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OSCILLATORY BEHAVIOUR OF THE SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED DELAY

Abstract. In this paper, the oscillatory behaviour of second-order nonlinear neutral differential equations with distributed delay was examined, some exciting results were obtained which developed some traditional results in history.

1. Introduction

Consider the nonlinear second-order neutral differential equation

$$(1) \qquad [x(t) + \int_a^b x(t-\theta)\,d\mu(t,\theta)]'' + \int_c^d f(x(t-s)\,d\eta(t,s) = 0, \qquad t \geq t_0,$$

where a, b, c, d are nonnegative constants, the functions $\mu(t,\theta)$ and $\eta(t,s)$ are continuous in $t \in [t_0, \infty)$, of bounded variation in $\theta \in [a,b]$ and $s \in [c,d]$, respectively.

Let us give the following two basic hypotheses:

$$(H_1) \qquad 0 \leq V_{\theta=a}^{b}\mu(t,\theta) \leq 1, \quad V_{s=c}^{d}\eta(t,s) \geq 0,$$

$$(H_2) \qquad f(x) \geq k > 0,$$

for some positive constant k and for $x \neq 0$.

The neutral differential equations have many applications in various fields such as dealing with vibrating masses attached to an elastic bar and in some variational problems (see [1]).

In [2], [3] there was discussed the oscillatory behaviour of the following delay differential equation

$$(E_1) \qquad x''(t) + q(t)x(t-\tau) = 0,$$

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in [4] the results of [2], [3] were extended to

$$(E_2) \qquad [x(t) + p(t)x(t-\tau)]'' + q(t)x(t-\sigma) = 0,$$

and in [5] to the second-order neutral delay differential equation

$$(E_3) \qquad [x(t) + p(t)x(t-\tau)]'' + q(t)f(x(t-\sigma)) = 0,$$

including (E_1), (E_2) and the ordinary differential equation

$$(E_4) \qquad x''(t) + q(t)x(t) = 0.$$

Obviously (E_1)-(E_4) are special cases of (1).

In this paper, we will discuss the more generalized form of Eq.(1) by using Riccati technique due to Kamenev [6] and Yu [7], and the method of the integral mean value due to Wintner [8], Hartman [9], Coles [10] and Willett [11].

2. Main results

Let

$$z(t) = x(t) + \int_a^b x(t-\theta)\, d\mu(t,\theta),$$

$c(t) \in C^1[t_0, \infty)$ be a given function, $a(t) = exp\{-2\int^t c(s)\, ds\}$ and

$$\psi(t) = a(t)\{k[V_{s=c}^d \eta(t,s) - \int_c^d V_{\theta=a}^b \mu(t-s,\theta) d\eta(t,s)] + c^2(t) - c'(t)\}.$$

At first we give the following five lemmas.

LEMMA 1. *Let $x(t)$ be an eventually positive (negative) solution of (1). Then $z(t)$ and $z'(t)$ are nonnegative.*

Proof. In view of (H_1), it is clear that $z(t) \geq x(t) > 0$, $t \geq T_0$, for some $T_0 \geq t_0$. In the following we will see that $z'(t)$ is nonnegative. Actually, according to (1), we have $z''(t) \leq 0$ for $t \geq t_0$, this indictates that $z'(t)$ is decreasing. Hence, if there is $T_1 \geq T_0$ such that $z'(T_1) < 0$, then

$$z(t) \leq z(T_1) + (t - T_1)z'(T_1) \to -\infty, \quad t \to \infty,$$

which contradicts to $z(t) > 0$ for $t \geq T_0$. So the proof is completed.

REMARK 1. From the proof of Lemma 1 we can obtain that, if $V_{s=c}^d \eta(t,s) > 0$ for $t \geq t_0$ and $x(t)$ is a nonoscillatory solution of (1), then $z(t)z'(t)$ is eventually positive.

LEMMA 2. *If* (1) *is nonoscillatory, then there exists a number* $T_1 \ge t_0$ *and a function* $v(t) \in C^1[T_1, \infty)$ *such that*

$$v'(t) + \psi(t) + \frac{v^2(t)}{a(t)} \le 0, \quad t \ge T_1. \tag{2}$$

Proof. Without loss of generality, assume that $x(t) > 0$ for $t \ge T_1$ and for some $T \ge t_0$. By Lemma 1 and (H_2) as well as (1), we can get

$$z''(t) + \int_c^d f(x(t-s))\, d\eta(t,s) = 0,$$

so

$$z''(t) + k\int_c^d x(t-s)\, d\eta(t,s) \le 0$$

and

$$z''(t) + k\int_c^d [z(t-s) - \int_a^b x(t-s-\theta)\, d\mu(t-s,\theta)]\, d\eta(t,s) \le 0,$$

hence

$$z''(t) + k\Big(\int_c^d [1 - V_{\theta=a}^b \mu(t-s,\theta)]\, d\eta(t,s)\Big) z(t-d) \le 0.$$

Now define

$$v(t) = a(t)\left[\frac{z'(t)}{z(t-\beta)} + c(t)\right], \quad t \ge T_1 \ge T,$$

for some positive number β, then

$$\begin{aligned} v'(t) \le\ & -2c(t)v(t) + a(t)\Big\{ -k\Big(\int_c^d [1 - V_{\theta=a}^b \mu(t-s,\theta)]\, d\eta(t,s)\Big) \\ & - \frac{z'(t)z'(t-\beta)}{z^2(t-\beta)} + c'(t)\Big\} \\ \le\ & -2c(t)v(t) + a(t)\Big\{ -k\Big(\int_c^d [1 - V_{\theta=a}^b \mu(t-s,\theta)]\, d\eta(t,s)\Big) \\ & - \Big(\frac{z'(t)}{z(t-\beta)}\Big)^2 + c'(t)\Big\} \\ \le\ & -2c(t)v(t) + a(t)\Big\{ -k\Big(\int_c^d [1 - V_{\theta=a}^b \mu(t-s,\theta)]\, d\eta(t,s)\Big) \end{aligned}$$

$$- \left(\frac{v(t)}{a(t)} - c(t)\right)^2 + c'(t)\Big\}$$
$$= -\psi(t) - \frac{v^2(t)}{a(t)}$$

and the proof is completed.

In what follows, let us use the conception of the integral mean value to introduce other lemmas. Let S be the set of all nonnegative locally integrable on $[t_0, \infty)$ functions g satisfying the condition

$$\overline{\lim_{t\to\infty}} \left(\int^t g(s)\,ds\right)^{1-\alpha} [G_\alpha(\infty) - G_\alpha(t)] > 0 \tag{3}$$

for some $\alpha \in [0,1)$, where

$$G_\alpha(t) = \int^t g(s) \left[\frac{(\int^s g(\delta)d\delta)^\alpha}{\int^s a(\delta) g^2(\delta)\,d\delta}\right] ds.$$

If $G_\alpha(\infty) = \infty$ in (3), we consider that $g \in S$. And let S_0 be the set of all nonnegative locally integrable on $[t_0, \infty)$ functions g satisfying

$$\lim_{t\to\infty} \frac{\int^t a(s) g^2(s)\,ds}{(\int^t g(s)\,ds)^2} = 0. \tag{4}$$

In order that either (3) or (4) holds for a nonnegative locally integrable function g, it is necessary that g is not integrable on $[t_0, \infty)$, i.e.

$$\int^\infty g(s)\,ds = \infty. \tag{5}$$

Every element in S or S_0 is called a weight function. And it is clear that $S_0 \subset S$.

Let $g \in S$ and define

$$A_g(s,t) = \frac{\int_s^t g(\delta) \int_s^\delta \psi(u)\,du\,d\delta}{\int_s^t g(u)\,du}.$$

LEMMA 3. *Let $v(t)$ be a solution of inequality (2). If there exists $g \in S$ satisfying*

$$\underline{\lim}_{t\to\infty} A_g(\cdot, t) > -\infty, \tag{6}$$

then

$$\int^\infty \frac{v^2(s)}{a(s)}\,ds < \infty.$$

LEMMA 4. *If inequality* (2) *has a solution* $v(t)$ *satisfying*

$$\int^{\infty} \frac{v^2(s)}{a(s)}\,ds < \infty,$$

then $\overline{\lim}_{t\to\infty} A_g(.,t) < \infty$ *for any* $g \in S_0$ *and*

$$\overline{\lim_{t\to\infty}} A_g(u,t) \le v(u) - \int_u^{\infty} \frac{v^2(s)}{a(s)}\,ds.$$

In addition, if

$$\int_T^{\infty} \psi(s)\,ds < \infty, \tag{7}$$

then

$$v(u) \ge \int_u^{\infty} \psi(s)\,ds + \int_u^{\infty} \frac{v^2(s)}{a(s)}\,ds. \tag{8}$$

The proofs of Lemmas 3 and 4 are similar to those of Lemmas 2.3 and 2.4 in [5].

LEMMA 5 ([11]). *Assume that* $B(t)$ *and* $Q(s,t)$ *are nonnegative continuous functions for* $T \le s, t < \infty$. *If there exists* $\varepsilon > 0$ *such that*

$$\int_t^{\infty} Q(s,t)B^2(s)\,ds \ge \frac{1}{4}(1+\varepsilon)B(t) \not\equiv 0, \quad t \ge T, \tag{9}$$

then the inequality

$$v(t) \ge B(t) + \int_t^{\infty} Q(s,t)v^2(s)\,ds, \quad t \ge T, \tag{10}$$

does not have a continuous solution $v(t)$.

Now we will give out our main results.

THEOREM 1. *If there exists* $g \in S$ *such that* (6) *holds, then either* (1) *is oscillatory or* $\overline{\lim}_{t\to\infty} A_h(\cdot,t) < \infty$ *for each* $h \in S_0$.

Proof. Suppose that (1) is nonoscillatory. According to Lemma 2, there is a function $v \in C^1$ such that (2) holds on $[T,\infty)$. Then, by Lemma 3, we have

$$\int^{\infty} \frac{v^2(s)}{a(s)}\,ds < \infty.$$

Hence, by Lemma 4, we have $\overline{\lim}_{t\to\infty} A_h(.,t) < \infty$ for each $h \in S_0$, and our proof is completed.

COROLLARY 1. *If*

$$\int^{\infty} \frac{1}{a(s)}\, ds = \int^{\infty} \psi(s)\, ds = \infty,$$

then (1) *is oscillatory.*

COROLLARY 2. *If*

$$\lim_{t\to\infty} \frac{1}{t^2} \int^{t} a(s)\, ds = 0, \tag{11}$$

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \int^{t}\int^{s} \psi(\theta)\, d\theta\, ds = \infty, \tag{12}$$

then (1) *is oscillatory.*

The proofs of Corollaries 1 and 2 are similar to those of Corollaries 3.4 and 3.5 in [5].

THEOREM 2. *Suppose that* $V_{s=c}^{d}\eta(t,s) > 0$ *on* $[t_0,\infty)$ *and that there is a nonnegative locally integrable function* g *on* $[t_0,\infty)$ *satisfying* (4). *If*

$$\int_{T}^{\infty} \psi(s)\, ds < \infty, \tag{13}$$

$$\overline{\lim_{t\to\infty}}\, t\{\frac{1}{a(t)} \int_{t}^{\infty} \psi(s)\, ds - c(t)\} > 1 \tag{14}$$

hold, *then* (1) *is oscillatory.*

Proof. Suppose that $x(t)$ is nonoscillatory solution of (1). Without loss of generality, $x(t) > 0$ for $t \geq T_0 \geq t_0$. Since $V_{s=c}^{d}\eta(t,s) > 0$, it follows from Remark 1 that $z'(t) > 0$ is decreasing on $[T,\infty)$ for some $T \geq T_0 + b + d$. Let

$$v(t) = a(t)\{\frac{z'(t)}{z(t-b-d)} + c(t)\}, \quad t \geq T, \tag{15}$$

then (2) holds. By (13) and Lemma 4, we can get the inequality

$$v(t) \geq \int_{t}^{\infty} \psi(s)\, ds + \int_{t}^{\infty} \frac{v^2(s)}{a(s)}\, ds, \quad t \geq T,$$

indicating that

$$v(t) \geq \int_{t}^{\infty} \psi(s)\, ds, \quad t \geq T.$$

Since $z'(t)$ is positive and decreasing on $[T, \infty)$, so

$$\frac{z(t-b-d)}{tz'(t)} = \frac{z(T) + \int_T^{t-b-d} z'(s)\,ds}{tz'(t)} \geq \frac{z(T) + z'(t-b-d)(t-b-d-T)}{tz'(t)} \geq \frac{z(T) + z'(t)(t-b-d-T)}{tz'(t)} \geq \frac{t-b-d-T}{t}.$$

This and (15) imply

$$\overline{\lim_{t\to\infty}}\, t\{\frac{1}{a(t)} \int_t^\infty \psi(s)\,ds - c(t)\} \leq \overline{\lim_{t\to\infty}}\, t\{\frac{v(t)}{a(t)} - c(t)\} = \overline{\lim_{t\to\infty}} \frac{tz'(t)}{z(t-b-d)} \leq \overline{\lim_{t\to\infty}} \frac{t}{t-b-d-T} = 1,$$

which contradicts (14). This contradiction proves that (1) is oscillatory.

THEOREM 3. *Assume that there is a $T \geq 0$ such that*

$$F(t) = \int_t^\infty \psi(s)\,ds \geq 0, \quad t \geq T. \tag{16}$$

Let

$$\Phi(t) = \int_t^\infty \frac{F^2(s)}{a(s)} E(s,t)\,ds \not\equiv 0, \quad t \geq T,$$

where $E(s,t) = \exp\{2\int_t^s (F(\theta))\,d\theta\}$ for $s \geq t \geq T$. If

$$\Phi(t) = \infty \quad or \quad \int_t^\infty \frac{\Phi^2(s)}{a(s)} E(s,t)\,ds \geq \frac{1}{4}(1+\varepsilon)\Phi(t), \tag{17}$$

for all $t \geq T$ and for some $\varepsilon > 0$, then (1) is oscillatory.

COROLLARY 3. *Let $F(t)$ and $\Phi(t)$ be as in Theorem 3. If*

$$\Phi(t) \geq \frac{1}{2}(1+\varepsilon)F(t), \tag{18}$$

for all $t \geq T$ and for some $\varepsilon > 0$, then (1) is oscillatory.

COROLLARY 4. *Let $F(t)$ and $\Phi(t)$ be as in Theorem 3. If*

$$\int_t^\infty \frac{F^2(s)}{a(s)}\,ds \geq \frac{1}{4}(1+\varepsilon)F(t), \tag{19}$$

for all $t \geq T$ and for some $\varepsilon > 0$, then Eq. (1) is oscillatory.

The proofs of Theorem 3 and Corollaries 3, 4 are similar to those in [5].

3. Examples

EXAMPLE 1. Let us consider the equation

$$(20)\qquad [x(t)+\int_0^{2\pi} x(t-\theta)\,d\mu(t,\theta)]''+\int_0^{\pi} x(t-s)\,d\eta(t,s)=0,\quad t\geq 2\pi,$$

where

$$\mu(t,s)=\begin{cases} A, & \text{for } \theta\in[0,2\pi),\\ \frac{1}{t+\pi}, & \theta=2\pi,\end{cases}\qquad \eta(t,s)=\begin{cases} B, & \text{for } s\in[0,\pi),\\ \frac{\lambda}{t^2}, & \text{for } s=\pi,\lambda>0\end{cases}$$

and A, B, λ are some constants. Now let $c(t)=-\frac{1}{2t}$, then $a(t)=t$ and $\psi(t)=\frac{4\lambda-1}{4t}-\frac{\lambda}{t^2}$. Let $g(t)=\frac{1}{t}$, then

$$\lim_{t\to\infty}\frac{\int^t a(s)g^2(s)\,ds}{(\int^t g(s)\,ds)^2}=0,$$

$$\overline{\lim_{t\to\infty}}\frac{\int_{2\pi}^t g(s)\int_{2\pi}^s \psi(\theta)\,d\theta ds}{\int_{2\pi}^t g(s)\,ds}=+\infty$$

for $\lambda>\frac{1}{4}$. Therefore, by Theorem 1, the equation(20) is oscillatory.

EXAMPLE 2. Consider the equation

$$(21)\qquad [x(t)+\int_0^{2} x(t-\theta)\,d\mu(t,\theta)]''+\int_0^{1} x(t-s)\,d\eta(t,s)=0,\quad t\geq 2,$$

where

$$\mu(t,\theta)=\begin{cases} M, & \text{for } \theta\in[0,2),\\ \frac{1}{(t+1)\log(t+1)}, & \theta=2,\end{cases}$$

$$\eta(t,s)=\begin{cases} N, & \text{for } s\in[0,1),\\ \frac{1}{4t^2}+\frac{\lambda}{t^2\log^2 t}, & \text{for } s=1,\lambda>0\end{cases}$$

and M, N, λ are some constants.

Now set $c(t)=-\frac{1}{2}(\frac{1}{t}+\frac{1}{t^2\log^2 t})$, then $a(t)=t\log t$ and $\psi(t)=\frac{4\lambda-1}{4t\log t}-\frac{1}{4t^2}-\frac{\lambda}{t^2\log^2 t}$. Let $g(t)=\frac{1}{t\log t}$, then

$$\lim_{t\to\infty}\frac{\int_2^t a(s)g^2(s)\,ds}{(\int_2^t g(s)\,ds)^2}=\lim_{t\to\infty}\frac{1}{\log\log t-\log\log 2}=0,$$

$$\overline{\lim_{t\to\infty}}\frac{\int_2^t g(s)\int_2^s \psi(\theta)\,d\theta\,ds}{\int_2^t g(s)\,ds}=\overline{\lim_{t\to\infty}}\int_2^t(\frac{4\lambda-1}{4s\log s}-\frac{1}{4s^2}-\frac{\lambda}{s^2\log^2 s})\,ds=\infty,$$

for $\lambda>\frac{1}{4}$. Therefore, by Theorem 1, the equation (21) is oscillatory if $\lambda>\frac{1}{4}$.

EXAMPLE 3. Consider the delay differential equation

$$x''(t)+\int_0^1 f(x(t-s))\,d\eta(t,s)=0,\quad t\geq 1, \tag{22}$$

where $f(x)=x+x^2 signx$,

$$\eta(t,s)=\begin{cases} G, & \text{for } s\in[0,1),\\ \frac{\lambda}{t^2}, & \text{for } s=1, \lambda>\frac{1}{4}\end{cases}$$

and G, λ are some constants. The equation (22) is a special case of (1) when $a=b=c=0$, $d=1$. We will examine the oscillatory behaviour of (22) in two cases.

Case 1: $\frac{1}{4}<\lambda<1$. Let $c(t)=-\frac{1}{\alpha t}$, $2<\alpha<\frac{1}{1-\sqrt{\lambda}}$. Then $a(t)=t^{\frac{2}{\alpha}}$, $\psi(t)=t^{\frac{2}{\alpha}}(\frac{\lambda}{t^2}+\frac{1}{\alpha^2t^2}-\frac{1}{\alpha t^2})$ and $\frac{\lambda\alpha^2-1}{\alpha(\alpha-2)}>1$. If $g(t)\equiv 1$, then

$$\lim_{t\to\infty}\frac{\int^t a(s)g^2(s)\,ds}{(\int^t g(s)\,ds)^2}=0,$$

$$\overline{\lim_{t\to\infty}}\, t\{\frac{1}{a(t)}\int_t^\infty \psi(s)\,ds-c(t)\}=\frac{\lambda\alpha^2-1}{\alpha(\alpha-2)}>1.$$

Hence, by Theorem 2, the equation (22) is oscillatory if $\frac{1}{4}<\lambda<1$.

Case 2: $\lambda\geq 1$. Let $c(t)=-\frac{1}{3t}$, then $a(t)=t^{\frac{2}{3}}$ and $\psi(t)=t^{\frac{2}{3}}(\frac{\lambda}{t^2}+\frac{1}{9t^2}-\frac{1}{3t^2})$. If $g(t)\equiv 1$, then

$$\lim_{t\to\infty}\frac{\int^t a(s)g^2(s)\,ds}{(\int^t g(s)\,ds)^2}=0,$$

$$\overline{\lim_{t\to\infty}}\, t\{\frac{1}{a(t)}\int_t^\infty \psi(s)\,ds-c(t)\}=\frac{9\lambda-1}{3}\geq\frac{8}{3}>1.$$

Therefore, from Theorem 2, we obtain that (22) is oscillatory for $\lambda\geq 1$. According to the two cases above, the equation (22) is oscillatory for $\lambda>\frac{1}{4}$.

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