

2. *$u, v \in PC_\sigma(D)$ and $(P_\star^i u)(t, x) \leq (P_\star^i v)(t, x)$ for $(t, x) \in \Omega_i := \{(t, x) \in D_\star : u^i(t, x) > v^i(t, x)\}$, (i=1,2,...,m);*

3. *$u - v$ is bounded from above in D and satisfies conditions $(BC)_\star$ with the constant $h = 0$ and $(LC)_\star$, and the impulsive inequalities*

$$u^i(t, x) - u^i(t^-, x) - h^i(t, x, u(t^-, x)) \leq v^i(t, x) - v^i(t^-, x) - h^i(t, x, v(t^-, x))$$

for $(t, x) \in \delta_\star$, where h^i $(i = 1, \ldots, m)$ are given functions satisfying Assumption D;

4. *$f_\star^i (i = 1, \ldots, m)$ are parabolic with respect to u or v in Ω_i $(i = 1, \ldots, m)$.*

Then

$$u(t, x) \leq v(t, x) \quad \text{for} \quad (t, x) \in D. \tag{4.1}$$

Proof. To prove inequality (4.1) consider the following problem:

(4.2) $(P_\star^i u)(t, x) \leq (P_\star^i v)(t, x)$ for $(t, x) \in \Omega_i \cap [(t_k, T_{k+1}] \times R^n]$ $(i = 1, \ldots, m)$; $\limsup_{\nu \to \infty} [u^i(t_\nu, x_\nu) - v^i(t_\nu, x_\nu)] \leq 0$ for $P_\nu = (t_\nu, x_\nu) \in D \cap [(t_k, T_{k+1}] \times R^n]$ such that $\lim_{\nu \to \infty} \operatorname{dist}(P_\nu, [\delta_{t_k} \cup (\Sigma_\star \setminus \sigma_\star^i)] \cap [[t_k, T_{k+1}] \times R^n]) = 0$ $(i = 1, \ldots, m)$; $\beta^i(t, x)(u^i - v^i)(t, x) - \alpha^i(t, x) \bar{D}_{l^i}(u^i - v^i)(t, x) \leq 0$ for $(t, x) \in \sigma_\star^i \cap [(t_k, T_{k+1}] \times R^n]$ $(i = 1, \ldots, m)$,

for $k = 0, 1, \ldots, s$, where $t_0 = 0$, $\delta_{t_0} = S_0$, $T_{k+1} (k = 0, 1, \ldots, s - 1)$ are the arbitrary numbers such that $t_k < T_{k+1} < t_{k+1}$ and $T_{s+1} = T$.

First, consider the problem (4.2) for $k = 0$. By the assumptions of Theorem 4.1, corresponding to this problem, we obtain, by Theorem 3.1 applied to the set $D \cap \left[(0, T_1] \times R^n\right]$, the inequality

$$(4.3) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,T_1] \times R^n\right].$$

Consequently,

$$(4.4) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,t_1) \times R^n\right].$$

By (4.4) and by the fact that $u, v \in PC_\sigma(D)$,

$$(4.5) \qquad u(t^-,x) \le v(t^-,x) \quad \text{for } (t,x) \in \delta_{t_1}.$$

From inequality (4.5) and Assumption D we obtain

$$(4.6) \qquad u^i(t^-,x) + h^i(t,x,u(t^-,x)) \le v^i(t^-,x) + h^i(t,x,v(t^-,x)) \quad \text{for } (t,x) \in \delta_{t_1} (i = 1,\dots,m).$$

Inequalities (4.6) and Assumption 3 of Theorem 4.1 imply that

$$(4.7) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in \delta_{t_1}.$$

By (4.4) and (4.7),

$$(4.8) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,t_1] \times R^n\right].$$

Now, consider the problem (4.2) for $k = 1$. By the assumptions of Theorem 4.1,corresponding to this problem, we obtain, by Theorem 3.1 applied to the set $D \cap \left[(t_1, T_2] \times R^n\right]$, the inequality

$$(4.9) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(t_1,T_2] \times R^n\right].$$

From (4.8) and (4.9) we have

$$(4.10) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,T_2] \times R^n\right].$$

Consequently,

$$(4.11) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,t_2) \times R^n\right].$$

By (4.11) and by the fact that $u, v \in PC_\sigma(D)$,

$$(4.12) \qquad u(t^-,x) \le v(t^-,x) \quad \text{for } (t,x) \in \delta_{t_2}.$$

From inequality (4.12) and Assumption D we obtain

$$(4.13) \qquad u^i(t^-,x) + h^i(t,x,u(t^-,x)) \le v^i(t^-,x) + h^i(t,x,v(t^-,x)) \quad \text{for } (t,x) \in \delta_{t_1} \quad (i = 1,\dots,m).$$

Inequalities (4.13) and Assumption 3 of Theorem 4.1 imply

$$(4.14) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in \delta_{t_2}.$$

By (4.11) and (4.14),

$$(4.15) \qquad u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \left[(0,t_2] \times R^n\right].$$

Repeating the above procedure $(s-2)$-times, we have

$$u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \big[(0,t_s] \times R^n\big]. \tag{4.16}$$

Finally, consider the problem (4.2) for $k = s$. By Theorem (3.1), we get the inequality

$$u(t,x) \le v(t,x) \quad \text{for } (t,x) \in D \cap \big[(t_s,T] \times R^n\big]. \tag{4.17}$$

Inequalities (4.16) and (4.17) imply $u(t,x) \le v(t,x)$, $(t,x) \in D$.
The proof of Theorem 4.1 is complete.

Consider the following mixed impulsive parabolic differential-functional problem:

$$(P^i_\star u)(t,x) = 0 \quad \text{for} \quad (t,x) \in D_\star \quad (i = 1,\dots,m); \tag{4.18}$$

$$u^i(t,x) = \phi^i_1(t,x) \quad \text{for} \quad (t,x) \in \Sigma_\star \backslash \sigma^i_\star \ (i = 1,\dots,m); \tag{4.19}$$

$$\beta^i(t,x)u^i(t,x) - \alpha^i(t,x)\frac{du^i}{dl^i}(t,x) = \phi^i_2(t,x) \quad \text{for} \quad (t,x) \in \sigma^i_\star \quad (i = 1,\dots,m); \tag{4.20}$$

$$u^i(t,x) - u^i(t^-,x) - h_i(t,x,u(t^-,x)) = \phi^i_3(t,x) \quad \text{for} \quad (t,x) \in \delta_\star \quad (i = 1,\dots,m); \tag{4.21}$$

where $\phi^i_1 : \Sigma_\star \backslash \sigma^i_\star \to R$, $\phi^i_2 : \sigma^i_\star \to R$ and $\phi^i_3 : \delta_\star \to R$ $(i = 1,\dots,m)$ are given functions.

A function $u \in PC_\sigma(D)$ is said to be a *solution* of problem (4.18)–(4.21) if:

(a) the derivatives $\frac{du^i}{dl^i}$ $(i = 1,\dots,m)$ are finite on $\sigma^i_\star$ $(i = 1,\dots,m)$, respectively;

(b) u satisfies formulas (4.18), (4.20) and (4.21) ;

(c) $\lim_{\substack{(t,x)\to(\bar t,\bar x)\\ (t,x)\in D}} u^i(t,x) = \phi^i_1(\bar t,\bar x)$ $(i = 1,\dots,m)$ for all $(\bar t,\bar x) \in \Sigma_\star \setminus \sigma^i_\star$ $(i = 1,\dots,m)$.

Let

$$LPC_\sigma(D) := \Big\{ u \in PC_\sigma(D) : \lim_{\substack{|x|\to\infty\\ (t,x)\in D_\star\cup\sigma^i_\star}} u^i(t,x) = 0 \quad (i = 1,\dots,m), \ \text{almost uniformly with respect to t} \Big\}.$$

As a consequence of Theorem 4.1 we obtain the following one:

THEOREM 4.2. *Suppose that Assumption B, where f and D are replaced by $f_\star$ and $D_\star$, respectively, is satisfied and the functions h^i $(i = 1,\dots,m)$ satisfy Assumption D. Then in the class of all functions u belonging to*

$LPC_\sigma(D)$, bounded in D and such that the functions $f^i_\star$ $(i = 1, \ldots, m)$ are parabolic with respect to u in D, there exists at most one function u satisfying the mixed impulsive parabolic problem (4.18)–(4.21).

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