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APPLICATIONS OF C-GROUPS TO EXISTENCE OF SOLUTIONS OF SOME MIXED PROBLEMS

Introduction

This paper consists of two parts. In the first, the existence of solutions of inhomogeneous and the semilinear abstract Cauchy problem have been proved. It has been done by C-semigroups which are generalisation of C_0-semigroups. The C-semigroups can be applied to the many differential and integral equations that may be modelled as an abstract Cauchy problem on a Banach space, where strongly continuous C_0-semigroups cannot be applied directly, for example to ill-posed problems. In the second part, the existence of the entire C-group for the hyperbolic type abstract problem has been proved. Finally, the theory of C-groups have been applied to generalised backward parabolic equation (ill-posed problem) and hyperbolic one in the space $L^p(\Omega)$, $p \geq 1$.

2. The abstract Cauchy problem

Let X be a Banach space with the norm $\| \|$, $B(X)$ the space of bounded linear operators from X to itself and $C \in B(X)$; A is an unbounded linear operator with the domain $D(A)$ and the image $\mathrm{Im}(A) \subset X$. In the book [4], the fundamental theories of the C-semigroups have been presented.

DEFINITION 1. Suppose that C is an injective linear, bounded operator on X. A strongly continuous family of bounded operators $\{W(t)\} : X \to X$, $t \geq 0$, is a C-regularised semigroup, if $W(0) = C$, and $W(t)W(s) = CW(t+s)$, for all $t, s \geq 0$.

DEFINITION 2. The operator A generates $\{W(t)\}$, $t \geq 0$, if

$$Ax = C^{-1} \lim_{t \to 0+} \frac{(W(t)x - Cx)}{t}$$

and has the domain $D(A) = \{x |$ limit exists and it is $\mathrm{Im}(C)\}$.

At first we will consider the following linear abstract Cauchy problem

$$u'(t) = Au(t) + f(t), t \in \langle 0, T\rangle, u(0) = x. \tag{1}$$

DEFINITION 3. If A generates a C-regularised semigroup $\{W(t)\}$, $t \geq 0$, $x \in \mathrm{Im}(C)$ and $f \in C(\langle 0,T\rangle, \mathrm{Im}(C))$, then we may define a mild solution of (1) by

$$u(t) = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s)ds. \tag{1'}$$

Next we will be considered the semilinear initial value problem

$$u'(t) = Au(t) + f(t, u(t)), t \in \langle 0, T\rangle, u(0) = x. \tag{2}$$

DEFINITION 4. A function $u : \langle 0, T\rangle\rangle \to X$ is said to be a mild solution of the problem (2), if $u \in C(\langle 0,T\rangle, X)$ for any $x \in \mathrm{Im}(C)$ and u satisfies the following integral equation

$$u(t) = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s, u(s))ds, 0 \leq t \leq T. \tag{3}$$

DEFINITION 5. A function $f : \langle 0, T\rangle \times X \to \mathrm{Im}(C)$ is said to satisfy Lipschitz condition with respect to w, uniformly in $t \in \langle 0, T\rangle$, with constant $L > 0$ if $\|C^{-1}(f(t, w_2) - f(t, w_1))\| \leq L\|w_2 - w_1\|$ for every $t \in \langle 0, T\rangle$, $w_1, w_2 \in X$.

THEOREM 1. *Suppose that:*

i) *A generates a C-regularised semigroup $\{W(t)\}$, $t \geq 0$ generated by A,*

ii) *$f : \langle 0, T\rangle \times X \to \mathrm{Im}(C)$ is continuous with respect to t,*

iii) *f satisfies Lipschitz condition,*

iv) *$u(0) = x \in \mathrm{Im}(C)$.*

Then there exists a unique mild solution of the problem (2).

Proof. Define the operator P, from $C(\langle 0,T\rangle, X)$ into itself, by

$$(Pu)(t) = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s, u(s))ds.$$

Let $K = \sup_{0 \leq t \leq T} \|W(t)\|$. It follows from the definition of P that

$$\begin{aligned}\|(Pu)(t) - (Pv)(t)\| &= \|\int_0^t W(t-s)C^{-1}[f(s, u(s)) - f(s, v(s))]ds\| \\ &\leq \int_0^t KL\|u - v\|_{C(\langle 0,T\rangle, X)} ds = tKL\|u - v\|_{C(\langle 0,T\rangle, X)}.\end{aligned}$$

We prove by induction on n that

$$\|(P^n u)(t) - (P^n v)(t)\| \leq \frac{(KLt)^n}{n!}\|u - v\|_{C(\langle 0,T\rangle,X)}.$$

Suppose that this inequality holds with $n-1$, i.e.,

$$\|(P^{n-1}u)(t) - (P^{n-1}v)(t)\| \leq \frac{(KLt)^{n-1}}{(n-1)!}\|u - v\|_{C(\langle 0,T\rangle,X)}.$$

Then

$$\begin{aligned}\|(P^n u)(t) - (P^n v)(t)\| &\leq \int_0^t \frac{(KLs)^{n-1}}{(n-1)!} KL\|u - v\|_{C(\langle 0,T\rangle,X)}ds \\ &= \frac{(KLt)^n}{n!}\|u - v\|_{C(\langle 0,T\rangle,X)}.\end{aligned}$$

For n large enough $\frac{(KLT)^n}{n!} < 1$ and by an extension of the contraction principle (see [3], Theorem 2.2, p.88) P has a unique fixed point in $C(\langle 0,T\rangle, X)$, being desired solution of the integral equation (3).

DEFINITION 6. By a classical solution of the abstract problem (1) we mean a function $v \in C(\langle 0,T\rangle, D(A)) \cap C^1(\langle 0,T\rangle, X)$ satisfying (1).

THEOREM 2. *Let A be a generator of the C-regularised semigroup $\{W(t)\}$, $t \geq 0$, $f \in C(\langle 0,T\rangle, \mathrm{Im}(C))$ and*

$$v(t) = \int_0^t W(t-s)C^{-1}f(s)ds, 0 \leq t \leq T.$$

The initial value problem (1) has a classical solution u on $\langle 0,T\rangle$ for every $x \in C(D(A))$, if one of the following conditions is satisfied:

(i) $$v \in C^1(\langle 0,T\rangle, X).$$

(ii) $$v \in C(\langle 0,T\rangle, D(A)).$$

Proof. For $h > 0$ the following identity holds

$$\frac{Cv(t+h) - Cv(t)}{h} = \frac{(W(h) - C)v(t)}{h} + \frac{C}{h}\int_t^{t+h} W(t+h-s)C^{-1}f(s)ds. \tag{4}$$

The continuity of f and closedness of C, C^{-1} imply that

$$\lim_{h\to 0} \int_t^{t+h} W(t+h-s)C^{-1}f(s)ds = Cf(t).$$

The assumption (i) and the indentity (4) guarantee that there exists $\lim_{h \to 0} \frac{(W(h)-C)v(t)}{h}$, i.e., $v(t) \in D(A)$ for $t \in \langle 0, T \rangle$ and that

$$\frac{d(Cv(t))}{dt} = CAv(t) + Cf(t) \text{ for } t \in \langle 0, T \rangle.$$

C is closed and injective operator, so

$$\frac{d(v(t))}{dt} = Av(t) + f(t) \text{ for } t \in \langle 0, T \rangle.$$

Since $v(0) = 0$, it follows that $u(t) = W(t)C^{-1}x + v(t)$ is the solution of the problem (1). Similarly the assumption (ii), the continuity of f, closedness of C, C^{-1} and existence of limit of the right-hand side of the identity (4) when h tends to zero, imply that there exists $\frac{d(Cv(t))}{dt}$ and v satisfies the equation $\frac{dv(t)}{dt} = Av(t) + f(t)$ for $t \in \langle 0, T \rangle$. It is clear that $u(t) = W(t)C^{-1}x + v(t)$ is the solution of the problem (1).

An idea of this proof is based on the proof of Theorem 2.4 from [6], (p. 107).

Remark 1. If there exists a classical solution of the problem (1) for some $x \in C(D(A))$ and some $f \in C(\langle 0, T \rangle, \mathrm{Im}(C))$, then v satisfies both (i) and (ii).

THEOREM 3. *Suppose that A generates a C-regularised semigroup $\{W(t)\}$, $t \geq 0$, and $x \in C(D(A))$, $f \in C(\langle 0, T \rangle, C(D(A)))$. Then* (1) *has a unique classical solution* (1′).

The proof follows from Theorem 2.

Remark 2. If $\{W(t)\}$, $t \geq 0$, is a C-regularised semigroup generated by A, then A is closed (cf. [5], Theorem 3.4, p.14) and $Y = [D(A)]$ with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$, $x \in D(A)$, is a Banach space and $W|_Y$ is a C-regularised semigroup (cf. [5], Theorem 3.5 and Definitions 2.4, 2.5, p. 4).

DEFINITION 7. A function $u : \langle 0, T \rangle \to X$ is said to be a classical solution of the problem (2), if $u \in C(\langle 0, T \rangle, Y) \cap C^1(\langle 0, T \rangle, X)$ and u satisfies (2) in X.

THEOREM 4. *Let the function $f : \langle 0, T \rangle \times Y \to C(Y)$ be a Lipschitz continuous in Y, uniformly in $t \in \langle 0, T \rangle$ and let $f(t, y)$ be a continuous from $\langle 0, T \rangle$ into $C(Y)$ for each $y \in Y$, then for any $x \in C(Y)$ the problem* (2) *has a unique classical solution on $\langle 0, T \rangle$.*

Proof. First we apply Theorem 1 in Y and obtain a mild solution $u \in C(\langle 0, T \rangle, Y)$ satisfying in Y (a fortiori in X) the integral equation

$$u(t) = W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s,u(s))ds, 0 \leq t \leq T.$$

Next, let $g(t) = f(t, u(t))$, $t \in \langle 0, T\rangle$. Then, by the assumption of our theorem, it follows that $g(t) \in C(Y)$ for $t \in \langle 0, T\rangle$ and $g \in C(\langle 0, T\rangle, C(Y))$. Theorem 2 guarantees existence of a unique classical solution v on $\langle 0, T\rangle$ for the linear problem

$$v'(t) + Av = g(t), v(0) = x, \tag{5}$$

if $g \in C(\langle 0, T\rangle, C(Y))$ and $x \in C(Y)$. This solution is then clearly also a mild solution of (5) and therefore

$$\begin{aligned} v(t) &= W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}g(s)ds \\ &= W(t)C^{-1}x + \int_0^t W(t-s)C^{-1}f(s,u(s))ds = u(t), \quad 0 \leq t \leq T. \end{aligned}$$

So $u = v$ and u is a classical solution of the problem (2) an $\langle 0, T\rangle$.

2. The entire C-group for abstract hyperbolic type equation

DEFINITION 8. The family of bounded operators $\{W(z)\}$, $z \in \mathbb{C}$, is an entire C-group, if

1) $W(z)$ is an entire $B(X)$-valued function of z,
2) $W(z)W(w) = CW(z+w)$ for all complex $z, w \in \mathbb{C}$,
3) $W(0) = C$.

DEFINITION 9. Suppose that B is closed densely defined operator such that $(-\infty, 0)$ is contained in the resolvent set $\varrho(B)$, with $\{r\|(r+B)^{-1}\}, r > 0\}$ bounded. The spectrum of B is contained in $S_\Theta = \{re^{i\phi}, r \geq 0, |\phi| < \Theta\}$ for some $\Theta < \pi$. For the operator B it can be defined the fractional power $\{-B^\alpha\}$. If zero is in $\varrho(B)$ and $0 < \alpha\Theta < \frac{\pi}{2}$, then $\{-B^\alpha\}$ can be defined as the generator of exponentially decaying strongly continuous analytic semigroup $\{T_\alpha(t)\}$, $t \geq 0$, given by

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_\phi} e^{-t\omega^\alpha}(\omega - B)^{-1}d\omega,$$

where $\Theta < \phi < \pi$, $\alpha\phi < \frac{\pi}{2}$, $t > 0$, and Γ_ϕ is the boundary of S_ϕ [1]. When zero is not in $\varrho(B)$, another formula [2] defines the fractional powers, with the same properties.

THEOREM 5. [4] *Suppose that $(-B)$ generates a strongly continuous analytic semigroup on a Banach space X and λ is complex. Then there exist* *real* k, $\alpha > 1$ *such that* $A = (\lambda B)$ *generates an entire* C_ε*-group* $T_\varepsilon(z)$, $z \in \mathbb{C}$ *for all* $\varepsilon > 0$ *where* $C_\varepsilon = e^{-\varepsilon[B-k]^\alpha}$.

Remark 3. Definition 9 and Theorem 5 guarantee that, if $-B$ generates an analytic semigroup in a Banach space X and $0 \in \varrho(B)$, then exists $B^{\frac{1}{2}}$, and $-B^{\frac{1}{2}}$ generates a strongly continuous analytic semigroup; both operators $B^{\frac{1}{2}}$, $-B^{\frac{1}{2}}$ will generate entire C-groups with the same $C = e^{-\varepsilon B^{\frac{\alpha}{2}}}$ (see [4]).

THEOREM 6. *Suppose that* $-B$ *generates strongly continuous analytic semigroup and* $0 \in \varrho(B)$, *then there exists* $\alpha > 1$ *such that for all* $\varepsilon > 0$ *the operator* $\Lambda = \begin{bmatrix} 0 & I \\ -B & 0 \end{bmatrix}$ *generates an entire* $W_\varepsilon(z)$, $z \in \mathbb{C}$, *group with* $C_\varepsilon = iB^{\frac{1}{2}} e^{-\varepsilon B^{\frac{\alpha}{2}}} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$.

Proof. The proof is similar to that of Theorem 4.1 in [4]. Since $\{\overline{e}^{\varepsilon B^{\frac{\alpha}{2}}}\}$, $\varepsilon \geq 0$, is a bounded strongly continuous analytic semigroup, $iB^{\frac{1}{2}} e^{-\varepsilon B^{\frac{\alpha}{2}}}$ is a bounded operator for all $\varepsilon > 0$. Since B, and therefore $B^{\frac{1}{2}}$, is injective, Lemma 3.1 in [4] guarantees that $iB^{\frac{1}{2}} e^{-\varepsilon B^{\frac{\alpha}{2}}}$ is injective for all $\varepsilon > 0$. By the Theorem 5 with the operator $B^{\frac{1}{2}}$ and $\lambda = i$, there exists $\alpha > 1$ such that $iB^{\frac{1}{2}}$ generates an entire group $\{S_\varepsilon(z)\}$, $z \in \mathbb{C}$, with C equal to $e^{-\varepsilon B^{\frac{\alpha}{2}}}$. Now the family of bounded operators $\{W_\varepsilon(z)\}$, $z \in \mathbb{C}$, on $X \times X$ will be defined as follows

$$W_\varepsilon(z) = \frac{1}{2} \begin{bmatrix} iB^{\frac{1}{2}}(S_\varepsilon(z) + S_\varepsilon(-z)) & S_\varepsilon(z) - S_\varepsilon(-z) \\ -B(S_\varepsilon(z) - S_\varepsilon(-z)) & iB^{\frac{1}{2}}(S_\varepsilon(z) + S_\varepsilon(-z)) \end{bmatrix}.$$

Making some routine calculations, we obtain

$$\begin{aligned} \frac{d}{dz} W_\varepsilon(z) &= \begin{bmatrix} -B(S_\varepsilon(z) - S_\varepsilon(-z)) & iB^{\frac{1}{2}}(S_\varepsilon(z) + S_\varepsilon(-z)) \\ -iB^{\frac{3}{2}}(S_\varepsilon(z) + S_\varepsilon(-z)) & -B(S_\varepsilon(z) - S_\varepsilon(-z)) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \Lambda W_\varepsilon(z) x \end{aligned}$$

for $x = (x_1, x_2) \in D(\Lambda)$, $z \in \mathbb{C}$. Theorem 2.8 from [4] implies that an extension of Λ generates an entire semigroup $\{W_\varepsilon(z)\}$, $z \in \mathbb{C}$, with $W_\varepsilon(0) = iB^{\frac{1}{2}} S_\varepsilon(0) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = iB^{\frac{1}{2}} e^{-\varepsilon B^{\frac{\alpha}{2}}} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. Let $G = \begin{bmatrix} 0 & (-B)^{-1} \\ I & 0 \end{bmatrix}$. Proposition 2.9 from [4] and Theorem 2.3 from [4] guarantee that Λ generates C_ε-group $\{W_\varepsilon(z)\}$, $z \in \mathbb{C}$.

3. Applications to generalised backward parabolic and hyperbolic equation $\operatorname{Im} L^p$, $p \geq 1$

Let Ω be a bounded domain in R^n with smooth boundary Γ. Consider the differential operator $A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$, $m \in N$, where the coefficients $a_\alpha(x)$ are sufficiently smooth complex-valued functions of $x \in \overline{\Omega}$. The principal part $A'(x, D)$ of $A(x, D)$ is the operator $A'(x, D) = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$.

DEFINITION 10. The operator $A(x, D)$ is strongly elliptic, if there exists a constant $c > 0$ such that

$$\Re(-1)^m A'(x, D) \geq c|\zeta|^{2m} \text{ for } x \in \overline{\Omega} \text{ and } \zeta \in R^n.$$

For $1 < p < \infty$ we associate with $A(x, D)$ an operator A_p in $L^p(\Omega)$ by

$$D(A_p) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \text{ and } A_p u = A(x, D)u \text{ for } u \in D(A_p).$$

It has been proved in [6], (Theorem 7.35) that $-A_p$ is the infinitesimal generator of analytic semigroup on $L^p(\Omega), p > 1$. By adding to $A(x, D)$, and hence to A_p, multiplied by enough large positive k, the identity operator, we obtain an infinitesimal generator $-(A_p + kI)$ of analytic semigroup, which is invertible. In the sequel we will tacity assume that this has been done and thus assume directly that A_p itself is invertible.

EXAMPLE 1. Let $X = L^p(\Omega)$, $p > 1$. Consider generalised backward parabolic problem

$$u'(t) = A_p u(t) + f(t), t \in \langle 0, T \rangle, u(0) = x. \tag{6}$$

It is clear that all assumptions of Theorem 5 are satisfied. Thus, operator A_p generates C-entire group on the space $L^p(\Omega)$, $p > 1$. So, if $C^{-1}x \in L^p(\Omega)$ and $C^{-1}f(t, x) \in L^p(\Omega)$ for $t \in \langle 0, T \rangle$, then there exists a unique mild solution of the problem (6) and $u \in C(\langle 0, T \rangle, L^p(\Omega))$.

Let $C^{-1}x \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ and $C^{-1}f(t, .) \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$ for $t \in \langle 0, T \rangle$. In this case, Theorem 3 guarantees the existence of a unique classical solution of the problem (6) and $u \in C(\langle 0, T \rangle, W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)) \cap C^1(L^p(\Omega))$.

EXAMPLE 2. Let as above $X = L^p(\Omega)$, $p > 1$. Consider generalised hyperbolic problem

$$u''(t) + A_p u(t) = f(t), t \in \langle 0, T \rangle, u(0) = x, u'(0) = y \tag{7}$$

which can be written as the following Cauchy problem in the space $X \times X$

$$w'(t) = \Lambda w + F(t), t \in \langle 0, T \rangle, w(0) = w_0,$$

where $\Lambda \begin{bmatrix} 0 & I \\ -A_p & 0 \end{bmatrix}, w = \begin{bmatrix} u \\ u' \end{bmatrix}, F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, w_0 = \begin{bmatrix} x \\ y \end{bmatrix}, D(\Lambda) = D(A_p) \times X.$

By Theorem 6, the operator Λ generates on the space $X \times X$ C-entire group. If $C^{-1}x$, $C^{-1}y$, $C^{-1}f(t,.) \in L^p(\Omega)$ for $t \in \langle 0,T \rangle$, then there exists a mild solution of the problem (7), i.e., $u \in C^1(\langle 0,T \rangle, L^p(\Omega))$.

If $C^{-1}x \in W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$, $C^{-1}y$, $C^{-1}f(t,.) \in L^p(\Omega)$ for $t \in \langle 0,T \rangle$, then there exists classical solution of the problem (7), i.e., $u \in C(\langle 0,T \rangle, W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega))$, $u \in C^2(\langle 0,T \rangle, L^p(\Omega))$.

Remark 4. In the homogeneous case ($f \equiv 0$) there exist global solutions for $t \in \langle 0, \infty)$.

Remark 5. Theorems 3.7 and 3.10 from [6], (pp.217, 218) guarantee that the restrictions of $(-A)$ generate analytic semigroups in the space of continuous functions ($p = \infty$) and in L^1 (the operator A are defined in these spaces by strongly elliptic operator $A(x,D)$ of order $2m$). So we can also obtain by theory of C-groups solutions of backward parabolic and hyperbolic equations in these spaces.

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