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Ali A. Mehemmed

A THEOREM DESCRIBING THE STRUCTURE OF SET MAPPINGS BETWEEN MEASURE ALGEBRAS

Abstract. The subject of this paper is the problem of describing the structure of set mappings h between two measure algebras $(\mathbb{A}_1, \mu_1)$ and $(\mathbb{A}_2, \mu_2)$ effectively. We shall use disjoint partitions of the σ-algebras $\mathbb{A}_1$ and $\mathbb{A}_2$, and we shall assume that the homomorphism $h : \mathbb{A}_1 \to \mathbb{A}_2$ is continuous. We shall construct a pairwise disjoint sequence $\langle A_n \rangle$ in the second σ-algebra $\mathbb{A}_2$ which covers the homomorphic image of the whole space X_1 and for $n \in \mathbb{N}$ we construct a partition $\{E_{1n}, E_{2n}, \dots, E_{k_n n}\}$ of the first space X_1 so that the set mapping

$$h_{in} : \mathbb{A}_1(E_{in}) \to \mathbb{A}_2(A_n \cap h(E_{in})) \text{ is a } \sigma\text{-homomorphism.}$$

Introduction

For all definitions and known results concerning measure theory and Boolean algebra theory, the reader is referred to the books of Halmos [1], Royden [4] and Sikorski [5]. We recall here only the most important terminology, definitions, and notations.

A Boolean algebra $\mathbb{A}$ is called a *Boolean σ-algebra* if, whenever $\langle A_n \rangle$ is a sequence in $\mathbb{A}$, there exists a smallest element B such that $A_n \leq B$, for all n, and $B \leq C$ for any element C satisfying $A_n \leq C$, and we denote B by $\bigvee_{n=1}^{\infty} A_n$. Therefore, every σ-algebra of sets is a Boolean σ-algebra.

A Boolean σ-algebra $\mathbb{A}$ together with a measure μ such that

i) $\mu(A) = 0$ iff $A = 0$, and

ii) $\mu\Big(\bigvee_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$, if $A_n \wedge A_m = 0$ for $n \neq m$,

is called a *measure algebra*.

If $\mathbb{A}$ is a measure algebra, and $A \in \mathbb{A}$, by $\underline{\mathbb{A}(A)}$, we mean the restriction of the measure algebra $\mathbb{A}$ to A, i.e.

$$\mathbb{A}(A) = \{B \in \mathbb{A} : B \leq A\}.$$

An element $A \neq 0$ in a measure algebra $\mathbb{A}$ is called an *atom* if $B \leq A$ implies $B = A$ or $B = 0$.

A measure μ as well as its corresponding measure algebra $\mathbb{A}$ is called *non-atomic* if whenever $F \in \mathbb{A}$ with $\mu(F) > 0$, there exists an element $E \in \mathbb{A}$ with $E \leq F$ such that $\mu(F) > \mu(E) > 0$.

Suppose $(X, \mathbb{F}_1, \mu_1)$ and $(Y, \mathbb{F}_2, \mu_2)$ are two measure spaces, and $\mathbb{A}_1, \mathbb{A}_2$ are the corresponding measure algebras. A set mapping $h : \mathbb{A}_1 \to \mathbb{A}_2$ is called a *homomorphism* if:

(i) $$h(X) = Y,$$

(ii) $$h(-E) = -h(E) \quad \text{for} \quad E \in \mathbb{A}_1,$$

(iii) $$h(E \cup F) = h(E) \cup h(F) \quad \text{for any} \quad E, F \in \mathbb{A}_1.$$

If, in addition, h has the property that for any sequence $\langle E_n \rangle$ in $\mathbb{A}_1$, $h\left(\bigcup_{n=1}^{\infty} E_n\right) = \bigcup_{n=1}^{\infty} h(E_n)$, then h is called a *σ-homomorphism*.

The Boolean homomorphism h is *continuous* if, in addition, it satisfies the condition $\mu_1(E_n) \to 0$ implies $\mu_2(h(E_n)) \to 0$.

The point mapping $\varphi : Y \to X$ is called *measurable* if $\varphi^{-1}(E) \in \mathbb{F}_2$ for each $E \in \mathbb{F}_1$.

By $L_0(\mu)$ we mean as usual the set of all measurable functions with respect to the measure μ.

LEMMA 1. *Let $(X, \mathbb{A}, \mu)$ be a finite measure space. If $\mathbb{B} = \{g_\alpha : \alpha \in I\} \subset L_0(\mu)$, then there exists an extended real-valued function h such that*

i) *$g_\alpha \leq h$ μ-a.e. for every $\alpha \in I$,*

ii) *if $g_\alpha \leq f$ μ-a.e. for some f in $L_0(\mu)$, then $h \leq f$ μ-a.e. and there is a sequence $\langle g_{\alpha_n} \rangle$ in $L_0(\mu)$ with $h = \bigvee_{n=1}^{\infty} g_{\alpha_n} = \sup\{g_{\alpha_n} : n \in \mathbb{N}\}$.*

Proof.

Case 1, if $g_\alpha : X \to [0, 1]$.

Let $\beta = \sup\{\int(g_{\alpha_1} \vee \ldots \vee g_{\alpha_n}) : g_{\alpha_i} \in \mathbb{B},\ 1 \leq i \leq n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, there is $h_n = g_{\alpha_{1n}} \vee g_{\alpha_{2n}} \vee \ldots \vee g_{\alpha_{kn}}$, with $\int h_n \uparrow \beta$. Put $h = \bigvee_{n=1}^{\infty} h_n$ (note that $\int(g_\alpha \vee h_n) \leq \beta$, $\forall n$).

Claim: $g_\alpha \leq h$ μ-a.e. $\forall \alpha \in I$.

Suppose $g_\alpha > h$ for some $\alpha \in I$ on a set of positive measure, i.e. $(g_\alpha - h) > \varepsilon$ on a set E with $\mu(E) > 0$, for some $\varepsilon > 0$.

Therefore,

$$\begin{aligned}\int (g_\alpha \vee h_n) &= \int_E (g_\alpha \vee h_n) + \int_{-E} (g_\alpha \vee h_n)\\ &\geq \int_E g_\alpha + \int_{-E} h_n\\ &> \int_E h_n + \int_E \varepsilon + \int_{-E} h_n\\ &= \int h_n + \varepsilon\mu(E) \to \beta + \varepsilon\mu(E) > \beta\end{aligned}$$

which contradicts the note above. Thus, $g_\alpha \leq h$ μ-a.e. for every $\alpha \in I$, and ii) is true obviously.

Case 2. If $\{g_\alpha : \alpha \in I\}$ is an arbitrary subset of $L_0(\mu)$, define the mapping $\varphi : (-\infty, \infty] \to (0, 1]$ by

$$\varphi(x) = \begin{cases} 1 & \text{if } x = \infty \\ \frac{1}{2}\left(\frac{x}{1+|x|}\right) + \frac{1}{2} & \text{otherwise.}\end{cases}$$

Let $h = \sup\{\varphi \circ g_\alpha : \alpha \in I\}$ so that, by Case 1:

(1) if $g_\alpha \leq f$ μ-a.e. for some f, then $\varphi^{-1} \circ h = \sup g_\alpha \leq f$ μ-a.e.,

(2) $\varphi^{-1} \circ h = \sup\{g_\alpha : \alpha \in I\} \geq g_\alpha$ μ-a.e., and also, there is a sequence $\langle g_{\alpha_n}\rangle$ such that $\bigvee_{n=1}^{\infty} \varphi \circ g_{\alpha_n} = h$.

Thus, $\bigvee_{n=1}^{\infty} g_{\alpha_n} = \varphi^{-1} \circ h$ and hence $\mathbb{B}$ has the required i) and ii).

LEMMA 2. *If the measure algebra $(\mathbb{A}, \mu)$ is finite non-atomic then for any $A \in \mathbb{A}$ with $\mu(A) > 0$, and for each $\varepsilon > 0$, there exists $B \subset A$ such that $\mu(B) < \varepsilon$.*

The proof of this lemma is evident and we omit it. The following technical lemma proves some combinatorial result we need in our main Theorem 4.

Although the proof of the next Lemma 3 has been published in [2] p. 169, we include the proof in our paper for completeness. Let us mention that the proof of this lemma was first included in the author's M. Sc. thesis [3] (this thesis has not been published).

LEMMA 3. *Suppose that $A_1, \ldots, A_n, E_0$ are sets in $\mathbb{A}$ such that $\sum_{i=1}^{n} \chi_{A_i} \geq M\chi_{E_0}$ $(n > M \in \mathbb{N})$. Then $\forall r$, $1 \leq r \leq n - M$, there exists a subsequence*

$A_{i_1}, A_{i_2}, \dots, \mathbb{A}_{i_r}$ *of* $\langle A_i\rangle_{i=1}^n$ *such that*

$$\mu\Big(\bigcup_{j=1}^{r} A_{i_j}\Big) \geq \left(1-\left(1-\frac{M}{n}\right)^r\right)(\mu(E_0)).$$

Proof. It is clear that $\sum_{i=1}^{n} \chi_{A_i}(x) \geq M$, $\forall x \in E_0$. Let us call E an r-set if $|E| = r$, hence the number of r-sets in $\{1,2,\dots,n\}$ is $\binom{n}{r}$, there is a set $F \subset \{1,\dots,n\}$ with $|F| = M$ such that $x \in \bigcap_{i\in F} A_i$. If $|E| = r$, $x \notin \bigcup_{i\in E} A_i$ only if $E \cap F = \varnothing$, i.e. $E \subset \{1,2,\dots,n\} \setminus F$. But the number of r-sets in $\{1,2,\dots,n\}\setminus F$ is $\binom{n-M}{r}$. Therefore, $\sum_{|E|=r} \chi_{\bigcup_{i\in E} A_i}(x) \geq \binom{n}{r} - \binom{n-M}{r}$. Since the right hand side of this inequality does not depend on x, it is true for every $x \in E_0$.

Thus, the average measure of a union of r-many sets A_i is

$$\begin{aligned}
\binom{n}{r}^{-1} \sum_{|E|=r} \mu\Big(\bigcup_{i\in E} A_i\Big) &= \binom{n}{r}^{-1} \int_X \Big(\sum_{|E|=r} \chi_{\bigcup_{i\in E} A_i}\Big) d\mu \\
&\geq \binom{n}{r}^{-1} \int_{E_0} \Big(\sum_{|E|=r} \chi_{\bigcup_{i\in E} A_i}\Big) d\mu \\
&\geq \binom{n}{r}^{-1} \left(\binom{n}{r} - \binom{n-M}{r}\right)(\mu(E_0)) \\
&= \left(1 - \frac{\binom{n-M}{r}}{\binom{n}{r}}\right)(\mu(E_0)) \\
&= \left(1 - \frac{(n-M)!(n-r)!}{n!(n-M-r)!}\right)(\mu(E_0)) \\
&\geq \left(1-\left(1-\frac{M}{n}\right)^r\right)(\mu(E_0)).
\end{aligned}$$

So there is a subsequence $A_{i_1}, \dots, A_{i_r}$ of $\langle A_i\rangle_{i=1}^n$ such that

$$\mu\Big(\bigcup_{j=1}^{r} A_{i_j}\Big) \geq \binom{n}{r}^{-1} \sum_{|E|=r} \mu\Big(\bigcup_{i\in E} A_i\Big) \geq \left(1-\left(1-\frac{M}{n}\right)^r\right)(\mu(E_0)).$$

Note that if $M \to \infty$, $\left(1-\frac{M}{n}\right)^r \to 0$, because $M < n$, then $\frac{M}{n} \to 1$ if $\langle n-M\rangle$ is bounded $\forall M$. □

Now, we state and prove our main theorem.

THEOREM 4 [3]. *Let $(X_1, \mathbb{F}_1, \mu_1)$ and $(X_2, \mathbb{F}_2, \mu_2)$ be two finite non-atomic measure spaces, and let $\mathbb{A}_1$ and $\mathbb{A}_2$ be the corresponding measure algebras of μ_1 and μ_2 respectively.*

Suppose $h : \mathbb{A}_1 \to \mathbb{A}_2$ is a mapping satisfying

(i) *$E_1, E_2 \in \mathbb{A}_1$ implies $h(E_1 \cup E_2) = h(E_1) \cup h(E_2)$,*
(ii) *if $\langle E_n \rangle$ is a decreasing sequence in $\mathbb{A}_1$ with $\mu_1(E_n) \to 0$, then $\mu_2(h(E_n)) \to 0$.*

Then there is a pairwise disjoint sequence $\langle A_n \rangle$ in $\mathbb{A}_2$ such that $h(X_1) = \bigcup_{n=1}^{\infty} A_n$ and for each $n \in \mathbb{N}$, there is a partition $\{E_{1n}, \ldots, E_{k_n n}\}$ of X_1, so that the mapping $h_{in}(E) = A_n \cap h(E)$, for $E \in \mathbb{A}_1(E_{in})$ is a σ-homomorphism of $\mathbb{A}_1(E_{in})$ into $\mathbb{A}_2(A_n \cap h(E_{in}))$.

Proof. Let $\mathbb{P}$ be the family of all partitions of the Boolean algebra $\mathbb{A}_1$. $\mathbb{P}$ is partially ordered by the following relation

$$P \leq Q \iff Q \text{ is a refinement of } P, \text{ for every } P, Q \in \mathbb{P}.$$

If $P = \{E_1, \ldots, E_n\} \in \mathbb{P}$ and $H(P) = \sum_{i=1}^{n} \chi_{h(E_i)}$, then $H(P) \in L_0(\mu_2)$. By (i), it is obvious that the operation H is monotonic.

$$P \leq Q \text{ implies } H(P) \leq H(Q), \text{ for all } P, Q \in \mathbb{P}. \tag{1}$$

Now, let $g = \sup\{H(P) : P \in \mathbb{P}\}$. It follows from Lemma 1 that by (1) and the definition of H, there is a monotonic increasing sequence $\langle Q_n \rangle$ in $\mathbb{P}$, such that

$$g = \lim_{n\to\infty} H(Q_n). \tag{2}$$

We claim that $g(x) < \infty$, μ_2-a.e.

In order to prove this, let us presume that, on the contrary, $g = \infty$ on a set $D \subset X$ with $\mu_2(D) > \alpha > 0$.

Let M be an arbitrary integer and let

$$L_n = \{x \in D : H(Q_n)(x) \leq M\}, \quad n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} H(Q_n)(x) = \infty$ for $x \in D$ and $L_1 \supset L_2 \supset \ldots$, we have $\lim_{n\to\infty} \mu_2(L_n) = 0$, so given any $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $\mu_2(L_k) < \varepsilon$ and $H(Q_k)(x) > M$ for every $x \in D \setminus L_k$. We can choose ε so that the set $Z_0 = D \setminus L_k$ has a positive measure. Hence $\mu_2(D \setminus L_k) \geq \alpha - \varepsilon > 0$.

Since, by the above considerations, $H(Q_k) \geq M\chi_{Z_0}$, we can apply here Lemma 3. For every $n > M$, let $Q'_n = \{F_1, \dots, F_n\}$ be a refinement of the partition $Q_k = \{E_1, \dots, E_m\}$ such that $\mu_1(F_j) \leq 2\mu_1(E_k)$ and $\mu_1(F_j) \leq \frac{2}{n}\mu_1(X_1)$, for $1 \leq j \leq n$ and $1 \leq k \leq m$, such a Q'_n exists by Lemma 2.

By Lemma 3, for every r, $1 \leq r \leq n$, there is a subsequence $F_{i_1}, \dots, F_{i_r}$ of $F_1, \dots, F_n$ such that

$$(3) \qquad \mu_2\Big(h\Big(\bigcup_{j=1}^{r} F_{i_j}\Big)\Big) = \mu_2\Big(\bigcup_{j=1}^{r} h(F_{i_j})\Big) \geq \left(1 - \left(1 - \frac{M}{n}\right)^r\right)(\alpha - \varepsilon)$$

and

$$(4) \qquad \mu_1\Big(\bigcup_{j=1}^{r} F_{i_j}\Big) < \frac{2r}{n}\beta, \text{ where } \beta = \mu_1(X_1).$$

Since, for $M > m$, n may be chosen in such a way that the sequence $\langle n - M\rangle$ is bounded, the conjunction of (3) and (4) contradicts the assumption (i). This contradiction proves that $g < \infty$ μ_2-a.e.

Now, if we recall the definition (2) $g = \lim_{n\to\infty} H(Q_n)$, where Q_k is an increasing sequence in $\mathbb{P}$, and if we define $B_n = g^{-1}([1, n])$, we obtain $\bigcup_{n=1}^{\infty} B_n = g^{-1}\Big(\bigcup_{n=1}^{\infty} [1, n]\Big) = h(X_1)$, since g is finite and integer-valued. For every pair of integers (k, n), let $C_{k,n} = \{x \in B_n : H\big(Q_k\big)(x) < g(x)\}$.

Since $H(Q_k) \uparrow g$ for $k \uparrow \infty$, we have $\lim_{k\to\infty} \mu_2(C_{k,n}) = 0$. So, for a fixed n, k may be chosen so that $\mu_2(C_{k,n}) < \frac{1}{n}$. In this case, $B_n \setminus C_{k,n} = \{x \in B_n : H\big(Q_k\big)(x) \geq g(x)\}$.

Since $H\big(Q_k\big)(x)$ cannot be greater than $g(x)$, we have

$$(5) \qquad B_n \setminus C_{k,n} = \{x \in B_n : H\big(Q_k\big)(x) = g(x)\}.$$

Now, let $A_0 = g^{-1}(\{0\})$, and for $n > 0$, let $A_n = B_n \setminus \Big(\bigcup_{i=0}^{n-1} A_i \cup C_{k,n}\Big)$ so that $B_n \setminus \bigcup_{i=0}^{n} A_i$ is a subset of $C_{k,n}$, and then $\mu_2\Big(B_n \setminus \bigcup_{i=1}^{n} A_i\Big) < \frac{1}{n}$.

Therefore, the sequence $\langle A_n\rangle$ is pairwise disjoint and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n = h(X_1)$ (with accuracy of a set of measure zero).

Now, consider the partition Q_k such that (5) holds.

If $Q'_k = \{E_{1,n}, \dots, E_{m,n}\}$ and $A, B \in \mathbb{A}_1(E_{in})$ such that $A \cup B = E_{in}$ and $A \cap B = \varnothing$, then we claim that $\mu_2(A_n \cap h(A) \cap h(B)) = 0$.

Presume that this is not the case. Then, we shall discuss the refinement

$$Q'_k = \{E_{1,n}, \ldots, E_{i-1,n}, A, B, E_{i+1,n}, \ldots, E_{m,n}\} \text{ of } Q_k.$$

For every $x \in A_n \cap h(A) \cap h(B)$ we have

$$\begin{aligned} H(Q'_k)(x) &= \sum_{j \neq i} \chi_{h(E_{jn})}(x) + \chi_{h(A)}(x) + \chi_{h(B)}(x) \\ &= \sum_{j \neq i} \chi_{h(E_{jn})}(x) + 2 \\ &= \sum_{j=1}^{m} \chi_{h(E_j)}(x) + 1 = g(x) + 1, \end{aligned}$$

because $h(A) \cup h(B) = h(A \cup B) = h(E_{in})$.

But this contradicts (5).

Finally, define the mapping $h_{in} : \mathbb{A}_1(E_{in}) \to \mathbb{A}_2(A_n \cap h(E_{in}))$, by $h_{in}(E) = A_n \cap h(E)$, for each $E \in \mathbb{A}_1(E_{in})$.

Since $h(E_{in} \setminus E) \cup h(E) = h(E_{in})$, we have $h_{in}(E_{in} \setminus E) \cup h_{in}(E) = A_n \cap h(E_{in}) \ldots$ by (i), which is the greatest element of $\mathbb{A}_2(A_n \cap h(E_{in}))$.

On the other hand, $h_{in}(E_{in} \setminus E) \cap h_{in}(E) = A_n \cap h(E_{in} \setminus E) \cap h(E)$ by definition of h_{in} and by the property of meet in Boolean algebra. But it was proved previously that $\mu_2(A_n \cap h(E_{in} \setminus E) \cap h(E)) = 0$, it follows that in the measure algebra $\mathbb{A}$ two elements $A_n \cap h(E_{in} \setminus E)$ and $A_n \cap h(E)$ are disjoint. With the final result, the second one is the complement of the first one in the algebra $\mathbb{A}_2(A_n \cap h(E_{in}))$, i.e. $h_{in}(E_{in} \setminus E) = -h_{in}(E)$, and since h_{in} preserves finite union, it is a homomorphism. Hence $\mu_1(E_n) \downarrow 0$ in $\mathbb{A}_1(E_{in})$ implies $\mu_2(h_{in}(E_n)) \downarrow 0$, by (ii) and the definition of h_{in}.

Thus, h_{in} is a σ-homomorphism for each $n \in \mathbb{N}$ and each $1 \le i \le k_n$. □

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WARSAW UNIVERSITY OF TECHNOLOGY
INSTITUTE OF MATHEMATICS
Plac Politechniki 1,
00661 WARSAW, POLAND

Mail address of the author:
THE UNIVERSITY OF EL-FATEH
P.O. BOX 13611
TRIPOLI–LIBYA

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