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THE DIRICHLET PROBLEM FOR BIHARMONIC EQUATION IN CASE OF THE HALF PLANE

Abstract. Using the representations of the solution of Dirichlet problem for the half plane, the basic biharmonic problem (BP) is solved. Applying the half plane theorem on Almansi type representation of the solution is given by direct or analytical methods. New formulas are proved and for special cases some applications are presented.

1. The Dirichlet problem for the half plane

Let $C = \{z = x + i\,y : x, y \in R\}$ be the complex plane, where the upper half plane $D_1(y > 0)$ and the lower half plane $D_2(y < 0)$ are delimited through the boundary $c = (-\infty, \infty)$; then $C = D_1 \cup D_2 \cup c$.

THEOREM 1.1. *The Dirichlet problem (DP) for D_1 requires to find a harmonic function $U(x,y)$ of $C_2(D_1)$ class, continuous on $\overline{D_1} = D_1 \cup c$ such that $U|_{y=0} = U^*(x)$. The solution is* [1], [5], [9]

$$U(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{U^*(t)}{(x-t)^2 + y^2} dt. \tag{1}$$

The problem has solutions in the case when $U^*(t)$ has a finite number of discontinuities or singularities, too. To assure the convergence of the integral, $U^*(t)$ will satisfy Hölder type conditions or regularity conditions at infinity, and for this purpose it is sufficient to exist $\alpha > 0$ with the condition [1], [9]

$$\lim_{x \to \infty} |x^\alpha U^*(x)| = M < \infty.$$

Let $V(x,y)$ be the harmonic conjugated function of $U(x,y)$ in C. Then the Cauchy-Riemann conditions

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} \tag{2}$$

are satisfied [1], [9].

In this case, we consider the function

$$f(z) = U(x,y) + iV(x,y) \tag{3}$$

which may be holomorphic in C, except a finite number of isolated singular points.

THEOREM 1.2. *If on the c axis the real part $\Re f(z) = U^*(x)$ and the imaginary part $\Im f(z) = V^*(x)$ are known in regularity conditions, then one can derive in D_1 the holomorphic function $f(z)$ with the Cisotti formula*

$$f(z) = \frac{1}{i\pi}\int_{-\infty}^{\infty} \frac{U^*(t)}{t-z}dt + i\,K_1 \quad \text{and} \quad f(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{V^*(t)}{t-z}dt + K_2, \tag{4}$$

respectively, where the real constants K_1, K_2 can be determined knowing the value of $f(z)$ in a single point of D_1.

It may be observed that if one determines the real part of $f(z)$ from (4), then (1) is obtained.

We suppose that, if $U^*(x)$ is given as a real rational function, then from $U^*(z) = \frac{p(z)}{q(z)}$ one performs directly the simple fraction decomposition, determining the principal part $P_p\{U^*(z)\} = P_1(z) + P_2(z)$, where $P_1(z)$ and $P_2(z)$ correspond to the poles from D_1 and D_2, respectively. The solution is obtained calculating the Cauchy integral and then applying the residue theorem.

THEOREM 1.3. *The solution of the Dirichlet problem with rational data on boundary c is, for the two domains D_l* [4], [7]

$$f_l(z) = U(x,y) + iV(x,y) = 2P_k(z) + iK, \qquad l,k = 1,2, \quad l \neq k. \tag{5}$$

There are situations when the solution of the Dirichlet problem is required, so that $f(z)$ must have a finite number of isolated singularities in C: poles, logarithmic singularities, essential singularities or poles at infinity [9], [11]. Such a case is furnished in hydrodynamics, electromagnetism or thermodinamics by the potential plane fields $\mathbf{E} = E_x(x,y)\mathbf{i} + E_y(x,y)\mathbf{j}$ with $div\mathbf{E} = 0$ and $rot\mathbf{E} = 0$ [2], [5], [9]; in this case $\mathbf{E} = gradU(x,y)(\mathbf{E} = gradV(x,y))$ and one derives the potential field $f(x,y) = U(x,y) + iV(x,y)$ with $E_x = \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, E_y = \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$, where $f(z)$ is holomorphic in C, except the specific singularities. The functions $U(x,y)$ =const., $V(x,y) =$ const. may present field lines or equipotential lines, with $f'(z) = E_x - iE_y$. These potential fields $f(z)$ can be generated by potentials of the form

$$f_0(z) = \sum_{k=1}^{n}\sum_{j=1}^{m} \frac{A_{kj}}{(z-a_j)^k} + \sum_{l=1}^{p} B_l \log(z - b_l) + \sum_{h=1}^{q} C_h z^h + g(z),$$

where $A_{kj}, B_l, C_h \in C$ and $g(z)$ has essential singularities. Similarly to the Milne-Thomson theorem for the circle [13], generalized by C. Iacob [5], [6], using the analytic extension and the Schwarz principle, one obtains the solution of the Dirichlet problem with given singularities in the half-plane [4], [6].

THEOREM 1.4. *(Half-plane theorem) Supposing that $f_0(z)$ has given singularities in D_1 one requires a holomorphic function $f(z)$ in D_1, having the same singularities as f_0 and moreover, $\Re f(z)|_c = 0, (\Im f(z)|_c = 0)$. As it is known* [2], [3], [4], [6] *one obtains immediately*

$$(6) \qquad f(z) = f_0(z) - \overline{f_0(\overline{z})} + iK_1, \qquad (f(z) = f_0(z) + \overline{f_0(\overline{z})} + K_2).$$

The proof is performed by verification. For example, in the second relation in (6), $f_0(z)$ has singularities in the principal part in D_1 and $\overline{f_0(\overline{z})}$ is the analytical extension with singularities in the principal part in D_2. Since $f(z) - f_0(z) - \overline{f_0(\overline{z})}$ is analytical and bounded in the whole plane C, therefore, according to the Liouville theorem, it reduces to a constant. One observes that $\Im f(z)|_c = \Im\{f_0(x) + \overline{f_0(x)}\}|_c = 0$ and hence the constant K_2 is real. The first relation in (6) may be verified similarly.

Combining Theorems 1.1 and 1.2, we get the following important result.

THEOREM 1.5. *The function $f(z)$ holomorphic in D_1, such that $\Re f(z)|_c = U^*(x)$, having the same singularities as $f_0(z)$ in D_1 and being the solution of DP problem with the specific singularities and real values given on the boundary c is of the form* [11]

$$(7) \qquad \begin{aligned} f(z) &= U(x,y) + iV(x,y) \\ &= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{U^*(t)}{t-z}dt + f_0(z) - \overline{f_0(\overline{z})} + iK, \quad K \in R. \end{aligned}$$

It is obviuos that, if $U^*(x)$ is a rational function, then we obtain according to (5), the solution of DP with given singularities in D_1 and rational data on the boundary c,

$$(8) \qquad f(z) = 2P_2(z) + f_0(z) - \overline{f_0(\overline{z})} + iK, \quad z \in D_1,$$

where $P_2(z) = P_p\{U(z)\}$ corresponding to the poles from D_2 [11].

Application 1. Let be the dipole $f_0 = \frac{ip}{z-ia}$ located in the point ia, $a > 0$, $p > 0$ and $\Re f(z)|_c = U^*(x) = \frac{1}{1+x^2}$. Then $U^*(z) = \frac{1}{1+z^2} = \frac{1}{2}(\frac{1}{z-i} - \frac{1}{z+i})$ with $P_2(z) = \frac{i}{2(z+i)}$ and, according to (8) the induced potential by $f_0(z)$ having the distribution U^* on c is $f(z) = \frac{ip}{z-ia} + \frac{ip}{z+ia} + \frac{i}{z+i}$.

THEOREM 1.6. [7] *Let $G(z)$ be holomorphic in D_1 and D_2, continuous and uniform; on c the function $G(x)$ may have at most singularities of first kind in $E = \{x_1, x_2, \dots, x_n\}$ and the rational function $f_0(z)$ has polar singularities in D_1. Holomorphic function $F_j(z)$ in D_j such that*

$$\Re F_j(z)|_c = \Re f_0(z) \cdot \Re G(z)|_c - E, \quad j = 1, 2, \tag{9}$$

is of the form

$$F_j(z) = f_0(z)G(z) - [P_j(z) - \overline{P_j(\overline{z})}] + iK, \quad j = 1, 2, \quad K \in R, \tag{10}$$

where $P_j(z)$ is the principal part of the function $f_0(z)G(z)$ in $D_j, j = 1, 2$.

Proof. According to Theorem 1.5, the parenthesis from (10) contains the principal parts for the singularities of $f_0(z)G(z)$ in D_j. Substracting the parenthesis from $f_0(z)G(z)$, the singularities are eliminated, getting $F_j(z)$ holomorphic in $D_1 \cup D_2$. By (9), we have $\Re\{F_j(z)\}|_c = \Re\{f_0(z)G(z) - [P_j(z) - \overline{P_j(\overline{z})}]\}|_c$ and hence $F_j(z)$, as solution of DP is (10). It may be observed that, if $G(z) \equiv 1$, then $\Re F_j(z)|_c = \Re f_0(z)|_c = \Re\{P_j(z) + \overline{P_j(\overline{z})} - [P_j(z) - \overline{P_j(\overline{z})}]\} = 2\Re \overline{P_j(\overline{z})}|_c = 2\Re P_k(z)$, $j \neq k$, and Theorem 1.3 is obtained particularly with (5), where $j = 1$, and $P_k(z) = P_2(z)$ is $P_p\{f_0(z)\}$ in D_2.

In some applications, for the specified domains, there are frequent cases when the conditions on OX axis are given with discontinuities on intervals. We give an application of this theorem.

Let c_1 be the interval (a, b) located on $0X$ axis, $a < b$. The complementary of c_1 on the axis is $\overline{c}_2 = c - c_1$.

The Dirichlet problem with rational piecewise data requires to determine in D_1 the holomorphic function $F(z) = U + iV$, knowing its real part on the boundary

$$U|_{c_1} = g(x), \qquad U|_{c_2} = 0 \tag{11}$$

where $g(x)$ is rational function without singularities on c.

THEOREM 1.7. [7]. *Let $g = g(z)$ and let*

$$G(z) = \frac{1}{\pi i} \ln \frac{z - b}{z - a}, \quad z \in D_1 \tag{12}$$

be a uniform function in D_1 and for $z = x > b$ the determination of the logarithm be real. If $P(z)$ is the principal part of the function $g(z)G(z)$ corresponding to the poles from D_1, then the solution of DP with the boundary conditions (11) is

$$F(z) = g(z)G(z) - [P(z) - \overline{P(\overline{z})}] + iK, \quad z \in D_1, \quad K \in R. \tag{13}$$

Proof. We introduce the rational function $g(z)$ with poles symmetric with respect to the x-axis, real for $y = 0$ and such that $g(\overline{z}) = \overline{g(z)}$. We notice that $\Re G(z)|_{c_1} = 1, \Re G(z)|_{c_2} = 0$ and $\Re F(z)|_c = \Re\{g(z)G(z)\}|_c = \Re\{g(z)G(z) - [P(z) - \overline{P(\overline{z})}]\}|_c$.

Using Theorem 1.6 or directly, as solution of DP, the formula (13) is derived, and then $U(x,y)$ is found.

As a general rule, the principal part of a function $F(z)$ is obtained expanding it in a Laurent series in the neighbourhood of the singular isolated points from D_1, i.e., $P(z)$. But there appear isolated poles x_k on c_2, the point at infinity included. For these points denote by $Q(z)$ the principal part of $F(z)$ and write

$$(14)\quad P_p\{F(z)\} = P(z) + Q(z) = \\ = \sum_{j=1}^{p}\sum_{n=1}^{\infty} \frac{a_n^{(j)}}{(z-z_j)^n} + \sum_{k=1}^{p}\sum_{m=1}^{\infty} \frac{C_m^{(k)}}{(z-x_k)^m} + \sum_{h=1}^{\infty} b_h z^h.$$

It is obvious that, if $g(z)$ has another poles x_k in c_2 or at infinity, then applying the residue and semiresidue theorems directly in (14) substracting the principal part $Q(z)$ from $g(z)G(z)$, the formula (13) becomes

$$(15)\qquad F(z) = g(z)G(z) - [P(z) - \overline{P(\overline{z})} + Q(z)] + iK, \quad K \in R.$$

Application 2. [7]. Let us consider $g(x) = x^n, n \geq 1$; in this case $\Re F(z) = x^n$, for $x \in (a,b), \Re F(z) = 0$ on c_2. The function

$$g(z)G(z) = \frac{z^n}{\pi i} \ln \frac{z-b}{z-a} = \frac{z^n}{\pi i}\left[\ln\left(1 - \frac{b}{z}\right) - \ln\left(1 - \frac{a}{z}\right)\right]$$

has a single pole at infinity. Hence, expanding the logarithms, we get the principal part $Q(z)$ in the form

$$Q(z) = \frac{i}{\pi}\left\{(b-a)z^{n-1} + \frac{1}{2}(b^2 - a^2)z^{n-2} + \ldots + \frac{1}{n-1}(b^{n-1} - a^{n-1})z\right\}$$

and $P(z) \equiv 0$. The solution (15) becomes

$$(16)\qquad F(z) = \frac{z^n}{\pi i} \ln \frac{z-b}{z-a} - \frac{i}{\pi}\sum_{p=1}^{n-1} \frac{1}{p}(b^p - a^p) z^{n-p}.$$

There are situations when DP must be solved to the half plane with boundary data as piecewise irrational functions [4]. Thus, one has to find in D_1 the harmonic function $U(x,y)$ with irrational data on boundary

$$(17)\qquad \begin{cases} U(x,+0) = (x-a)^{\mu-1}(b-x)^{-\mu}R(x), & x \in (a,b), \\ U(x,0) = 0 \quad \text{on} \quad c_2, \end{cases}$$

where $R(x)$ is a rational function without singularities on c_1 and $\mu \in (0,1)$. Here the complex plane C has a cut on $[AB]$. Then, using (4), we have

$$F(z) = \frac{1}{\pi i} \int_a^b \frac{R(t)}{(x-a)^{1-\mu}(b-x)^{\mu}} \frac{1}{t-z} dt + iK. \tag{18}$$

According to [4], this integral is evaluated by using the residue theorem and the methodology of Theorem 1.6. So, for $\zeta \in D_1$ one considers the function

$$F(\zeta) = \left(\frac{\zeta - a}{\zeta - b}\right)^{\mu} \frac{R(\zeta)}{(\zeta - a)} \tag{19}$$

and the contour Γ formed by the semicircle $C_R^+(0)$ of $|z| = R$ and the semicircles $C_\epsilon^+(a)$ and $C_\epsilon^+(b)$ of $|z-a| = \epsilon$ and $|z-b| = \epsilon$ isolating the critical points a and b, respectively. Computing

$$\begin{aligned} \oint_\Gamma F(\zeta)d\zeta = \int_{C_R^+} F(\zeta)d\zeta + \int_{C_\epsilon(a)} F(\zeta)d\zeta - \\ -(e^{i\mu\pi} - e^{-i\mu\pi}) \int_a^b \left(\frac{t-a}{b-t}\right)^{\mu} \frac{R(t)}{(t-a)(t-z)} dt + \int_{C_\epsilon(b)} F(\zeta)d\zeta \\ = 2\pi i \mathrm{Res}\{F(\zeta), \zeta = z\} + 2\pi i \sum_{k=1}^{n} \mathrm{Res}\{F(\zeta), \zeta = z_k\}, \end{aligned}$$

where z_k are the singularities of $F(z)$ at a finite distance in D_1. The conditions of the Jordan theorem are satisfied, hence for $R \to \infty, \epsilon \to 0$, the respective integrals tends to zero. Distinguishing $P(z) = P_p\{F(z)\}$ for D_1 and $Q(z) = P_p\{F(z)\}$ for the poles from c_2 the point at infinity included, one obtains

$$\begin{aligned} F(z) &= U + iV \\ &= \frac{1}{\sin \mu\pi}\{i\left(\frac{z-a}{z-b}\right)^{\mu} \frac{R(z)}{z-a} - [P(z) - \overline{P(\overline{z})} + Q(z)] + iK\}. \end{aligned} \tag{20}$$

THEOREM 1.8. [4]. *The solution of DP with irrational data* (17) *on boundary is* $U(x,y) = \Re F(z), z \in D_1$, *where* $F(z)$ *is given by (20).*

The above theorems can be used for the solution of the Dirichlet problems with these special boundary data, for single connected domains which can be conformaly mapped onto the half plane through rational functions.

The problems have been solved also for the inner or outer domains of the circle in many studies, but with outstanding contributions of the romanian school, leaded by C. Iacob [3], [4], [11]. Recently, C. Marinescu [12] performed a method for construction of some holomorphic functions $f(z)$

with singularities in the inner or outer circle, when on the same arc of circle there are given simultaneously $U(x,y)$ and $V(x,y)$.

2. The fundamental biharmonic problem for half-plane

The fundamental biharmonic problem (FBP) requires to find a function $U = U(x,y), U \in C'(\overline{D}_1) \cup C^4(D_1)$ biharmonic in D_1 [2], [4], [6], i.e.,

$$\triangle^2 U = \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0, \tag{21}$$

and satisfying the boundary conditions

$$U|_c = R_1(x), \quad \left.\frac{\partial U}{\partial \eta}\right|_c = -\left.\frac{\partial U}{\partial y}\right|_c = R_2(x), \tag{22}$$

where R_1, R_2 are continuous or Lebesgue integrable given functions. In this case Hölder type conditions are satisfied. The derivate direction is of the outer normal.

THEOREM 2.1. *Let R_i satisfy the smoothness conditions specified above. Then the solution of the FBP for the half plane of an Almansi type is of the form*

$$U(x,y) = U_1(x,y) + yU_2(x,y) \tag{23}$$

where U_1, U_2 are harmonic functions in D_1 and precisely

$$U(x,y) = \frac{2y^3}{\pi} \int_{-\infty}^{\infty} \frac{R_1(t)}{[(t-x)^2 + y^2]^2} dt - \frac{y^2}{\pi} \int_{-\infty}^{\infty} \frac{R_2(t)}{(t-x)^2 + y^2} dt. \tag{24}$$

Proof. We obtain U_1, U_2 from (23) solving two Dirichlet problems in D_1. For U_1, by the first condition of (22), we get

$$\Delta U_1(x,y) = 0 \quad \text{in } D_1, \quad U|_{y=0} = U_1|_c = R_1(x), \tag{25}$$

and, applying (1) we obtain

$$U_1(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{R_1(t)}{(t-x)^2 + y^2} dt. \tag{26}$$

By the second condition of (22), we get $\frac{\partial U}{\partial \eta}|_c = -\frac{\partial U}{\partial y}|_{y=0} = -[\frac{\partial U_1}{\partial y} + U_2]|_{y=0} = R_2(x)$; The function $\frac{\partial U_1}{\partial y} + U_2$ is harmonic in D_1. Using (1), we obtain

$$\frac{\partial U_1}{\partial y} + U_2(x,y) = -\frac{y}{\pi} \int_{-\infty}^{\infty} \frac{R_2(t)}{(t-x)^2 + y^2} dt. \tag{27}$$

The relations (26), (27) give (24). The uniqueness of the solution is garanteed by the uniqueness of the solution of a Dirichlet problem.

In order to give a complex representation of this solution we consider the following analytical functions in D_1

$$(28) \qquad A(z) = U_1(x,y) + iV_1(x,y), \quad B(z) = U_2(x,y) + iV_2(x,y),$$

where $U_i, V_i, i = 1,2$, are harmonic conjugate functions and

$$(29) \qquad U(x,y) = \Re\{A(z) + yB(z)\} = \Re F(z) = U_1(x,y) + yU_2(x,y).$$

For $F(z) = A(z) + yB(z)$ the conditions (22) become

$$(30) \qquad \begin{cases} U|_c = \Re F(z)|_c = U_1(x,y)|_c = R_1(x), \\ \dfrac{\partial U}{\partial \eta}|_c = -\dfrac{\partial}{\partial y}\{\Re F(z)\}|_c = -\Re[B(z) + iA'(z)]_{y=0} = R_2(x). \end{cases}$$

THEOREM 2.2. *By the smoothness conditions presented above, the function*

$$(31) \qquad \begin{aligned} F(z) &= A(z) + yB(z) \\ &= \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{R_1(t) - yR_2(t)}{t-z}dt - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yR_1(t)}{(t-z)^2}dt + K_1 + iK_2 \end{aligned}$$

is the solution of FBP with conditions (30).

Proof. Using (4), we shall follow the reasoning from the proof of Theorem 1.9. By virtue of the first condition (30), we have $U|_c = \Re F(z)|_{y=0} = \Re A(z)|_{y=0} = R_1(x)$ and the analytic function $A(z)$ is

$$(32) \qquad A(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{R_1(t)}{t-z}dt.$$

By the second condition (30), using (4), we get the analytic function

$$(33) \qquad B(z) + iA'(z) = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{R_2(t)}{t-z}dt.$$

The relations (32), (33) lead to (31), where the constants K_1, K_2 may be determined, knowing a priori $F(z)$ in one point of the domain D_1.

Hence we have obtained the solution (31) of FBP similar to the Cisotti formula (4). Also we notice that writing $\Re F(z)$ in (31) under the integral sign, we get the solution (24).

Let us consider now that $R_1(x), R_2(x)$ are rational functions satisfying the conditions of Theorem 1.3. We shall denote by $P_p\{R_j(z)\} = P_j^*(z)$, $j = 1,2$, the principal parts corresponding to the poles from D_2. After a short calculation, using the residue theorem in (31), we obtain the solution of (FBP) in D_1 with rational functions on boundaries

$$(34) \qquad F(z) = A(z) + yB(z) = 2P_1^*(z) - y[2P_2^*(z) + 2iP_1'^*(z)] + K_1 + iK_2.$$

We are going to generalize the result (34), according to Theorem 1.6. We shall consider again the FBP with the special conditions

$$(35) \qquad U|_{y=0} = R_1(x)\Re H_1(z)|_c, \quad \frac{\partial U}{\partial \eta}|_c = R_2(x)\Re H_2(z)|_c,$$

where R_1, R_2 are rational functions and H_1, H_2 are holomorphic functions in the domain D_1. On the c axis they can have a finite number of isolated singular points of the first kind.

THEOREM 2.3. *The solution of FBP with the conditions* (35) *in the domain* D_1 *is*

$$(36) \qquad \begin{cases} A(z) = R_1(z)H_1(z) - [L_1(z) - \overline{L_1(\overline{z})}] + iK_1, \\ B(z) = -iA'(z) - \{R_2(z)H_2(z) - [L_2(z) - \overline{L_2(\overline{z})}]\} + iK_2. \end{cases}$$

We have noted the principal parts $P_p\{R_j(t)H_j(z)\} = L_j(z)$ *in* D_1, $j = 1, 2$.

P r o o f. By the first condition (35), we have

$$\Re F(z)|_c = \Re A(z)|_c = \Re\{R_1(z)H_1(z)\}_c.$$

The hypotheses of Theorem 1.6 are satisfied by the holomorphic function $A(z)$ and, using (10), we find (36). In the same way, using the second condition (35), we have $\frac{\partial U}{\partial \eta}|_c = -\Re[iA'(z) + B(z)]|_c = \Re\{R_2(z)H_2(z)\}|_c$; and, by (10) for the analytic function $-[iA'(z)+B(z)]$, we obtain the solution in D_1. If the functions $R_j(z)H_j(z)$ have poles at infinity, then the formulae (36) must be modified in the sense of (15) by adding $[L_j(z) - \overline{L_j(\overline{z})} + Q_j(z)], j = 1, 2$, where $Q_j(z)$ are the principal part for $R_j(z)H_j(z)$ corresponding to the poles at infinity. We note that, if $H_j(z) \equiv 1$, then $L_j(z) \equiv P_j(z)$ and from (36) we obtain the solution (34).

As an application of this theorem we shall give the solution of the FBP with piecewice rational boundary value and also with piecewice irrational boundary value.

Let $H_j = R_j(z)G(z), j = 1, 2$, where $G(z) = \frac{1}{\pi i} \ln \frac{z-b}{z-a}$ as in (12), and let $R_j(z)$ be rational functions with poles in the domain D_1 or in the exterior of $[a, b]$ or at infinity, just like in the hypothesis of Theorem 1.7. In this case the biharmonic function $U(x, y)$ in D_1 is to be determined with the piecewice rational boundary conditions

$$(37) \qquad \begin{cases} U|_c = \Re H_1(z)|_c = \begin{cases} R_1(x), & x \in (a, b) \\ 0, & x \in c_2, \end{cases} \\ \frac{\partial U}{\partial \eta}|_c = \Re H_2(z)|_c = \begin{cases} R_2(x), & x \in (a, b) \\ 0, & x \in c_2. \end{cases} \end{cases}$$

THEOREM 2.4. *The solution of the FBP in the domain D_1 is $U(x,y) = \Re F(z) = \Re\{A(z) + yB(z)\}$, where*

$$(38)\quad \begin{cases} A(z) = H_1(z) - [P_1(z) - \overline{P_1(\overline{z})} + Q_1(z)] + iK_1, \\ B(z) = -iA'(z) - \{H_2(z) - [P_2(z) - \overline{P_2(\overline{z})} + Q_2(z)]\} + iK_2, \end{cases}$$

$P_1(z)$ and $P_2(z)$ are the principal parts of $H_1(z)$ and $H_2(z)$ in D_1, respectively; and $Q_j(z) = P_p\{H_y(z)\}, j = 1,2$ for the poles on c or at infinity.

Proof. Thanks to the solution (13) given by Theorem 1.7, the proof is similar to the proofs of Theorems 2.1 and 2.2.

Application 3. Let us solve the FBP in the domain D_1 with the following boundary conditions

$$U|_c = \frac{1}{1+x^2}, \quad \frac{\partial U}{\partial \eta}|_c = -\frac{\partial U}{\partial y}|_c = U_2(x) = \begin{cases} x^n, & x \in (-1,1) \\ 0, & x \in c_2. \end{cases}$$

We note that these conditions satisfy the hypotheses of Theorems 2.2 and 2.3. The function in the first condition is rational on c and in order to apply the formulae (36) with $H_1(x) \equiv 1$, we must take in (34) with $A(z) = 2P_1^*(z)$, where $P_1^*(z)$ is the principal part of the function $R_1(z) = \frac{1}{1+z^2}$ for the pole $z = i$ in D_2. We obtain $A(z) = \frac{i}{z+i}$. For the second condition we note that the hypotheses of Theorem 2.3 are satisfied with $H_2(z) = \frac{z^n}{\pi i} \ln \frac{z-1}{z+1}$. Hence, in the formulae (38) $P_2(z) \equiv 0$ and $Q_2(z)$ is the principal part for the pole at infinity calculated in Application 2 and (35), so

$$B(z) = -iA'(z) - \frac{z^n}{\pi i} \ln \frac{z-1}{z+1} + \frac{i}{\pi} \sum_{p=1}^{n-1} \frac{2}{\pi} z^{n-p}.$$

Let us consider the FBP with irrational piecewise boundary value. Let $H_j(z) = R_j(z)G(z)$, $j = 1,2$, where $G(z) = (z-a)^{\mu-1}(b-z)^{-\mu}$, $\mu \in (0,1)$ and R_1, R_2 are rational functions with poles in D_1 or on c_2 the pole at infinity included, as it was mentioned in Theorem 1.8. It is required to determine the function $U(x,y)$ biharmonic in the domain D_1 and satisfying the conditions

$$(39)\quad \begin{cases} U|_c = \Re H_1(z)|_c = \begin{cases} R_1(x)G(x), & x \in c_1, y = +0, \\ 0, & x \in c_2, \end{cases} \\ \frac{\partial U}{\partial \eta}|_c = \Re H_2(z)|_c = \begin{cases} R_2(x)G(x), & x \in c_1, y = +0, \\ 0, & x \in c_2, \end{cases} \end{cases}$$

where $G(x) = (x-a)^{\mu-1}(b-x)^{-\mu}$ is the value of $G(z)$ for $x \in (a,b)$, $y = +0$.

THEOREM 2.5. *The solution of FBP with condition* (39) *for the domain* D_1 *is* $U(x,y) = \Re F(z) = \Re\{A(z) + yB(z)\}$, *where*

$$(40)\quad \begin{cases} A(z) = \frac{1}{\sin\mu\pi}\{i(\frac{z-a}{z-b})^{\mu}\frac{R_1(z)}{z-a} - [P_1(z) - \overline{P_1(\overline{z})} + Q_1(z)]\} + iK_1, \\ B(z) = -iA'(z) - \frac{1}{\sin\mu\pi}\{i(\frac{z-a}{z-b})^{\mu} - [P_2(z) - \overline{P_2(\overline{z})} + Q_2(z)]\} + iK_2, \end{cases}$$

where $P_j(z), Q_j(z), j = 1,2$, *are the principal parts of*

$$R_j(z)G(z) = i\left(\frac{z-a}{z-b}\right)^{\mu}\frac{R_j(z)}{z-a},$$

respectively, corresponding to the poles in D_1, *at infinity or on* c_2.

P r o o f. Following the proof of the Theorem 2.2, imposing the first condition (39) i.e., $U|_c = \Re F(z)|_c = \Re A(z)|_c = \Re\{H_1(z)\}|_c$, using Theorem 1.8 and (20), we get $A(z)$ from (40). From the second condition (39), $\frac{\partial U}{\partial \eta}|_c = -\frac{\partial U}{\partial y}|_c = -\Re\{B(z) + iA'(z)\}|_c = \Re H_2(z)|_c$ we find for $B(z) + iA'(z)$ the solution (20) and then we find $B(z)$ from (40).

We note that for all theorems proved in Section 2 we had to solve two Dirichlet problems which are generally independent. Therefore we can combine Theorems 2.2, 2.3, regarding the boundary conditions and we can consider that for any logarithm or radical function a cut $[AB]$ was made on axis $X'OX$ in order to compute the complex integrals.

Application 4. Let us solve the FBP with the following conditions

$$(41)\qquad U|_c = \frac{1}{1+x^2}\exp(-x^{4p}),\quad \left.\frac{\partial U}{\partial \eta}\right|_{c_1} = \frac{x^4+a^2}{x^2+1}\frac{1}{\sqrt{1-x^2}},\quad \left.\frac{\partial U}{\partial \eta}\right|_{c_2} = 0,$$

where $p = 1,2,3,\dots$ and $a \in R$. As it is seen, the conditions of Theorems 2.3, 2.4 are fulfilled. Considering the function $\frac{1}{1+z^2}\exp(-z^{4p})$ with the pole $z = i$ in D_1 and using (36), from the first condition (41) we get

$$A(z) = \frac{1}{1+z^2}\exp(-z^{4p}) + \frac{iz}{e(z^2+1)}.$$

In order to satisfy the second condition (41) we consider the function $i(\frac{z-1}{z+1})^{\frac{1}{2}}\frac{z^4+a^2}{z^2+1}\frac{1}{z-1}$ having a pole $z = i$ in D_1 and a pole of first order at infinity. Here $\mu = \frac{1}{2}$ and c is cut along the segment $[-1,1]$. From $\lim_{z\to\infty}(\frac{z-1}{z+1})^{\frac{1}{2}} = 1$ it results $P_2(z) = -\frac{a^2+1}{2}\frac{i}{z-i}$ and $Q_2(z)|_{z=\infty} = iz$, because

$$\frac{z^4+a^2}{z^2+1} = i[z + \frac{a^2}{z^3}][1 - \frac{1}{z^2} + \frac{1}{z^4} + \dots][1 + \frac{1}{z} + \frac{1}{z^2} + \dots] = i[z + \frac{a^2}{z^3} + \dots].$$

We get for $B(z)$ the following expression

$$B(z) = iA'(z) + i\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\frac{z^4+a^2}{z^2+1}\frac{1}{z-1} + \frac{(a^2+1)iz}{z^2+1} - iz.$$

The above theorems may be used to solve FBP for simply-connected domains conformal mappable by means of rational functions to the half plane [1], [6]. The FBP for the circle was solved by Caius Iacob [6] in the sense state by Theorem 2.2.

Remark. It is well known that a FBP is to be solved when the permanent plane flows of viscous fluid [2], [5], [8] or the plane problem of elasticity [6], [14] is studied. For the viscous fluid flows the function $U(x,y)$ is the stream function for the velocity field $w = u + iv, u = \frac{\partial \Psi}{\partial x}, v = -\frac{\partial \Psi}{\partial y}$ with $\triangle^2 \Psi = 0$. Imposing boundary conditions on the solid frontiers, we must preserve bounded or null velocity at infinity. In the case of the plane elasticity problem the function $U(x,y)$ is the Airy function [2], [10] with $\triangle^2 U = 0$ and the stress tensor components are $T_{11} = \frac{\partial^2 U}{\partial y^2}, T_{12} = -\frac{\partial^2 U}{\partial x \partial y}, T_{22} = \frac{\partial^2 U}{\partial x^2}$. The boundary conditions must preserve null stress at infinity. For an Almansi type (23) representation of the solution we can derive Kolosov-Mushelishvili type formulae [2], [4].

For these problems it is known the Goursat complex representation of biharmonic equation $\frac{\partial^4 U}{\partial z^2 \partial \overline{z}^2} = 0$ with the solution $U(x,y) = \Im\{\overline{z}\varphi(z) + \chi(z)\}$, where φ, χ are analytic functions in D_1 [2], [4], [11]. For elasticity theory there are known the results of Mushelishvili [2], [4]. For viscous fluids the theory of Mushelishvili has been adapted by D. G. Ionescu [8], who has extended it on FBP for the circle and half-plane with given singularities for $\varphi(z)$ and $\chi(z)$.

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