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Sr. Arworn, K. Denecke

LEFT–EDGE SOLID VARIETIES OF DIFFERENTIAL GROUPOIDS

Abstract. An identity $s \approx t$ is called a hyperidentity in a variety V if by substituting terms of appropriate arity for the operation symbols in $s \approx t$, one obtains an identity satisfied in V. Such substitutions are called hypersubstitutions. In the paper we consider hyperidentities and hypersubstitutions in the variety of differential groupoids, certain idempotent and medial groupoids. Differential groupoids are modes as defined in [Rom-S; 85]. We show that this variety and all its subvarieties are left-edge solid.

1. Introduction

Algebras in the variety defined by the medial law, the idempotent law, and the identity $x(yz) \approx xy$ are called differential groupoids. Romanowska and Smith pointed out some interconnections between differential groupoids, differentials and differentiation ([Rom-S; 91]). From $x(yz) \approx xy$ one obtains $x(xy) \approx xx \approx x$ and if in a medial and idempotent variety the equation $x(xy) \approx x$ is satisfied one has also $x(yz) \approx (x(xy))(yz) \approx (xy)((xy)z) \approx xy$. That means, we can also consider the variety $DG = Mod\{(xy)(uv) \approx (xu)(yv),\ x^2 \approx x,\ x(xy) \approx x\}$.

The variety of differential groupoids (also called LIR-groupoids in [Rom-R; 87]) satisfies also the identity

$$(xy)z \approx (xz)y$$

of left normality ([Rom-S; 91]) since $(xy)z \approx (xy)(zz) \approx (xz)(yz) \approx ((xz)((xz)y))(yz) \approx ((xz)y)(((xz)y)z) \approx (xz)y$.

In this paper we consider the dual variety $RR = Mod\{(xy)(uv) \approx (xu)(yv),\ xx \approx x,\ (xy)y \approx y\}$ (see [Dud; 94], [Rom-R; 87], [Rom-R; 89]) and some of its subvarieties, for instance varieties which are dual to varieties of n-cyclic groupoids introduced by Płonka ([Pło; 85]). All results are

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dualizable and give us dual results for DG. There is no particular reason to prefer one or the other variety.

To these varieties we apply the theory of M-hyperidentities and M-solid varieties ([Den-R; 95], [Den-W; 98]). Our main result is that the variety of differential groupoids as well as all its subvarieties are left–edge solid and dually, that the variety RR and all its subvarieties are right–edge solid.

2. Basic concepts

An identity $s \approx t$ is called a hyperidentity in a variety V if for every substitution of terms of V of appropriate arity for the operation symbols in $s \approx t$, the resulting identity holds in V. In this paper we consider groupoids (binars), i.e. algebras with one binary operation f^A and varieties of groupoids.

Our informal definition of a hyperidentity shows that we are interested in a map which associates to the binary operation symbol f a binary term $\sigma(f)$. Any such map is called a hypersubstitution. Let $W(X)$ be the set of all terms of type 2 on an alphabet $X = \{x_1, x_2, x_3, \ldots\}$ and let $W(X_2)$ be the set of all binary terms, i.e. terms built up from $X_2 = \{x, y\}$. Using a hypersubstitution σ we can define a uniquely determined mapping $\hat{\sigma}$ defined on terms by

(i) $\hat{\sigma}[x] := x$ for any $x \in X$,
(ii) $\hat{\sigma}[f(t_1, t_2)] := \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By Hyp we denote the set of all these hypersubstitutions. If we define a multiplication $\circ_h$ on the set Hyp by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is the usual composition of functions together with σ_{id} defined by $\sigma_{id}(f) = f(x, y)$ we obtain a monoid $\underline{Hyp} = (Hyp;\ \circ_h, \sigma_{id})$. If $\underline{M}$ is a submonoid of $\underline{Hyp}$ then we define an equation $s \approx t$ built up by terms s, t of type 2 to be an M-hyperidentity in the variety V of groupoids if for all $\sigma \in M$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are satisfied as identities in V. Hyperidentities are M-hyperidentities for $M = Hyp$. A variety V of groupoids is called M-solid if every identity satisfied in V is an M-hyperidentity in V. For $M = Hyp$ we speak of solid varieties. All M-solid varieties of groupoids form a complete sublattice $\mathcal{S}_M(2)$ of the lattice $\mathcal{L}(2)$ of all varieties of groupoids with

$$M_1 \subseteq M_2 \Rightarrow \mathcal{S}_{M_1}(2) \supseteq \mathcal{S}_{M_2}(2) \ .$$

For more background on the theory of M-hyperidentities and M-solid varieties see e.g. [Den-R; 95] and [Den; 97].

Particular submonoids of Hyp are the monoids $\underline{Left}$ and $\underline{Right}$ of all leftmost and rightmost hypersubstitutions. A hypersubstitution $\sigma \in Hyp$ is called leftmost if the left most variable in the term $\sigma(f) = t$ is x.

The varieties corresponding to $\mathcal{S}_{Left}(2)$ and to $\mathcal{S}_{Right}(2)$ are called left–edge–solid and right–edge–solid, respectively.

Let V be a variety of groupoids and let M be a monoid of hypersubstitutions. To test whether an identity $s \approx t$ of V is an M-hyperidentity of V, our definition requires that we check, for each hypersubstitution σ in M, that $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity of V.

Indeed, we can restrict our testing to certain "special" hypersubstitutions. To make this restriction precise we recall of two concepts, both introduced by J. Płonka in [Pło; 94] for arbitrary types of algebras.

DEFINITION 2.1. Let V be a variety of groupoids. A hypersubstitution σ is called a V-proper hypersubstitution if for every identity $s \approx t$ of V, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V. We use $P(V)$ for the set of all V-proper hypersubstitutions.

Clearly, $(P(V); \circ_h, \sigma_{id})$ is a submonoid of $(Hyp; \circ_h, \sigma_{id})$, a variety V is M-solid for the monoid $M = P(V)$ and $P(V)$ is the largest M for which V is M-solid.

DEFINITION 2.2. Let V be a variety of groupoids. Two hypersubstitutions σ_1 and σ_2 are called V-equivalent if $\sigma_1(f) \approx \sigma_2(f)$ is an identity in V. In this case we write $\sigma_1 \sim_V \sigma_2$.

By induction on the complexity of term definition one shows

$$\sigma_1 \sim_V \sigma_2 \Leftrightarrow \forall t \in W(X)\ (\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in Id\,V)$$

(here $Id\,V$ denotes the set of all identities satisfied in V).

Using this proposition one proves also: If $\sigma_1 \sim_V \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V (see e.g. [Den-W; 97]).

It is clear from the definition that the relation $\sim_V$ is always an equivalence relation on Hyp and its restriction $\sim_V|M$ to M is always an equivalence relation on M.

DEFINITION 2.3. Let V be a variety of groupoids and let M be a monoid of hypersubstitutions. Let Φ be a choice function which chooses from M one hypersubstitution from each equivalence class of the relation $\sim_V|M$ and with $\Phi([\sigma_{id}]_{\sim_V}) = \sigma_{id}$ and let $M_\Phi^N(V)$ be the set of hypersubstitutions so chosen. The elements of $M_\Phi^N(V)$ are called $\Phi - V$-normal form hypersubstitutions of M . We will say that the variety V is $M_\Phi^N(V)$-solid if for every identity $s \approx t \in Id\,V$ and for every hypersubstitution $\sigma \in M_\Phi^N(V)$, the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ belongs to IdV.

Then from the remark before Definition 2.3 it follows that the variety V is M-solid iff it is $M_\Phi^N(V)$-solid ([Arw-D; 97]).

On the set $M_\Phi^N(V)$ we can define a multiplication by

$$\sigma_1 \circ_N \sigma_2 = \Phi[(\sigma_1 \circ_h \sigma_2)]_{\sim_V}$$

and obtain a groupoid $(M_\Phi^N(V); \circ_N, \sigma_{id})$ with identity.

It is easy to see that $(M_\Phi^N(V); \circ_N, \sigma_{id})$ is a monoid if V is M-solid ([Den-W; 97]). The converse is not true ([Wis; 97]). Note that if $P(V)$ is the monoid of V-proper hypersubstitutions, then $P(V)_\Phi^N(V)$ is also a monoid and V is $P(V)$-solid.

3. *RR*-normal form hypersubstitutions

Our first aim is to calculate the normal form hypersubstitutions for the variety $RR = Mod\{(xy)(uv) \approx (xu)(yv),\ xx \approx x,\ (xy)y \approx y\}$. We need some more identities satisfied in RR.

In a similar way as in the introduction one can show

PROPOSITION 3.1 ([Dud; 94],[Rom-R; 87]). *A medial idempotent groupoid satisfies* $(xy)y \approx y$ *iff it satisfies* $(xy)z \approx yz$ *and* $x(yz) \approx y(xz)$. ■

In the sequel we write $x^n y$ for $x(\ldots(x(xy))\ldots)$. An inductive definition of $x^n y$ is given by $x^1 y := xy$, $x^n y = x(x^{n-1}y)$. Further we set $x^0 y := y$.

We recall of several identities satisfied in the variety RR.

PROPOSITION 3.2 ([Rom-R; 87] for the dual identities). *The variety RR satisfies the following identities:*

(i) $x^k(yx) \approx yx,\ k \in \mathbb{N}, k \geq 1,$

(ii) $(x^k y)^l y \approx y,\ k, l \in \mathbb{N},\ k, l \geq 1,$

(iii) $(y^k x)y \approx xy,\ k \in \mathbb{N},$

(iv) $(x^l y)(x^k y) \approx x^k y,\ k, l \in \mathbb{N},\ k, l \geq 1,$

(v) $(y^l x)(x^k y) \approx x^{k+1} y,\ k, l \in \mathbb{N},\ k, l \geq 1,$

(vi) $x^{k_n}(y^{l_n}(x^{k_{n-1}}(\ldots(x^{k_1}(y^{l_1}(xy))\ldots)))) \approx x^{k+1}y,$
$k_1, \ldots, k_n,\ l_1, \ldots, l_n \in \mathbb{N}$
$k_1, k_2, \ldots, k_n \geq 1, l_1, \ldots l_n \geq 1$ *and* $k = k_1 + \ldots + k_n$. ■

THEOREM 3.3 ([Rom-R; 87], see also [Dud; 95]). *The set of all binary terms of RR is* $\{x^k y \mid k \geq 0\} \cup \{y^k x \mid k \geq 0\}$. ■

Each of these terms is in fact an equivalence class, i.e. an element of the quotient set $W(X_2)/Id\,RR$ and there is a function Φ which selects from each of the classes the given binary term in $\{x^k y \mid k \geq 0\} \cup \{y^k x \mid k \geq 0\}$. But this means $Hyp_\Phi^N(RR) = \{\sigma_{x^k y} \mid k \geq 0\} \cup \{\sigma_{y^k x} \mid k \geq 0\}$ is the groupoid of normal form hypersubstitutions w.r.t. the choice function Φ. The elements of $Hyp_\Phi^N(RR)$ are pairwise different. Otherwise, if there are natural numbers k, l with $k \neq l$ and $\sigma_{x^k y} \sim_{RR} \sigma_{x^l y}$ then $x^k y \approx x^l y \in IdRR$, but there are algebras in RR which doesn't satisfy this identity. If $\sigma_{x^k y} \sim_{RR} \sigma_{y^l x}$ i.e. if

$x^k y \approx y^l x \in IdRR$ then by Proposition 3.2 $x^k y \approx x^{k+1} y \in IdRR$ and we get a contradiction as in the previous case. To describe the operation $\circ_N$ on $Hyp_\Phi^N(RR)$ we prove the following relations

PROPOSITION 3.4. (i) $\sigma_{x^k y} \circ_h \sigma_{x^l y} \sim_{RR} \sigma_{x^{k\cdot l} y}$, $k, l \geq 0$,

(ii) $\sigma_{x^k y} \circ_h \sigma_{y^m x} \sim_{RR} \sigma_{y^{k\cdot m} x}$, $k, m \geq 0$,

(iii) $\sigma_{y^m x} \circ_h \sigma_{y^n x} \sim_{RR} \sigma_y$, $m, n \geq 2$,

(iv) $\sigma_x \circ_h \sigma_{y^n x} \sim_{RR} \sigma_y$, $n \geq 1$,
$\sigma_{yx} \circ_h \sigma_{y^n x} \sim_{RR} \sigma_y$, $n \geq 2$,
$\sigma_{y^m x} \circ_h \sigma_x \sim_{RR} \sigma_x$, $m \geq 0$,
$\sigma_{y^m x} \circ_h \sigma_{yx} \sim_{RR} \sigma_{x^m y}$, $m \geq 0$,

(v) $\sigma_{y^m x} \circ_h \sigma_{x^k y} \sim_{RR} \sigma_x$, $m, k \geq 2$,

(vi) $\sigma_x \circ_h \sigma_{x^k y} \sim_{RR} \sigma_x$, $k \geq 0$,
$\sigma_{yx} \circ_h \sigma_{x^k y} \sim_{RR} \sigma_x$, $k \geq 2$,
$\sigma_{y^m x} \circ_h \sigma_{xy} \sim_{RR} \sigma_{y^m x}$, $m \geq 0$,
$\sigma_{y^m x} \circ_h \sigma_y \sim_{RR} \sigma_y$, $m \geq 0$.

Proof. (i) We prove this relation by induction on l. Assume that $l = 0$. Then $(\sigma_{x^k y} \circ_h \sigma_y)(f) = \hat{\sigma}_{x^k y}[y] \approx y$ and thus $\sigma_{x^k y} \circ_h \sigma_y \sim_{RR} \sigma_y$ and (i) is satisfied for $l = 0$. Assume now that (i) is satisfied for $l - 1$, i.e. $\sigma_{x^k y} \circ_h \sigma_{x^{l-1} y} \sim_{RR} \sigma_{x^{k(l-1)} y}$. Then $(\sigma_{x^k y} \circ_h \sigma_{x^l y})(f) = \hat{\sigma}_{x^k y}[x^l y] = \hat{\sigma}_{x^k y}[x(x^{l-1} y)] = x^k(x^{k(l-1)} y) = x^{k+k(l-1)} y = x^{k\cdot l} y$. This proves (i).

(ii) We give a proof by induction on m. For $m = 0$ we have $(\sigma_{x^k y} \circ_h \sigma_x)(f) = \hat{\sigma}_{x^k y}[x] = x$ and therefore $\sigma_{x^k y} \circ_h \sigma_x \sim_{RR} \sigma_x$. Assume that $\sigma_{x^k y} \circ_h \sigma_{y^{(m-1)} x} \sim_{RR} \sigma_{y^{k(m-1)} x}$. Then $(\sigma_{x^k y} \circ_h \sigma_{y^m x})(f) = \hat{\sigma}_{x^k y}[y(y^{m-1} x)] = y^k(y^{(m-1)k} x) = y^{k+(m-1)k} x = y^{k\cdot m} x$ and (ii) is satisfied.

(iii) We prove this by induction on n. For $n = 2$ we obtain $(\sigma_{y^m x} \circ_h \sigma_{y(yx)})(f) = \hat{\sigma}_{y^m x}[y(yx)] = (x^m y)^m y \approx y$ by Proposition 3.2 (ii) and thus $\sigma_{y^m x} \circ_h \sigma_{y^2 x} \sim_{RR} \sigma_y$. Assume that $\sigma_{y^m x} \circ_h \sigma_{y^{n-1} x} \sim_{RR} \sigma_y$. Then $(\sigma_{y^m x} \circ_h \sigma_{y^n x})(f) = \hat{\sigma}_{y^m x}[y^n x] = \hat{\sigma}_{y^m x}[y(y^{n-1} x)] = y^m y \approx y$ and therefore, $\sigma_{y^m x} \circ_h \sigma_{y^n x} \sim_{RR} \sigma_y$ for all $m, n \geq 2$.

(iv) The first relation is clear. We prove the second one by induction on $n \geq 2$. For $n = 2$ we get $(\sigma_{yx} \circ_h \sigma_{y(yx)})(f) = \hat{\sigma}_{yx}[y(yx)] = (xy)y \approx y$ and thus $\sigma_{yx} \circ_h \sigma_{y^2 x} \sim_{RR} \sigma_y$. Assume that $\sigma_{yx} \circ_h \sigma_{y^{n-1} x} \sim_{RR} \sigma_y$. Then $(\sigma_{yx} \circ_h \sigma_{y^n x})(f) = \hat{\sigma}_{yx}[y^n x] = \hat{\sigma}_{yx}[y(y^{n-1} x)] = \sigma_{yx}(f)(y, \hat{\sigma}_{yx}[y^{n-1} x]) \approx \sigma_{yx}(f)(y, y) = yy \approx y$ and then $\sigma_{yx} \circ_h \sigma_{y^n x} \sim_{RR} \sigma_y$ for all $n \geq 2$.

The relations $\sigma_{y^m x} \circ_h \sigma_x \sim_{RR} \sigma_x$ and $\sigma_{y^m x} \circ_h \sigma_{yx} \sim_{RR} \sigma_{x^m y}$ for all $m \geq 0$ are obvious.

(v) We prove this by induction on k. For $k = 2$ one has $(\sigma_{y^m x} \circ_h \sigma_{x(xy)})(f) = \hat{\sigma}_{y^m x}[x(xy)] = (y^m x)^m x \approx x$ by Proposition 3.2 (ii) and then $\sigma_{y^m x} \circ_h \sigma_{x^2 y} \sim_{RR} \sigma_x$.

Assume now that $\sigma_{y^m x} \circ_h \sigma_{x^{k-1}y} \sim_{RR} \sigma_x$. Then $(\sigma_{y^m x} \circ_h \sigma_{x^k y})(f) = \hat{\sigma}_{y^m x}[x^k y] = \hat{\sigma}_{y^m x}[x(x^{k-1}y)] = \sigma_{y^m x}(f)(x,x) = x^m x \approx x$ and so we obtain $\sigma_{y^m x} \circ_h \sigma_{x^k y} \sim_{RR} \sigma_x$.

(vi) The first, the third, and the fourth relation need no proof.

We show $\sigma_{yx} \circ_h \sigma_{x^k y} \sim_{RR} \sigma_x$ for all $k \geq 2$ by induction on k.

We begin with $k = 2$, then $\hat{\sigma}_{yx}[x(xy)] = (yx)x \approx x$ and so $\sigma_{yx} \circ_h \sigma_{x^2 y} \sim_{RR} \sigma_x$. Assume that $\sigma_{yx} \circ_h \sigma_{x^{k-1}y} \sim_{RR} \sigma_x$. Then $\hat{\sigma}_{yx}[x^k y] = \hat{\sigma}_{yx}[x(x^{k-1}y)] = \sigma_{yx}(f)(x, \hat{\sigma}_{yx}[x^{k-1}y]) = xx \approx x$. This completes the proof. ■

Proposition 3.4 shows that the multiplication $\circ_N$ in $Hyp^N_\Phi(RR)$ can be given by the following Cayley–table:

$\circ_N$	σ_x	σ_y	σ_{xy}	σ_{yx}	$\sigma_{x^k y}$	$\sigma_{x^l y}$	$\sigma_{y^m x}$	$\sigma_{y^n x}$
σ_x	σ_x	σ_y	σ_x	σ_y	σ_x	σ_x	σ_y	σ_y
σ_y	σ_x	σ_y	σ_y	σ_x	σ_y	σ_y	σ_x	σ_x
σ_{xy}	σ_x	σ_y	σ_{xy}	σ_{yx}	$\sigma_{x^k y}$	$\sigma_{x^l y}$	$\sigma_{y^m x}$	$\sigma_{y^n x}$
σ_{yx}	σ_x	σ_y	σ_{yx}	σ_{xy}	σ_x	σ_x	σ_y	σ_y
$\sigma_{x^k y}$	σ_x	σ_y	$\sigma_{x^k y}$	$\sigma_{y^l x}$	$\sigma_{x^{k^2} y}$	$\sigma_{x^{k \cdot l} y}$	$\sigma_{y^{m \cdot k} x}$	$\sigma_{y^{n \cdot k} x}$
$\sigma_{x^l y}$	σ_x	σ_y	$\sigma_{x^l y}$	$\sigma_{y^l x}$	$\sigma_{x^{k \cdot l} y}$	$\sigma_{x^{l^2} y}$	$\sigma_{y^{m \cdot l} x}$	$\sigma_{y^{n \cdot l} x}$
$\sigma_{y^m x}$	σ_x	σ_y	$\sigma_{y^m x}$	$\sigma_{x^m y}$	σ_x	σ_x	σ_y	σ_y
$\sigma_{y^n x}$	σ_x	σ_y	$\sigma_{y^n x}$	$\sigma_{x^n y}$	σ_x	σ_x	σ_y	σ_y

4. Hyperidentities and RR–proper hypersubstitutions

Now we want to answer the following two questions:

1. Which identities are preserved by all hypersubstitutions, i.e. which identities satisfied in RR are hyperidentities?
2. Which hypersubstitutions satisfy all identities of RR, i.e. which hypersubstitutions are RR–proper?

To check the medial law we need some more identities satisfied in RR.

PROPOSITION 4.1. *The following identities are satisfied in RR:*

$$(x^k y)^k (u^k v) \approx (x^k y)^{k-j}((x^j u)((x^{k-1}y)^j (u^{k-1}v))), \quad 1 \leq j \leq k-1, k \geq 2.$$

Proof. We give a proof by induction on j and begin with $j = 1$:

$$\begin{aligned}(x^k y)^k (u^k v) &= (x^k y)^{k-1}((x(x^{k-1}y))(u(u^{k-1}v))) \\ &\approx (x^k y)^{k-1}((xu)((x^{k-1}y)(u^{k-1}v))).\end{aligned}$$

Assume now that $(x^k y)^k (u^k v) \approx (x^k y)^{k-(j-1)}((x^{j-1}u)((x^{k-1}y)^{j-1}(u^{k-1}v)))$. Then

$$\begin{aligned}
&(x^k y)^{k-(j-1)}((x^{j-1}u)((x^{k-1}y)^{j-1}(u^{k-1}v))) \\
&\quad\approx (x^k y)^{k-j}((x^k y)((x^{j-1}u)((x^{k-1}y)^{j-1}(u^{k-1}v)))) \\
&\quad\approx (x^k y)^{k-j}((x(x^{k-1}y))((x^{j-1}u)((x^{k-1}y)^{j-1}(u^{k-1}v)))) \\
&\quad\approx (x^k y)^{k-j}((x(x^{j-1}u))((x^{k-1}y)((x^{k-1}y)^{j-1}(u^{k-1}v)))) \\
&\quad\approx (x^k y)^{k-j}((x^j u)((x^{k-1}y)^j (u^{k-1}v)))
\end{aligned}$$

and then $(x^k y)^k (u^k v) \approx (x^k y)^{k-j}((x^j u)((x^{k-1}y)^j (u^{k-1}v)))$. ■

Especially with $j = k-1$ we obtain:

COROLLARY 4.2. *The following identity holds in RR:*

$$(x^k y)^k (u^k v) \approx (x^k y)((x^{k-1}u)((x^{k-1}y)^{k-1}(u^{k-1}v)))$$

Proof. This follows from Proposition 4.1 with $j = k-1$. ■

THEOREM 4.3. *The idempotent and the medial law are hyperidentities in RR.*

Proof. For every term $v \in W_x$ (where W_x is the set of all terms built up only by using of x) we have $v \approx x \in Id\,RR$ (applying the idempotent law). If we apply an arbitrary normal form hypersubstitution σ of RR to $x^2 \approx x$, on the left hand side we get a term $v \in W_x$ and on the right hand side we get x, so $\hat{\sigma}[x^2] = v \approx x = \hat{\sigma}[x] \in Id\,RR$ and the idempotent law is satisfied as a hyperidentity in V.

Now we turn our attention to the medial law. It is routine matter to check that for every $\sigma \in \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\}$ we have $\hat{\sigma}[(xy)(uv)] \approx \hat{\sigma}[(xu)(yv)] \in Id\,RR$. Now we show by induction on k that $\hat{\sigma}_{x^k y}[(xy)(uv)] \approx \hat{\sigma}_{x^k y}[(xu)(yv)] \in Id\,RR$ for all $k \in \mathbb{N}$, $k \geq 2$ and begin with $k = 2$. Then

$$\begin{aligned}
\hat{\sigma}_{x^2 y}[(xy)(uv)] &= (x^2 y)^2 (u^2 v) \approx (x^2 y)((xu)((xy)(uv))) \\
&\approx (x^2 y)((xu)((xu)(yv))) \approx (x(xy))((xu)((xu)(yv))) \\
&\approx (x(xu))((xy)((xu)(yv))) \approx (x(xu))((x(xu))(y(yv))) \\
&\approx (x^2 u)^2 (y^2 v) = \hat{\sigma}_{x^2 y}[(xu)(yv)]
\end{aligned}$$

by Corollary 4.2 applying the medial identity several times. Assume that

$$\begin{aligned}
\hat{\sigma}_{x^{k-1} y}[(xy)(uv)] &= (x^{k-1}y)^{k-1}(u^{k-1}v) \approx (x^{k-1}u)^{k-1}(y^{k-1}v) \\
&= \hat{\sigma}_{x^{k-1} y}[(xu)(yv)].
\end{aligned}$$

Then

$$\begin{aligned}
\hat{\sigma}_{x^k y}[(xy)(uv)] &= (x^k y)^k (u^k v) \\
&\approx (x^k y)((x^{k-1}u)((x^{k-1}y)^{k-1}(u^{k-1}v))) \quad \text{(by Corollary 4.2)} \\
&\approx (x^k y)((x^{k-1}u)((x^{k-1}u)^{k-1}(y^{k-1}v))) \quad \text{(by hypothesis)}
\end{aligned}$$

$$\approx (x^k y)((x^{k-1}u)^k(y^{k-1}v)) \quad \text{and}$$

$$\begin{aligned}\hat{\sigma}_{x^k y}[(xu)(yv)] &= (x^k u)^k (y^k v)\\ &\approx (x^k u)((x^{k-1}y)((x^{k-1}u)^{k-1}(y^{k-1}v))) \quad \text{by Corollary 4.2}\\ &\approx (x(x^{k-1}u)((x^{k-1}y)((x^{k-1}u)^{k-1}(y^{k-1}v)))\\ &\approx (x(x^{k-1}y))((x^{k-1}u)((x^{k-1}u)^{k-1}(y^{k-1}v)))\\ &\approx (x^k y)((x^{k-1}u)^k(y^{k-1}v))\ .\end{aligned}$$

This shows that $\hat{\sigma}_{x^k y}[(xy)(uv)] \approx \hat{\sigma}_{x^k y}[(xu)(yv)] \in Id\,RR$ for all $k \geq 2$. To show that $\sigma_{y^k x}$ also preserves the medial identity we apply the equation $\sigma_{y^k x} = \sigma_{x^k y} \circ_N \sigma_{yx}$, i.e. $\sigma_{y^k x} \sim_{RR} \sigma_{x^k y} \circ_h \sigma_{yx}$ and therefore we have

$$\begin{aligned}\hat{\sigma}_{y^k x}[(xy)(uv)] &= \hat{\sigma}_{x^k y}[\hat{\sigma}_{yx}[(xy)(uv)]] = \hat{\sigma}_{x^k y}[(vu)(yx)] = (v^k u)^k (y^k x)\\ &\approx (v^k y)^k (u^k x) = \hat{\sigma}_{x^k y}[(vy)(ux)] = \hat{\sigma}_{x^k y}[\hat{\sigma}_{yx}[(xu)(yv)]]\\ &\approx (\sigma_{x^k y} \circ_h \sigma_{yx})\hat{}\ [(xy)(uv)] = \hat{\sigma}_{y^k x}[(xy)(uv)]\ .\end{aligned}$$

This finishes the proof. ■

Note that the medial law is a hyperidentity in the variety of all medial and idempotent groupoids. So Theorem 4.3 follows from this more general fact. But we wanted to give an independent proof.

The variety RR is not solid since the identity $(xy)y \approx y$ fails to be a hyperidentity as we can see by $\hat{\sigma}_x[(xy)y] = x \not\approx y = \hat{\sigma}_x[y]$.

If we want to check which hypersubstitutions preserve all identities of RR we have only to consider the identity $(xy)y \approx y$.

By RZ we denote the variety of right–zero semigroups, i.e. $RZ = Mod\{xy \approx y\}$. Clearly RZ is a subvariety of RR.

PROPOSITION 4.4. *$\{\sigma_{x^l y} \mid l \geq 0\}$ is the monoid of all RR–proper normal form hypersubstitutions.*

Proof. We have only to check the equation $(xy)y \approx y$. Applying $\sigma_{x^l y}$ for arbitrary $l > 0$ we get $\hat{\sigma}_{x^l y}[(xy)y] = (x^l y)^l y \approx y = \hat{\sigma}_{x^l y}[y]$ by Proposition 3.2(ii). For $l = 0$ we have $\sigma_y[(xy)y] = y = \hat{\sigma}_y[y]$. Now we have to prove that no normal form hypersubstitution different from $\sigma_{x^l y}$ preserves the equation $(xy)y \approx y$. Clearly, σ_x is not proper since $\hat{\sigma}_x[(xy)y] = x$ and $\hat{\sigma}_x[y] = y$. For $\sigma_{y^m x}$, $m \geq 1$ we have $\hat{\sigma}_{y^m x}[(xy)y] = y^m(y^m x)$ and $\hat{\sigma}_{y^m x}[y] = y$. Assume that $y^m(y^m x) \approx y \in Id\,RR$. Then $(y^m(y^m x))((xy)y) \approx yy \approx y \in Id\,RR$ and by the medial identity also $(y(xy))((y^{2m-1}x)y) \approx (y(xy))(xy) \approx (x(xy))(xy) \approx xy$.

But this means $xy \approx y \in IdRR$ and RR is the variety of right–zero–semigroups. Since RR is different from RZ we have a contradiction. ■

Further we have

COROLLARY 4.5. *The monoid of all RR-proper normal form hypersubstitutions is isomorphic to the monoid* $(\mathbb{N};\cdot,1,0)$ *of all natural numbers with zero.*

Proof. We consider the map $\varphi : \{\sigma_{x^l y} \mid l \geq 0,\ l \in \mathbb{N}\} \to \mathbb{N}$ defined by $\varphi(\sigma_{x^l y}) = l$. This map is one–to–one and onto since hypersubstitutions for different l are different. Further $\varphi(\sigma_{x^l y} \circ_N \sigma_{x^k y}) = \varphi(\sigma_{x^{l\cdot k} y}) = l \cdot k = \varphi(\sigma_{x^l y}) \cdot \varphi(\sigma_{x^k y})$ by Proposition 3.4(i). ■

5. Right–edge solid varieties of entropic groupoids

In section 2 we introduced the concept of left– and rightmost hypersubstitutions. Let $Right^N_\Phi(RR)$ be a set of all rightmost normal form hypersubstitutions of RR.

A variety V is right–edge–solid iff the set of all V–proper normal form hypersubstitutions agrees with the set of all rightmost normal form hypersubstitutions.

Checking $Hyp^N_\Phi(RR)$ and $Right^N_\Phi(RR)$ we notice that $P(RR)^N_\Phi = Right^N_\Phi(RR)$. So we have

PROPOSITION 5.1. *The variety RR is right–edge–solid.* ■

In [Rom-R; 87] also subvarieties of DG were considered. In the same way we obtain the subvarieties of RR. If in a subvariety of RR not all terms of the form $x^k y$ are pairwise different, i.e. if there are repetitions among these terms, then we consider the least $m \in \mathbb{N}$ such that there exists an $l \in \mathbb{N}$ with $x^m y \approx x^l y \in Id\,RR$. Let r be the least natural number such that $x^m y \approx x^{m+r} y \in Id\,RR$. ($m$ is called index and r is called period.) Then the $Id\,RR$–classes of $x, xy, x^2y, \ldots, x^m y, x^{m+1}y, \ldots, x^{m+(r-1)}y$ are all distinct and $x^{m+u}y \approx x^{m+v}y \in Id\,RR$ if and only if $u \equiv v \ mod\ r$. Clearly, $x^p y \approx x^q y \in Id\,RR$ iff $y^p x \approx y^q x \in Id\,RR$. Let $RR_{m,r} = Mod\{(xy)(uv) \approx (xu)(yv),\ x^2 \approx x, (xy)y \approx y,\ x^m y \approx x^{m+r} y,\ m \geq 0,\ r \geq 1,\ m, r \text{ minimal}\}$. Then we have

THEOREM 5.2. *Every variety* $RR_{m,r}$ *is right–edge–solid.*

Proof. From $x^m y \approx x^{m+r} y \in Id\,RR_{m,r}$ it follows that $y^m x \approx y^{m+r} x \in Id\,RR_{m,r}$. Therefore all binary terms over $RR_{m,r}$ are $\{x^l y \mid 0 \leq l \leq k+r-1\} \cup \{y^l x \mid 0 \leq l \leq k+r-1\}$. The hypersubstitutions $\{\sigma_{x^l y} \mid 0 \leq l \leq k+r-1\}$ preserve $(xy)(uv) \approx (xu)(yv), x^2 \approx x, (xy)y \approx y$. We check the identities $x^m y \approx x^{m+r} y,\ m \geq 0,\ r \geq 1, m, r$ minimal and obtain $\hat{\sigma}_{x^l y}[x^m y] = x^s y$ with $s \equiv m \cdot l \ mod\ r$ and $\hat{\sigma}_{x^l y}[x^{m+r} y] = x^{s'} y$ with $s' \equiv (m+r)l \ mod\ r$. Because of $(m+r)l \equiv m \cdot l \ mod\ r$ we have $x^s y \approx x^{s'} y \in Id\,RR_{m,r}$ and thus the equations $x^m y \approx x^{m+r} y$ are right hyperidentities and therefore the varieties $RR_{m,r}$ are right–edge–solid. ∎

The variety $RR_{0,1}$ agrees with the variety RZ of right–zero–semigroups. We have infinitely many different varieties between $RR_{0,1}$ and RR. Each variety $RR_{m,r}$ contains the variety of right–zero–semigroups and each variety $RR_{m,r}$ is contained in the variety RR.

For the join and the meet of two varieties $RR_{m,r}$, $RR_{m',r'}$ we have $RR_{m,r} \vee RR_{m',r'} = RR_{\max(m,m'),l.c.m.(r,r')}$ and $RR_{m,r} \wedge RR_{m',r'} = RR_{m,r} \cap RR_{m',r'} = RR_{\min(m,m'),g.c.d.(r,r')}$. The meet of all these varieties is clearly the variety $RR_{0,1}$ of right–zero–semigroups and the join is the variety RR.

If V is a variety of groupoids and if Σ is an equational basis for the equational theory of V, i.e. a system of axioms for V, then by V^d we denote the dual variety $V^d = Mod\ \sigma_{yx}[\Sigma] = \{\hat{\sigma}_{yx}[s] \approx \hat{\sigma}_{yx}[t] \mid s \approx t \in \Sigma\}$. Clearly, the variety RR^d is the variety DG of differential groupoids.

Then dually we obtain

THEOREM 5.3. *The variety* $DG = Mod\{(xy)(uv) \approx (xu)(yv),\ x^2 \approx x,\ x(xy) \approx x\}$ *is left–edge–solid. The varieties* $DG_{m,r} = Mod\{(xy)(uv) \approx (xu)(yv),\ x^2 \approx x,\ x(xy) \approx x,\ xy^k \approx xy^{k+r},\ \ k \geq 0,\ r \geq 1, k, r \text{ minimal}\}$ *are also left–edge–solid.* ∎

In [Rom-R ;87] Romanowska and Roszkowska proved that the subvariety lattice $\mathcal{L}(DG)$ of DG consists of exactly the varieties $DG_{m,r}$, of DG, and of the trivial variety and that $\mathcal{L}(DG)$ is isomorphic to a lattice $\mathcal{L}_{0,1}$ which is defined in the following way:
Let $\mathcal{N}_d$ be the lattice of all natural numbers with *l.c.m.* and *g.c.d.* as lattice operations, let $\mathcal{N}_0$ be the lattice of all natural numbers with *min* and *max* as lattice operations, let $\mathcal{L}$ be the direct product of N_d and N_0 and let $\mathcal{L}_{0,1}$ be the lattice arising from $\mathcal{L}$ by adding a new greatest element **1** and a new least element **0**.

A similar proposition holds for the dual variety RR. If we define $\mathcal{S}_{Left}(DG) := \mathcal{S}_{Left}(2) \cap \mathcal{L}(DG)$ and $\mathcal{S}_{Right}(RR) := \mathcal{S}_{Right}(2) \cap \mathcal{L}(RR)$ then we obtain:

COROLLARY 5.4. $\mathcal{S}_{Left}(DG) = \mathcal{L}(DG)$, $\mathcal{S}_{Right}(RR) = \mathcal{L}(RR)$ ∎

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Sr. Arworn
CHIANG MAI UNIVERSITY
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCES
50002 CHIANG MAI, THAILAND
e-mail:scmti002@chiangmai.ac.th

K. Denecke
UNIVERSITY OF POTSDAM
INSTITUTE OF MATHEMATICS
14415 POTSDAM, GERMANY
e-mail: Kdenecke@rz.uni-potsdam.de

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