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# ON THE OPENNESS OF A MULTIVALUED FUNCTION. OPEN DISCRETE MULTIVALUED FUNCTIONS

We extend to multivalued functions a result of S. Stoilow which says that if $f : X \to Y$ is a continuous map between two n-manifolds, $X$ with a countable base, then $f(D_f)$ is of the first Baire category in $Y$, where $D_f = \{x \in X | f \text{ is not open at } x\}$, and using the same ideas, we study the set of lower semicontinuity of a multivalued function. We also extend to multivalued functions a known result which says that if $X, Y$ are locally compact complete metric spaces, $X$ with a countable base and $f : X \to Y$ is continuous, open and discrete, then $\text{int}\, B_f = \phi$ and $\text{int}\, f(B_f) = \phi$, where $B_f = \{x \in X | f \text{ fails to be a local homeomorphism at } x\}$.

## 1. Notations and terminology

We remind that if $X, Y$ are Hausdorff spaces, $f : X \to Y$ is a multivalued function if for every $x \in X, f(x)$ is a nonempty subset of $Y$. For $B \subset Y$, we denote by $f^+(B) = \{x \in X | f(x) \subset B\}$ and by $f^-(B) = \{x \in X | f(x) \cap B \neq \phi\}$. We say that a multivalued function $f : X \to Y$ is upper semicontinuous if $f^+(B)$ is open for every $B \subset Y$ open, we say that $f$ is lower semicontinuous if $f^-(B)$ is open for every $B \subset Y$ open, and we say that $f$ is continuous if it is both lower semicontinuous and upper semicontinuous. Since $f^-(B) = Cf^+(CB)$ and $f^+(B) = Cf^-(CB)$ for every $B \subset Y$, it results that if $f : X \to Y$ is a multivalued upper semicontinuous function, then $f^-(B)$ is a closed set in $X$ for every closed $B \subset Y$, and if $f$ is lower semicontinuous, then $f^+(B)$ is closed for every closed $B \subset Y$. We say that a multivalued function $f : X \to Y$ is lower semicontinuous at a point $x \in X$ if for every open $V \subset Y$ such that $f(x) \cap V \neq \phi$, there exists $U \in V(x)$ such that $U \subset f^-(V)$, and we say that $f$ is upper semicontinuous at a point $x \in X$, if for every open $V \subset Y$ such that $f(x) \subset V$, there exists

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$U \in V(x)$ such that $f(U) \subset V$. It is obvious that a multivalued function $f : X \to Y$ is lower semicontinuous (upper semicontinuous) if and only if $f$ is lower semicontinuous (upper semicontinuous) at every point from $X$. If $f : X \to Y$ is a multivalued function and $A \subset X$ we say that $f$ is lower semicontinuous (upper semicontinuous) on $A$ if $f$ is lower semicontinuous (upper semicontinuous) at every point from $A$.

We say that a multivalued function $f : X \to Y$ is nonsingular if $\operatorname{int} f(U) \neq \phi$ for every $U \subset X$ open, $U \neq \phi$.

We say that a multivalued function $f : X \to Y$ is locally closed if for every $x \in X$, there exists $U \in V(x)$ such that for every $A \subset \bar{U}$, the set $f(\bar{A})$ is a closed in $Y$.

We say that a multivalued function $f : X \to Y$ is lower proper if for every compact $K \subset Y$, we have that $f^{-}(K)$ is compact.

We say that a multivalued function $f : X \to Y$ is discrete if for every $x \in X, f^{-}(f(x))$ is a discrete set.

We say that a multivalued function $f : X \to Y$ is injective if $f(x) \cap f(y) = \phi$ for every $x, y \in X, x \neq y$, and we say that $f : X \to Y$ is injective arround the point $x \in X$ if there exists $U \in V(x)$ such that $f|U : V \to Y$ is injective. We denote by $B_f = \{x \in X | f$ is not injective arround $x\}$ and observe that $B_f$ is closed in $X$.

We say that a multivalued function $f : X \to Y$ is open at $x \in X$ if for every $U \in V(x)$, we have $f(x) \cap \operatorname{int} f(U) \neq \phi$ and we say that $f : X \to Y$ is strongly open at $x \in X$ if for every $U \in V(x)$, we have that $f(x) \subset \operatorname{int} f(U)$. If $A \subset X$, we say that $f$ is open (strongly open) on $A \subset X$ if $f$ is open (strongly open) at every point from $A$ and we denote by $D_f = \{x \in X | f$ is not open at $x\}$. Our definition of the strong openness coincides with interiority definition given by P. Maritz [2]. Since the notion of interiority was introduced in mathematics by Simion Stoilow for open and light mappings, we choose the present definition.

**2.** An old and quite elementary result of S. Stoilow [3] says that if $X$ is a separable locally compact metric space, $Y$ is a complete metric space, $f : X \to Y$ is continuous, then $f(D_f)$ is of the first Baire category in $Y$, i. e. $f(D_f)$ is a "small" set from the point of view of the category classification, and this theorem may be regarded as a topological version of the well-known Sard's lemma. Another interesting result of S. Stoilow deals with what he called "nonsingular" mappings, i. e. with mappings $f : X \to Y$ such that for every open $Q \subset X$, $Q \neq \phi$, we have that $\operatorname{int} f(Q) \neq \phi$ (for instance a continuous light map between two n-manifolds is nonsingular). S. Stoilow proved in [3] that if $X$ is a separable locally compact metric space , $Y$ a complete metric space, $f : X \to Y$ is continuous and nonsingular and $f(X)$

is of the second Baire category in $Y$, then there exists $A \subset X, A \neq \phi$ such that $f$ is open at every point from $A$ and such that $CA$ is of the first Baire category in $X$, and $B \subset Y$ such that $CB$ is of the first Baire category in $Y$ and $f|A : A \to B$ is an open map.

In [1] I generalized this results to quite general topological spaces. The purpose of my paper is to show that some result remains true if we work with a multivalued function $f : X \to Y$.

We shall now use the following result from [5] (Theorem 1.2.28 and Theorem 1.2.33):

LEMMA 1. *Let $X$ be a Hausdorff space, $Y$ a regular space, $f : X \to Y$ a multivalued upper semicontinuous function such that $f(x)$ is closed for every $x \in X$ and let $K \subset X$ be compact. Then $f(K)$ is closed.*

THEOREM 1. *Let $X$ be a regular space with a countable base $(S_i)_{i \in \mathrm{N}}, Y$ a Hausdorff space, $f : X \to Y$ a multivalued locally closed function. Then $f(D_f)$ is of the first Baire category in $Y$. If $f$ is lower semicontinuous and nonsingular, there exists $A \subset X$ such that $f$ is open on $A$ and $CA$ is of the first Baire category in $X$. If $f$ is upper semicontinuous and for every open $U \subset X$, $U \neq \phi$ and $B \subset Y$ such that $B \cap f(x) \neq \phi$ for every $x \in U$, implies that $B$ is of the second Baire category in $Y$, then there exists $D \subset X$ such that $f$ is strongly open on $D$ and $CD$ is of the first Baire category in $X$.*

Proof. Since $X$ is a regular space and $f : X \to Y$ is locally closed, we can choose a base $(S_i)_{i \in \mathrm{N}}$ such that $f|\bar{S}_i : \bar{S}_i \to Y$ is a closed map for every $i \in \mathrm{N}$. Denote by $H_i = \partial f(\bar{S}_i)$ for $i \in \mathrm{N}$ and observe that $CH_i = Y \setminus \partial f(\bar{S}_i) = Y \setminus (f(\bar{S}_i) \setminus \mathrm{int} f(\bar{S}_i)) = Cf(\bar{S}_i) \cup \mathrm{int} f(\bar{S}_i)$. Hence $H_i$ is a closed set and obviously $intH_i = \phi$ for $i \in \mathrm{N}$. Let $A = X \setminus \bigcup_{i=1}^{\infty} f^+(H_i)$. If $A = \phi$, then $X = \bigcup_{i=1}^{\infty} f^+(H_i)$, and therefore $f(X) \subset \bigcup_{i=1}^{\infty} H_i$, hence $f(D_f) \subset \bigcup_{i=1}^{\infty} H_i$. If $A \neq \phi$, then we show that $A \subset CD_f$. Indeed, fix $x \in A$ and let $i, j \in \mathrm{N}$ be such that $x \in \bar{S}_i \subset S_j$. Then $f(x) \subset f(\bar{S}_i)$ and since $x \in A$, we have $f(x) \cap CH_i \neq \phi$. Observe that if $y \in f(x) \cap CH_i$, then $y \in f(\bar{S}_i) \cap CH_i = f(\bar{S}_i) \cap (Cf(\bar{S}_i) \cup \mathrm{int}\ f(\bar{S}_i)) = \mathrm{int} f(\bar{S}_i) \subset \mathrm{int} f(S_j)$. Therefore if $x \in A$, then $f(x) \cap \mathrm{int} f(S_j) \neq \phi$ for every $j \in \mathrm{N}$ such that $x \in S_j$. Since $(S_j)_{j \in \mathrm{N}}$ is a countable base in $X$, that $f$ is open in $x$, hence $x \in CD_f$. We have just proved that $A \subset CD_f$, or, equivalently $D_f \subset CA$. Hence $f(D_f) \subset f(CA) = f(\bigcup_{i=1}^{\infty} f^+(H_i)) = \bigcup_{i=1}^{\infty} f(f^+(H_i)) \subset \bigcup_{i=1}^{\infty} H_i$. So, in both cases $f(D_f)$ is of the first Baire category in $Y$.

Now, if we suppose that $f$ is lower semicontinuous and nonsingular, then we show that $\mathrm{int} f^+(H_i) = \phi$ for every $i \in \mathrm{N}$. Indeed, if there exists $i \in \mathrm{N}$ and open $Q \subset X$, $Q \neq \phi$ such that $Q \subset f^+(H_i)$, then $f(Q) \subset H_i$. Since $f$ is nonsingular, then there exists open $V \subset f(Q)$, $V \neq \phi$, which represents a contradiction, since $\mathrm{int} H_i = \phi$. Using the fact that $f$ is lower semicontinuous, we conclude that $f^+(H_i)$ is a closed set with empty interior for every $i \in \mathrm{N}$. Since $CA \subset \bigcup_{i=1}^{\infty} f^+(H_i)$, we obtain that $CA$ is of the first Baire category in $X$, (we proved before that $f$ is open on $A$).

Suppose now that $f$ is upper semicontinuous and for every open $U \subset X$, $U \neq \phi$ and $B \subset Y$ such that $B \cap f(x) \neq \phi$ for every $x \in U$, $B$ is of the second Baire category in $Y$. Let $D = X \setminus \bigcup_{i=1}^{\infty} f^-(H_i)$. Using the fact that $f$ is upper semicontinuous, we see that the sets $f^-(H_i)$ are closed for every $i \in \mathrm{N}$. Suppose that there exists $i \in \mathrm{N}$ and open $U \subset X$, $U \neq \phi$ such that $U \subset f^-(H_i)$. Then $f(x) \cap H_i \neq \phi$ for every $x \in U$, and from the hypothesis, we see that $H_i$ is of the second Baire category in $Y$, which represents a contradiction. This shows that $\mathrm{int} f^-(H_i) = \phi$ and $f^-(H_i)$ is closed for every $i \in \mathrm{N}$, hence $CD = \bigcup_{i=1}^{\infty} f^-(H_i)$ is of the first Baire category in $X$. We shall now prove that $f$ is strongly open on $D$. Indeed, fix $x \in D$ and let $i, j \in \mathrm{N}$ be such that $x \in \bar{S}_i \subset S_j$. Then $f(x) \subset f(\bar{S}_i) \cap f(D) \subset f(\bar{S}_i) \cap CH_i = \mathrm{int} f(\bar{S}_i) \subset \mathrm{int} f(S_j)$ and since $(S_j)_{j\in \mathrm{N}}$ is a base in $X$, then $f$ is strongly open in $x$. This shows $f$ is strongly open on $D$ and completes the proof.

Using Lemma 1 and Theorem 1, we obtain:

THEOREM 2. *Let $X$ be a locally compact space with a countable base, $Y$ a regular space, $f : X \to Y$ a multivalued upper semicontinuous function such that $f(x)$ is closed for every $x \in X$. Then $f(D_f)$ is of the first Baire category in $Y$. If $f$ is continuous and nonsingular, there exists $A \subset X$ such that $CA$ is of the first Baire category in $X$ and such that $f$ is open on $A$. If for every $U \subset X, U \neq \phi$ and $B \subset Y$ such that $B \cap f(x) \neq \phi$ for every $x \in U$, we have that $B$ is of the second Baire category in $Y$, then there exists $D \subset X$ such that $CD$ is of the first Baire category in $X$ and $f$ is strongly open on $D$.*

**3.** As in [1], we can use the preceeding ideas to study the set of lower semicontinuity of a multivalued function.

THEOREM 3. *Let $X$ be a Hausdorff space, $Y$ a regular space with a countable base $(S_i)_{i\in \mathrm{N}}$ and $f : X \to Y$ a multivalued function such that $f^-(\bar{S}_i)$ is closed for every $i \in \mathrm{N}$. Then $f$ is lower semicontinuous on $X$ with the possible exception of a set of the first Baire category in $X$.*

Proof. Let $H_i = \partial f^-(\bar{S}_i)$ for $i \in \mathrm{N}$ and $A = X \setminus \bigcup_{i=1}^{\infty} H_i$. Then $H_i$ is closed and $\mathrm{int} H_i = \phi$ for every $i \in \mathrm{N}$. We shall show that $f$ is lower semicontinuous at every point from $A$. Indeed, let $x \in A$ and $V \subset Y$ open, $V \neq \phi$ such that $f(x) \cap V \neq \phi$ and let $y \in f(x) \cap V$ and $i \in \mathrm{N}$ such that $S_i \in V(y)$ and $\bar{S}_i \subset V$. Then $x \in f^-(\bar{S}_i) \cap A \subset f^-(\bar{S}_i) \setminus \partial f^-(\bar{S}_i) = \mathrm{int} f^-(\bar{S}_i) \subset \mathrm{int} f^-(V)$, hence we can find $U \in V(x)$ such that

$U \subset f^{-}(V)$, and this shows that $f$ is lower semicontinuous at $x$. We proved that $f$ is lower semicontinuous on $A$, and since $CA = \bigcup_{i=1}^{\infty} H_i$ is of the first Baire category in $X$, the proof is complete.

We obtain now immediately the following:

COROLLARY 1. *Let $X$ be a Hausdorff space, $Y$ a regular space with a countable base and let $f : X \to Y$ be a multivalued upper semicontinuous function. Then $f$ is continuous on $X$ with the possible exception of a set of the first Baire category in $X$.*

COROLLARY 2. *Let $X$ be a Hausdorff space, $Y$ a locally compact space with a countable base and let $f : X \to Y$ be a multivalued lower proper function. Then $f$ is lower semicontinuous on $X$ with the possible exception of a set of the first Baire category in $X$.*

4. We shall generalize now some classical facts from the theory of continuous, open and discrete mappings between two n-manifolds (see for instance [4]).

PROPOSITION 1. *Let $X, Y$ be Hausdorff spaces and $f : X \to Y$ a lower semicontinuous and strongly open multivalued function, $i \in \mathrm{N}$ and $K_i(f) = \{x \in X |$ Card $f^{-}(f(x)) \leq i\}$. Then $K_i(f)$ is a closed set.*

Proof. Let $x \in \overline{K_i(f)}$ and suppose that we can find $a_1, \ldots, a_i \in K_i(f)$ such that $x \neq a_k, a_p \neq a_k$ for $p \neq k, p, k \in \{1, \ldots, i\}$ and $f(a_p) \cap f(x) \neq \phi$ for $p = 1, \ldots, i$. Let $y_p \in f(x) \cap f(a_p)$ for $p = 1, \ldots, i$ and $Q_p \in V(a_p), p = 1, .., i, Q \in V(x)$ disjoint. Since $f$ is strongly open at $a_p$, then $f(a_p) \subset \operatorname{int} f(Q_p)$, and hence there exist $V_p \in V(y_p)$ such that $V_p \subset f(Q_p)$ for $p = 1, \ldots, i$. Using the fact that $f$ is lower semicontinuous, we see that the sets $f^{-}(V_p)$ are open, $p = 1, \ldots, i$, and since $y_p \in f(x) \cap V_p$, we obtain that $x \in f^{-}(V_p)$ for $p = 1, \ldots, i$. Let $W = \bigcap_{p=1}^{i} f^{-}(V_p) \cap Q$. Since $x \in \overline{K_i(f)}$, there exist $z \in K_i(f)$ such that $z \in W$, hence $f(z) \cap V_p \neq \phi$ and let $z_p \in f(z) \cap V_p$ for $p = 1, \ldots, i$. We have $z_p \in V_p \subset f(Q_p)$, hence there exists $b_p \in Q_p$ such that $z_p \in f(b_p)$, hence $z_p \in f(b_p) \cap f(z)$ for $p = 1, \ldots, i$. Since $b_p \in Q_p$ for $p = 1, \ldots, i, z \in Q$ and $Q_1, \ldots, Q_p, Q$ are disjoint, then $b_p \neq b_k, b_p \neq z$ for $p \neq k, p, k \in \{1, \ldots, i\}$ and $f(b_p) \cap f(z) \neq \phi$ for $p = 1, \ldots, i$. We obtained that $\{b_1, \ldots, b_i, z\} \subset f^{-}(f(z))$, therefore Card$f^{-}(f(z)) \geq i + 1$, what represents a contradiction, since $z \in K_i(f)$. It shows that $x \in K_i(f)$ and since $x$ was arbitrarily chosen from $\overline{K_i(f)}$, we conclude that $\overline{K_i(f)} \subset K_i(f)$, i. e. $K_i(f)$ is a closed set.

THEOREM 4. *Let $X$ be locally compact, $Y$ a Hausdorff space and $f : X \to Y$ a multivalued continuous, discrete and strongly open function such that $f(x)$ is closed for every $x \in X$. Then* $\operatorname{int} B_f = \phi$ .

P r o o f. Suppose that $B_f \neq \phi$ and $\mathrm{int}B_f \neq \phi$. Then we can find $Q \neq \phi, Q$ open such that $\bar{Q}$ is compact and $Q \subset B_f$. If $x \in X, f(x)$ is closed, and since $f$ is upper semicontinuous, then $f^-(f(x))$ is closed and discrete. So $f^-(f(x)) \cap \bar{Q}$ is compact and discrete, hence a finite set. We therefore have $\bar{Q} = \partial Q \cup \bigcup_{i=1}^{\infty}(K_i(f|Q))$ and since $\bar{Q}$ with the induced topology is a Baire space then using Proposition 1, we can find $i \in \mathbf{N}$ such that $\mathrm{int}K_i(f|Q) \neq \phi$. Hence we can find open $B \subset Q$, $B \neq \phi$ such that $\mathrm{Card}f^-(f(x)) \cap Q \leq i$ for every $x \in B$, and let $j \leq i$ be maximal such that $\mathrm{Card}f^-(f(x)) \cap Q \leq j$ for every $x \in B$. Then there exist $x_1, \ldots, x_j \in Q$ such that $x_1 \in B$, the points $x_1, \ldots, x_j$ are distinct and $f(x_1) \cap f(x_p) \neq \phi$ for $p = 2, \ldots j$. Pick $y_p \in f(x_1) \cap f(x_p), p = 2, \ldots, j$ and $Q_p \in V(x_p), p = 1, \ldots, j$ disjoint such that $Q_1 \subset B, Q_p \subset Q, p = 2, \ldots, j$. Since $f$ is strongly open at $x_p$, then $f(x_p) \subset \mathrm{int}f(Q_p)$ for $p = 2, \ldots, j$ and hence there exist $V_p \in V(y_p)$ such that $V_p \subset f(Q_p)$ for $p = 2, \ldots, j$. Because $f$ is lower semicontinuous, so that the sets $f^-(V_p)$ are open for $p = 2, \ldots, j$ and since $y_p \in f(x_1) \cap f(x_p)$, we obtain that $x_1 \in f^-(V_p)$ for $p = 2, \ldots, j$. Let $W = \bigcap_{p=2}^{j} f^-(V_p) \cap Q_1$. We shall show that $f$ is injective on $W$. Indeed, if it is not true, there exists $a_1, a_2 \in W, a_1 \neq a_2$ such that $f(a_1) \cap f(a_2) \neq \phi$. Take any $z \in f(a_1) \cap f(a_2)$. Since $a_1 \in W$, we have $f(a_1) \cap V_p \neq \phi$, and hence there exist $z_p \in f(a_1) \cap V_p$ for $p = 2, \ldots, j$ and $z_p \in V_p \subset f(Q_p)$. There exist $b_p \in Q_p$ such that $z_p \in f(b_p)$ for $p = 2, \ldots, j$. Then $z_p \in f(a_1) \cap f(b_p)$ for $p = 2, \ldots, j$, hence the points $a_1, a_2, b_2, b_3, \ldots, b_j$ are distinct and $\{a_1, a_2, b_2, \ldots, b_j\} \subset f^-(f(a_1))$, what represents a contradiction, since $a_1, a_2, b_2, \ldots, b_j$ are points from $Q$ and $a_1 \in B$. It shows that $f$ is injective on $W$, and $W \subset Q_1 \subset B \subset Q \subset B_f, W \neq \phi$, what contradicts with the definition of $B_f$. We obtained this contradiction since we presumed that $\mathrm{int}B_f \neq \phi$, hence $\mathrm{int}\, B_f = \phi$, q. e. d.

THEOREM 5. *Let $X$ be a regular space with a countable base $(S_i)_{i \in \mathbf{N}}, Y$ a locally compact or a complete metric space, $f : X \to Y$ a multivalued function which is locally closed, strongly open, lower semicontinuous and such that there exists $H \subset Y$ of the first Baire category in $Y$ such that for every $y \in Y \setminus H$, either $f^-(y)$ is empty or for every $\alpha \in f^-(y)$ there exists $U_\alpha \in V(\alpha)$ such that $f^-(f(\alpha)) \cap U_\alpha = \{\alpha\}$. Then* $\mathrm{int}\, f(B_f) = \phi$.

P r o o f. Since $f$ is a locally closed function, we can suppose that the base $(S_i)_{i \in \mathbf{N}}$ is chosen such that $f|\bar{S}_i : \bar{S}_i \to Y$ is a closed map for every $i \in \mathbf{N}$. Let $F_i = \{x \in S_i \cap B_f | f^-(f(x)) \cap S_i = \{x\}\}$. We show that for $i \in \mathbf{N}$ fixed, $f(F_i)$ is closed in $f(S_i)$. Indeed, let $y \in f(S_i)$ and $y_n \in f(x_n)$ be such that $y_n \to y$, with $x_n \in F_i$, for $n \in \mathbf{N}$. Suppose that there exists $a, b \in S_i$, $a \neq b$ such that $y \in f(a) \cap f(b)$. Choose $U_a \in V(a), U_b \in V(b)$ be such that $U_a \cup U_b \subset S_i$ and $U_a \cap U_b = \phi$. Since $f$ is strongly open, we have $f(a) \subset \mathrm{int}f(U_a), f(b) \subset \mathrm{int}f(U_b)$. Thus $y \in f(a) \cap f(b) \subset \mathrm{int}f(U_a) \cap \mathrm{int}f(U_b)$, hence there exists $V \in V(y)$ such that $V \subset f(U_a) \cap f(U_b)$. Using the fact that $y_n \to y$, we can find $n_0 \in \mathbf{N}$ such that $y_n \in V$ for $n \geq n_0$. Hence we can find $a_n \in U_a, b_n \in U_b$ such that $y_n \in f(a_n) \cap f(b_n)$ for $n \geq n_0$. Since $y_n \in f(a_n) \cap f(b_n) \cap f(x_n), a_n \neq b_n, a_n, b_n \in S_i, x_n \in F_i$ for $n \geq n_0$, we obtain a contradiction. We proved that there exists a single point $x \in S_i$ such that $y \in f(x)$. Let us denote by $A = \overline{\{x_n\}_{n \in \mathbf{N}}}$. Since $A \subset \bar{S}_i$ and $f|\bar{S}_i : \bar{S}_i \to Y$ is a closed map, then $f(A)$ is closed in $Y$, hence $y \in \overline{f(A)} = f(A)$. This implies that we can find $\alpha \in A$ such that $y \in f(\alpha)$. If $\alpha \in S_i$, from what we have proved before, we see that $x = \alpha$. If $\alpha \in \partial S_i$, then $x \neq \alpha$, and let $U_x \in V(x), U_\alpha \in V(\alpha)$ be such that $U_x \subset S_i$ and $U_x \cap U_\alpha = \phi$. As before, using the strong openness of the multivalued function $f$, we can find $W \in V(y)$ such that $W \subset f(U_x) \cap f(U_\alpha)$. Since $f$ is lower semicontinuous at $\alpha$, we can find $Q_\alpha \in V(\alpha)$ such that $Q_\alpha \subset U_\alpha \cap f^-(W)$. Using the fact that $\alpha \in \overline{\{x_n\}_{n \in \mathbf{N}}}$, we can find $x_n \in Q_\alpha$, and since $Q_\alpha \subset f^-(W)$, we can find $w_n \in W$ such that $w_n \in f(x_n)$. Now $w_n \in W \subset f(U_x)$, hence we can find $c_n \in U_x$ such that $w_n \in f(c_n)$. We have obtained that $w_n \in f(c_n) \cap f(x_n)$, hence $f(c_n) \cap f(x_n) \neq \phi, c_n \neq x_n, c_n \in S_i$ and $x_n \in F_i$, what represents a contradiction. It means that we can not have $\alpha \in \partial S_i$, hence $x = \alpha$. Since $\alpha \in A \subset \bar{B}_f = B_f$, we obtained that $x \in F_i$, hence $y \in f(F_i)$. We therefore proved that $f(F_i)$ is closed in $f(S_i)$ for every $i \in \mathbf{N}$.

Let $J_i = \{j \in \mathbf{N} | \bar{S}_j \subset S_i \text{ for } i \in \mathbf{N}\}$. We have $f(F_i) = f(F_i) \cap f(S_i) = f(F_i) \cap (\bigcup_{j \in J_i} f(\bar{S}_j)) = \bigcup_{j \in J_i} f(F_i) \cap f(\bar{S}_j)$ for $i \in \mathbf{N}$.

If $i \in \mathbf{N}$ and $j \in J_i$ are fixed, we shall show that the set $f(F_i) \cap f(\bar{S}_j)$ is closed in $Y$. Indeed, let $y_n \in f(F_i) \cap f(\bar{S}_j)$ be such that $y_n \to y$. Since $f(\bar{S}_j)$ is closed in $Y$, then $y \in f(\bar{S}_j) \subset f(S_i)$, and using the fact that $f(F_i)$ is closed in $f(S_i)$, we obtain that $y \in f(F_i)$, hence $y \in f(F_i) \cap f(\bar{S}_j)$ and this proves that the sets $f(F_i) \cap f(\bar{S}_j)$ are closed sets in $Y$ for $j \in J_i, i \in \mathbf{N}$. Suppose now that $f(B_f)$ contains an open, nonempty set $S$. We have $S \setminus H \subset \bigcup_{i=1}^{\infty} f(F_i) = \bigcup_{i=1}^{\infty} \bigcup_{j \in J_i} (f(F_i) \cap f(\bar{S}_j))$, hence $\bar{S} = \partial S \cup (H \cap S) \cup \bigcup_{i=1}^{\infty} \bigcup_{j \in J_i} (f(F_i) \cap f(\bar{S}_j) \cap S)$, and using the fact that $\bar{S}$ with the induced topology is a Baire space and $\partial S$ and $H$ are of the first Baire category in $Y$ and in $\bar{S}$, we can find $i \in \mathbf{N}$ and $j \in J_i$ such that there exists $Q \subset Y$ open, $Q \neq \phi$ and $Q \subset f(F_i) \cap f(\bar{S}_j) \cap S$. Let $C = S_i \cap f^-(Q)$ and observe that $C$ is open and nonempty. For any $c \in C$ we have $Q \cap f(c) \neq \phi$, hence there exists $z \in Q \cap f(c)$. Since $Q \subset f(F_i)$, we can find $b \in F_i$ such that $z \in f(b)$. We use now the fact that $f(b) \cap f(c) \neq \phi, c \in S_i, b \in F_i$, to see that $b = c$, and this implies that $c = b \in F_i$.

Since we used an arbitrary point $c \in C$, we proved that $C \subset F_i \subset B_f \cap S_i$. From the definition of $F_i$, we conclude that $f$ is injective on $F_i$, hence $f$ is injective on $C$, and since $C$ is open, $C \neq \phi$, then $C \cap B_f = \phi$, what represents a contradiction, since $C \subset B_f \cap S_i$. It results that $f(B_f)$ cannot contain an open, nonempty set, hence int$f(B_f) \neq \phi$, q. e. d.

COROLLARY 3. *Let $X$ be a locally compact space with a countable base, $Y$ a locally compact or a complete metric space, $f : X \to Y$ a multivalued function such that $f$ is continuous, strongly open, $f(x)$ is closed for every $x \in X$ and there exists $H \subset Y$ of the first Baire category in $Y$ such that for every $y \in Y \setminus H$, either $f^{-}(y)$ is empty, or for every $\alpha \in f^{-}(y)$ there exists $U_\alpha \in V(\alpha)$ such that $f^{-}(f(\alpha)) \cap U_\alpha = \{\alpha\}$. Then int$f(B_f) = \phi$.*

COROLLARY 4. *Let $X$ be a locally compact space with a countable base, $Y$ a locally compact or a complete metric space, $f : X \to Y$ a multivalued function such that $f$ is continuous, discrete, strongly open and $f(x)$ is closed for every $x \in X$. Then int$B_f = \phi$ and int$f(B_f) = \phi$.*

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