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E. Magnucka-Blandzi, J. Popenda

ON THE BEHAVIOUR OF RATIONAL FIRST ORDER DIFFERENCE EQUATION WITH CONSTANT COEFFFICIENTS

In this note we consider difference equations with constant coefficients of the form

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in N \tag{E1}$$

where $a, b, c, d \in R$ are such that

$$\kappa = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \quad \text{and} \quad c \neq 0.$$

Here by N, R we denote set of positive integers or real numbers respectively.

We prove that almost each of the trajectories (except countable set of finite trajectories) tends to one of stationary points.

This work is motivated by the results contained in the monograph Kocic and Ladas [5], where the authors based on the paper [3] have presented asymptotic properties of solutions of Riccati equations. As it is noticed in [3] (see also papers quoted therein), Riccati equations appear in mathematical biology, optics, chemistry, and are applied in other branches of mathematics. Therefore, we have decided to present correct results on asymptotic properties of solutions of this equation. Our results are similar (not the same!) to these contained in [5], furhermore we have to notice that proofs contained in [5] are not correct because of wrong formulas for general solution of (E1) (Theorem A.1 (f) and essential applications of it in Theorems A.3, and A.4). (Also the result for Riccati equation with variable coefficients contained in Theorem A.3 are not correct because of the same mistake. This will be shown in separated paper).

The case of (E1) for which $\kappa = 0$ leads to the equation $x_{n+1} = \frac{a}{c}$, $n \in N$ while for $c = 0$ to the equation $x_{n+1} = \frac{a}{d}x_n + \frac{b}{d}$, $n \in N$ which is well described in several monographs devoted to difference equations e.g. [4], some asymptotic properties can be found in [1], and so we shall not consider it here.

Notice that if $c \neq 0$, then (E1) can be transformed to the form

$$x_{n+1} = \frac{a}{c} + \frac{-\frac{\kappa}{c^2}}{x_n + \frac{d}{c}}.$$

Therefore, instead of studying (E1), we can examine

$$\text{(E2)} \qquad x_{n+1} = \alpha + \frac{\beta}{x_n + \delta}, \quad n \in N.$$

Before we present main result related to (E1), we consider its particular cases. We start with the case

$$\text{(E3)} \qquad x_{n+1} = \alpha + \frac{\beta}{x_n + \alpha}, \quad n \in N.$$

DEFINITION. (see [2]) Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of elements of some set X and $\{X_\gamma : \gamma \in \Gamma\}$ be any family of disjoint subsets of X (that is $X_\tau \cap X_\sigma = \emptyset$ for all $\tau, \sigma \in \Gamma$, $\tau \neq \sigma$). We say that this sequence oscillates in relation to the family X_γ if

$$\forall \gamma \in \Gamma \ \exists \{n_{\gamma,k}\}_{k=1}^{\infty} : \forall \ k \in N \ w_{n_{\gamma,k}} \in X_\gamma.$$

We can suppose that $\bigcup_{\gamma \in \Gamma} X_\gamma = X$.

For periodic (regular) oscillation between family of sets (that is in the case all $\{\Delta n_{\gamma,k}\}_{k=1}^{\infty}$, $\gamma \in \Gamma$ are periodic (regular type) sequences) we can say about the cycles that is the string of $m = \sum_{\gamma \in \Gamma} \tau_\gamma$ consecutive elements of the sequence $\{w_n\}_{n=1}^{\infty}$, where τ_γ is the fundamental period of the sequence $\{\Delta n_{\gamma,k}\}_{k=1}^{\infty}$.

X_γ — semicycle consists of these consecutive terms of the cycle for which $w_n \in X_\gamma$.

The number σ such that $w_{n-1} \in X_\gamma$ (if any), $w_n, w_{n+1}, \cdots, w_{n+\sigma-1} \in X_\gamma$, and $w_{n+\sigma} \notin X_\gamma$ we will call the measure of X_γ — semicycle (in the point n).

LEMMA 1. *Let $\alpha, \beta \in R_+$. Then every solution $x = \{x_n\}_{n=1}^{\infty}$ (defined on the set N) of* (E3) *tends to $\sqrt{\alpha^2 + \beta}$.*

Proof. Notice at first that the function

$$\phi(x) = \alpha + \frac{\beta}{x + \alpha}$$

is defined on $R \setminus \{-\alpha\}$ and possesses two stationary points

$$\overline{x}_1 = -\sqrt{\alpha^2 + \beta}, \quad \overline{x}_2 = \sqrt{\alpha^2 + \beta}.$$

If $x_k = \overline{x}_1$ or $x_k = \overline{x}_2$ for some $k \in N$ then $x_n = \overline{x}_1$ respectively $x_n = \overline{x}_2$ for all $n \geq k$. Furthermore if $x_k = -\alpha$ then x_{k+1} does not exist. That is this solution consists of finite k terms and vanishes starting from $n = k+1$. Such type of solutions are excluded from our considerations in this lemma.

Let us denote :

$$D_1 = (-\infty, \overline{x}_1), \quad D_2 = (\overline{x}_1, -\alpha), \quad D_3 = (-\alpha, \overline{x}_2), \quad D_4 = (\overline{x}_2, \infty),$$

furthermore

$$D_{1,1} = \left(-\infty, -\alpha - \frac{\beta}{2\alpha} \right), \quad D_{1,2} = \left(-\alpha - \frac{\beta}{2\alpha}, \overline{x}_1 \right),$$
$$D_{3,1} = (-\alpha, \alpha), \quad D_{3,2} = (\alpha, \overline{x}_2).$$

Examining the function $\phi(x)$ on the sets D_i one can observe that

(r1) $\phi(D_1) \subset D_2 \cup D_{3,1}$,
(r2) $\phi(D_2) \subset D_1$,
(r3) $\phi(D_3) \subset D_4$,
(r4) $\phi(D_4) \subset D_{3,2}$.

We prove (r4) (proofs of the rest relations are similar).

Let $x \in D_4$, then

$$0 < \overline{x}_2 + \alpha < x + \alpha \text{ yields } \frac{1}{\overline{x}_2 + \alpha} > \frac{1}{x + \alpha},$$

and consequently

$$\phi(x) = \alpha + \frac{\beta}{x + \alpha} < \alpha + \frac{\beta}{\overline{x}_2 + \alpha} = \overline{x}_2.$$

Suppose that $\phi(x) \leq \alpha$. This can be written as $\alpha + \frac{\beta}{x+\alpha} \leq \alpha$. Therefore $\frac{\beta}{x+\alpha} \leq 0$, however it is impossible because $x + \alpha > 0$ and $\beta > 0$. Consequently $\phi(x) > \alpha$. The relation (r4) is proved.

The next function we have to consider is

$$\Psi(x) = \phi(\phi(x)) = \alpha + \frac{\beta}{\alpha + \phi(x)} = \alpha + \frac{\beta}{\alpha + \left(\alpha + \frac{\beta}{x+\alpha}\right)}.$$

The domain of the function ϕ is the set $R \setminus \{-\alpha\}$. Therefore Ψ does not exist for $x = -\alpha$. As the composition $\phi \circ \phi$ of two functions ϕ, the function Ψ does not exist also when $\phi(x) = -\alpha$. Solving the equation

(1) $$w = \alpha + \frac{\beta}{z + \alpha}, \quad \text{for } w \neq \alpha$$

with respect to z, we obtain

(2) $$z = \frac{\alpha^2 + \beta - \alpha w}{w - \alpha}$$

and this solution is unique. Solving (1) for $w = -\alpha$ we obtain from (2) $z = -\alpha - \frac{\beta}{2\alpha}$.

In the considered case $\alpha, \beta \in R_+$, there is

$$-\alpha - \frac{\beta}{2\alpha} < \overline{x}_1.$$

Following above, the domain of the function Ψ is $R \setminus \{-\alpha - \frac{\beta}{2\alpha}, -\alpha\}$. The function Ψ can be transformed to the form

$$\Psi(x) = \alpha + \frac{\beta(x+\alpha)}{2\alpha^2 + 2\alpha x + \beta}.$$

It can be checked that the function Ψ possesses the only $\overline{x}_1, \overline{x}_2$ as stationary points. Furthermore

(m1) $x < \Psi(x)$ for $x \in D_2 \cup D_3 \cup D_{1,1}$,
(m2) $x > \Psi(x)$ for $x \in D_{1,2} \cup D_4$.

We show condition (m1) for $x \in D_2 \cup D_3$. For $x \in D_2 \cup D_3$ we have $x^2 < \alpha^2 + \beta$. From there (because $\alpha > 0$) we obtain

$$2\alpha x^2 < 2\alpha^3 + 2\alpha\beta,$$
$$2\alpha x^2 + 2\alpha^2 x + \beta x < 2\alpha^3 + 2\alpha\beta + 2\alpha^2 x + \beta x,$$

that is

(3) $$x(2\alpha x + 2\alpha^2 + \beta) < \alpha(2\alpha x + 2\alpha^2 + \beta) + \beta(x+\alpha).$$

On the other hand for $x \in D_2 \cup D_3$ there is $x > -\alpha - \frac{\beta}{2\alpha}$, from there we get

(4) $$2\alpha x + 2\alpha^2 + \beta > 0$$

Combining (3) with (4) we have

$$x < \alpha + \frac{\beta(x+\alpha)}{2\alpha^2 + 2\alpha x + \beta} = \Psi(x)$$

that is (m1), in the case $x \in D_2 \cup D_3$.

In the rest of the proof we will examine solutions with initial point x_1 in each of the sets D_i.

(p1) Let $x_1 \in D_4$.

Then by (r4) and (r3) we have $x_2 \in D_{3,2}, x_3 \in D_4$. Consequently by (m2) $x_3 < x_1$. Following this way, we get $x_{2k+1} < x_{2k-1}$ for all $k \in N$. Moreover, $D_4 \ni x_{2k-1} > \overline{x}_2$. Therefore, the sequence $\{x_{2k-1}\}_{k=1}^{\infty}$ is decreasing and bounded from below by $\overline{x}_2$, hence convergent to some $g \geq \overline{x}_2$. Let us observe that since $x_{2k+1} = \Psi(x_{2k-1})$, the limit g has to be stationary point of the function Ψ. The only such a point in $[\overline{x}_2, \infty)$ is $\overline{x}_2$, so $g = \overline{x}_2$. Since $x_{2k-1} \in D_4$ then $x_{2k} \in D_{3,2}$ for all $k \in N$, by (r4). Furthermore, by (m1) there is $x_{2k+2} > x_{2k}$ for all $k \in N$. This yields the sequence $\{x_{2k}\}_{k=1}^{\infty}$ is increasing and bounded from above by $\overline{x}_2$ hence convergent to some $g \in (\alpha, \overline{x}_2]$. Again from equality $x_{2k+2} = \Psi(x_{2k})$, continuity of Ψ on the set $(\alpha, \overline{x}_2]$ we deduce that $g = \overline{x}_2$. The lemma for the case $x_1 \in D_4$ has been proved.

(p2) Let $x_1 \in D_3$.

Then by (r3) we have $x_2 \in D_4$ and we come back to the case (p1) with

$$D_4 \ni x_{2k} > x_{2k+2} \to \overline{x}_2, \quad D_{3,2} \ni x_{2k+1} < x_{2k+3} \to \overline{x}_2, \quad k \in N.$$

(p3) Let $x_1 \in D_1$.

Then by (r1) $x_2 \in D_2, x_2 \in D_{3,1}$ or $x_2 = -\alpha$.

$x_2 = -\alpha$ leads to the finite trajectory (this holds if $x_1 = -\alpha - \frac{\beta}{2\alpha}$),

$x_2 \in D_{3,1}$ leads to the just considered case (p2),

$x_2 \in D_2$ leads by (r2) to $x_3 \in D_1$ and we come back to the starting position with x_3 instead of x_1.

Following this, by (r1) and (r2) three possible cases should be considered:

(p3,1) $x_{2k} = -\alpha$ for some $k \in N$ which gives finite trajectory, out of our considerations,

(p3,2) $x_{2k} \in D_{3,1}$ for some $k \in N$ which in turns leads to the case (p2) and hence gives a solution convergent to stationary point $\overline{x}_2$,

(p3,3) the solution $\{x_n\}_{n=1}^{\infty}$ oscillates between the sets D_1 and D_2 and measures of D_1-semicycle and D_2-semicycle are equal 1.

Because of properties we have observed above, we need to study the case (p3,3).

Since $x_2 \in D_2$ then $x_{2k} \in D_2$ for all $k \in N$. Furthermore, by (m1), $x_{2k} < x_{2k+2}$ what means that the sequence $\{x_{2k}\}_{k=1}^{\infty}$ is increasing and bounded from above by $-\alpha$ and hence convergent to some $g \in (\overline{x}_1, -\alpha]$. Suppose that $x_{2n} \to -\alpha$ as $n \to \infty$ Then from the equality

$$x_{2n+2} = \Psi(x_{2n}) = \alpha + \frac{\beta}{\alpha + (\alpha + \frac{\beta}{x_{2n}+\alpha})}$$

we obtain $\lim_{n\to\infty} x_{2n+2} = -\alpha$, while

$$\lim_{n\to\infty} \left(\alpha + \frac{\beta}{\alpha + \left(\alpha + \frac{\beta}{x_{2n}+\alpha}\right)} \right) = \lim_{x\to-\alpha_-} \left(\alpha + \frac{\beta}{\alpha + \left(\alpha + \frac{\beta}{x+\alpha}\right)} \right) = \alpha.$$

This contradiction shows that the relation $g = -\alpha$ is impossible. If $g \neq -\alpha$ then the continuity of the function Ψ yields that g has to be stationary point of Ψ. However, in the interval we consider such a point does not exist.

The case (p3,3) can not hold. So the only possible for the solution defined on the whole N is (p3,2).

(p4) Let $x_1 \in D_2$.

Then by (r2) $x_2 \in D_1$ and we come to the case (p3) with x_2 instead of x_1. This completes the proof.

REMARK 1. In Lemma 1 we do not consider finite trajectories. However considering the initial value problem (related to (2)):

$$w_{n+1} = \frac{\alpha^2 + \beta - \alpha w_n}{w_n - \alpha}, \quad n \in N$$
$$w_1 = -\alpha$$

by (r1),(r2),(m1),(m2) we obtain $\{D_{1,2}, D_2\}$-oscillating infinite sequence with the property

$$\lim_{n\to\infty} w_n = \overline{x}_1$$

and such that the trajectory of the equation (E3) starting from the point w_k is finite and consists of k terms.

LEMMA 2. *Let $\alpha \in R_+$, $\beta \in R_-$, $\alpha^2 + \beta > 0$. Then every solution $x = \{x_n\}_{n=1}^{\infty}$ (defined on the set N) of* (E3) *tends monotonically to $\sqrt{\alpha^2 + \beta}$.*

Proof. In this case let us denote by

$$D_1 = (-\infty, -\alpha), \quad D_2 = (-\alpha, \overline{x}_1), \quad D_3 = (\overline{x}_1, \overline{x}_2), \quad D_4 = (\overline{x}_2, \infty).$$

As in Lemma 1 the stationary points of the function $\phi(x) = \alpha + \frac{\beta}{x+\alpha}$ are

$$\overline{x}_1 = -\sqrt{\alpha^2 + \beta}, \quad \overline{x}_2 = \sqrt{\alpha^2 + \beta}.$$

We can check that now

$$\phi(D_1) \subset D_4, \quad \phi(D_2) \subset D_1 \cup D_2, \quad \phi(D_3) \subset D_3, \quad \phi(D_4) \subset D_4. \tag{5}$$

Furthermore,

$$x > \phi(x) \text{ for } x \in D_2 \text{ and } x \in D_4, \tag{6}$$
$$x < \phi(x) \text{ for } x \in D_3. \tag{7}$$

Let $x = \{x_n\}_{n=1}^{\infty}$ be a solution of (E3) such that

(p1) $x_1 \in D_4$.

Then by (5) and (6) this solution is strictly decreasing and bounded from below by $\overline{x}_2$. Hence convergent. As in the proof of Lemma 1 we can show that $lim_{n\to\infty} x_n = \overline{x}_2$. Now let

(p2) $x_1 \in D_1$.

Then by (5) $x_2 \in D_4$ and we return to the case (p1) with x_2 instead of x_1. Therefore in this case $\lim_{n\to\infty} x_n = \overline{x}_2$. Let

(p3) $x_1 \in D_3$.

Now, by (5) and (7) the sequence $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded from above by $\overline{x}_2$. Hence also convergent. It can be shown that $\lim_{n\to\infty} x_n = \overline{x}_2$. Finally let

(p4) $\quad x_1 \in D_2$.

For infinite solution $\{x_n\}_{n=1}^{\infty}$ by (5) we have two possibilities:

(p4,1) $\quad x_n \in D_2$ for all $n \in N$,
(p4,2) $\quad$ there exists $k \in N$ such that $x_k \in D_1$.

In the case (p4,1), by (6), the sequence $\{x_n\}_{n=1}^{\infty}$ is decreasing, bounded from below by $-\alpha$ and hence convergent to some $g \in [-\alpha, \overline{x}_1)$. The limit can not be $-\alpha$ because we would have

$$-\alpha = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \phi(x_n) = \lim_{x\to-\alpha+} \phi(x) = -\infty$$

what is a contradiction. The limit can not be any other number from $[-\alpha, \overline{x}_1)$ because this would lead to the fact that $\phi(x)$ possesses stationary point in $(-\alpha, \overline{x}_1)$, what is not true. So (p4,2) holds but this gives the case (p2) with x_k instead of x_1.

LEMMA 3. *Let $\alpha \in R_-$, $\beta \in R_+$. Then every solution $\{x_n\}_{n=1}^{\infty}$ (defined on the set N) of* (E3) *in the oscillating mode tends to $-\sqrt{\alpha^2+\beta}$.*

LEMMA 4. *Let $\alpha, \beta \in R_-$, $\alpha^2+\beta > 0$. Then every solution $x = \{x_n\}_{n=1}^{\infty}$ (defined on the set N) of* (E3) *monotonically tends to $-\sqrt{\alpha^2+\beta}$.*

However, this is not the case we consider in this paper, we give one example of the asymptotic properties of the solutions of (E3) in the case $\alpha^2+\beta < 0$, to show different type of the behaviour.

LEMMA 5. *Let $\alpha \in R_+$, $\beta \in R_-$, $\alpha^2+\beta < 0$. Then every trajectory of* (E3) *is finite or non-convergent, infinite $\{D_1, D_2\}$-oscillatory, where $D_1 = (-\infty, -\alpha)$, $D_2 = (-\alpha, \infty)$. The measure of each D_1-semicycle is equal* 1, *and moreover $sup_{k,l\geq n}||x_k| - |x_l|| \geq 2\alpha$ for all $n \in N$.*

Proof. In this case the inequality $x^2 - \alpha^2 - \beta > 0$, which holds for all $x \in R$, gives

$$\begin{cases} x > \phi(x) & \text{if } x > -\alpha, \\ x < \phi(x) & \text{if } x < -\alpha. \end{cases} \tag{8}$$

Furthermore, $\phi((-\infty, -\alpha)) \subset (\alpha, \infty)$.

Therefore, if for some term x_k of the solution $\{x_n\}_{n=1}^{\infty}$ there is $x_k < -\alpha$, then $x_{k+1} > \alpha$. Suppose that $x_n \in (-\alpha, \infty)$ for all $n > k$. Then, by the condition (8), the sequence $\{x_n\}_{n=k+1}^{\infty}$ is decreasing and bounded from below. Hence convergent to some $g \in \big[-\alpha, \infty\big)$. However this leads to the contradiction

$$-\alpha = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \phi(x_n) = \lim_{x\to-\alpha_+} \phi(x) = -\infty$$

or states that the function ϕ possesses stationary point what is not true. Hence, for some $l > k+1$ there is $x_l < -\alpha$ and so on.

REMARK 2. Generalizing the observations from Remark 1 we can say that solving the initial problem

$$w_{n+1} = \frac{\alpha^2 + \beta - \alpha w_n}{w_n - \alpha} \quad n \in N,$$
$$w_1 = -\alpha,$$

we obtain countable set of points which are initial points for finite trajectories of (E3). The trajectory starting at point w_k consists of k terms and $x_k = -\alpha$.

For the sequence $\{w_n\}_{n=1}^{\infty}$ we have

$$\lim_{n\to\infty} w_n = -\sqrt{\alpha^2+\beta} \text{ (in the case } \alpha > 0),$$
$$\lim_{n\to\infty} w_n = \sqrt{\alpha^2+\beta} \text{ (in the case } \alpha < 0).$$

REMARK 3. Notice, that if $\beta \neq 0$, the two stationary points of the function $\phi(x)$: $\overline{x}_1 = -\sqrt{\alpha^2+\beta}$, $\overline{x}_2 = \sqrt{\alpha^2+\beta}$ form constant solutions of (E3), that is the sequences

$$x_n = -\sqrt{\alpha^2+\beta}, \quad n \in N,$$
$$x_n = \sqrt{\alpha^2+\beta}, \quad n \in N$$

are solutions of (E3). If $\alpha = 0$, then every solution of (E3) (except for the case $x_1 = 0$ which leads to finite trajectory) is 2-periodic sequence $\{x_1, \beta/x_1, x_1, \beta/x_1, \dots\}$. Basing on Lemmas 1–4 we can obesrve the following bifurcation phenomena passing over zero on the α – axis. If

$\alpha < 0$ then each infinite (nonconstant) solution is attracting by left stationary point $\overline{x}_1$;

$\alpha = 0$ then $\overline{x}_1$ lose his attractive property and each solution stay 2-periodical;

$\alpha > 0$ then each (nonconstant) solution stays attracting by the right stationary point $\overline{x}_2$.

Let us observe that any solution of the equation

$$(\text{E}_k) \qquad x_{n+k} = \alpha + \frac{\beta}{x_n + \alpha}, \quad n \in N,$$

where k is arbitrary but fixed positive integer can be considered as the composition of k independent solutions of equations

$$y_{i,n+1} = \alpha + \frac{\beta}{y_{i,n} + \alpha}, \quad n \in N, \ i = 1, \dots, k$$

where $y_{i,n} = x_{i+(n-1)k}$ and all $\{y_{i,n}\}_{n=1}^{*}$ are infinite sequences if infinite is the solution $x = \{x_n\}_{n=1}^{\infty}$. Basing on the above remarks and Lemmas 1–4 we can prove :

THEOREM 1. *Let $\alpha, \beta \in R \setminus \{0\}$ be such that $\alpha^2 + \beta > 0$. Then every solution $x = \{x_n\}_{n=1}^{\infty}$ (which has no nonconstant subsequences and is defined on the set N) of* (E_k) *tends to* $\operatorname{sgn}(\alpha)\sqrt{\alpha^2 + \beta}$.

REMARK 4. The case $\alpha = 0$, $\beta \neq 0$, leads to 2 k-periodic solutions of (E_k), while $\beta = 0$ to constant solutions. In the general case (E_k) possesses solutions which has constant subsequences equal to one of stationary points and the rest which tends to $sgn(\alpha)\sqrt{\alpha^2 + \beta}$.

Now we turn our attention to the equation (E2).

THEOREM 2. *Let $\alpha, \beta, \delta \in R$ be such that $\beta \neq 0$, $(\alpha + \delta)^2 + 4\beta > 0$. Then every nonconstant solution $x = \{x_n\}_{n=1}^{\infty}$ (defined on the set N) of* (E2) *has the property*

$$\lim_{n \to \infty} x_n = \left(\frac{\alpha - \delta}{2} + \operatorname{sgn}(\alpha + \delta) \frac{1}{2} \sqrt{(\alpha + \delta)^2 + 4\beta} \right) \quad \text{if } \alpha + \delta \neq 0,$$

$$x \text{ is 2-periodic} \qquad \text{if } \alpha + \delta = 0.$$

Proof. Let $x = \{x_n\}_{n=1}^{\infty}$ be any solution of (E2). Then the sequence $y = \{y_n\}_{n=1}^{\infty}$ defined by

$$y_n = x_n - \frac{\alpha - \delta}{2} \tag{9}$$

is defined for all $n \in N$ and satisfies the equation

$$y_{n+1} = \widetilde{\alpha} + \frac{\beta}{y_n + \widetilde{\alpha}}, \quad n \in N \tag{10}$$

where $\widetilde{\alpha} = \frac{\alpha+\delta}{2}$. Let us observe that

$$\widetilde{\alpha}^2 + \beta = \left(\frac{\alpha + \delta}{2} \right)^2 + \beta = \frac{1}{4}((\alpha + \delta)^2 + 4\beta) > 0.$$

If $\alpha + \delta \neq 0$ then $\widetilde{\alpha} \neq 0$ and all assumptions of the Theorem 1 are satisfied (in relation to the equation (10)). Therefore

$$\lim_{n \to \infty} y_n = \operatorname{sgn}(\widetilde{\alpha})\sqrt{\widetilde{\alpha}^2 + \beta} = \operatorname{sgn}(\alpha + \delta)\frac{1}{2}\sqrt{(\alpha + \delta)^2 + 4\beta}$$

and our result follows from (9).

If $\alpha + \delta = 0$ then $\widetilde{\alpha} = 0$ and we have the equation $y_{n+1} = \frac{\beta}{y_n}$ which corresponds to the equation $x_{n+1} = \alpha + \frac{\beta}{x_n - \alpha}$. Now by Remark 4 we see that y is 2-periodic sequence, what by (9) yields that $x_n = y_n + \alpha$ is 2-periodic as well.

To get the result for (E1) it suffices to put in (E2)

$$\alpha = \frac{a}{c}, \ \beta = -\frac{\kappa}{c^2}, \ \delta = \frac{d}{c},$$

and we obtain the following theorem.

THEOREM 3. *Let $a, b, c, d \in R$ be such that $c \neq 0$, $\kappa \neq 0$, $(a+d)^2 - 4\kappa > 0$. Then every (nonconstant) solution $x = \{x_n\}_{n=1}^{\infty}$ (defined on the set N) of (E1) has the property*

$$\lim_{n\to\infty} x_n = \left(\frac{a-d}{2c} + sgn(a+d)\frac{1}{2c}\sqrt{(a+d)^2 - 4\kappa}\right) \quad \text{if } a+d \neq 0,$$

$$x \text{ is 2-periodic} \qquad \text{if } a+d = 0.$$

REMARK 5. In [3] Brand has considered (E1), and has presented results on asymptotic behaviour of the solutions of equation

$$\text{(E4)} \qquad y_{n+1} = p - \frac{q}{y_n}$$

to which (E1) can be transformed putting $p = \frac{a+d}{c}$, $q = \frac{\kappa}{c^2}$ and substituting $x_n = y_n - \frac{d}{c}$. Next, using Riccati substitution $y_n = \frac{z_{n+1}}{z_n}$, he has got second order linear difference equation $z_{n+2} - pz_{n+1} + qz_n = 0$, which in the case $p^2 + 4q \neq 0$ has general solution of the form $z_n = C_1\lambda_1^n + C_2\lambda_2^n$, where C_1, C_2 are arbitrary constants and λ_1, λ_2 different roots of the respective characteristic equation. Consequently general solution of (E4) can be written as follows

$$\text{(11)} \qquad y_n = \frac{C_1\lambda_1^{n+1} + C_2\lambda_2^{n+1}}{C_1\lambda_1^n + C_2\lambda_2^n}.$$

Using the above formula Brand has studied asymptotic properties of (E4) (also in the case $p^2 + 4q < 0$). The error made by Kocic and Ladas in [5] consists in this, that (11) does not valid for (E1), and should be modified for (E1). Namely, we have the following. If $c \neq 0$ and $(a+d)^2 - 4\kappa \neq 0$, then the general solution of (E1) can be presented in the form

$$\text{(12)} \qquad x_n = \frac{(\lambda_1 c - d)C_1\lambda_1^n + (\lambda_2 c - d)C_2\lambda_2^n}{cC_1\lambda_1^n + cC_2\lambda_2^n}, \quad n \in N.$$

Formula (12) gives the same solution of (E1) starting from the given x_1 each time C_1, C_2 are taken such that

$$(\lambda_1 c - d - cx_1)C_1 + (\lambda_2 c - d - cx_1)C_2 = 0.$$

EXAMPLE. Consider the equation

$$x_{n+1} = 4 + \frac{4}{x_n - 1}, \quad n \in N.$$

Let us observe that all conditions of Theorem 2 are satisfied. Therefore, we get because of $\alpha + \delta = 4 - 1 \neq 0$, that every solution defined on N has the asymptotic behaviour

$$\lim_{n\to\infty} x_n = \left(\frac{\alpha - \delta}{2} + \operatorname{sgn}(\alpha + \delta)\frac{1}{2}\sqrt{(\alpha + \delta)^2 + 4\beta}\right)$$
$$= \left(\frac{4+1}{2} + \operatorname{sgn}(4-1)\frac{1}{2}\sqrt{(4-1)^2 + 4\cdot 4}\right) = 5.$$

Let us quote part a) of Theorem A.4 from [5 p.182].

"Consider the Riccati difference equation with constant coefficients $x_{n+1} = \frac{ax_n+b}{cx_n+d}$, where a, b, c, d are real numbers such that $ad - bc \neq 0$ and $c \neq 0$. Let λ_1 and λ_2 be the roots of the quadratic equation $\lambda^2 - p\lambda - q = 0$, where p and q are given by $p = d + a$ and $q = bc - ad$. Then the following statements are true:

(a) If λ_1 and λ_2 are real and either $|\lambda_1| > |\lambda_2|$ or $\lambda_1 = \lambda_2$ then $\lim_{n\to\infty} x_n = \lambda_1$. (A.22)

Let us transform our equation to the form (E1) (considered by Kocic and Ladas):

$$x_{n+1} = \frac{4x_n}{x_n - 1}. \tag{13}$$

Here $a = 4$, $b = 0$, $c = 1$, $d = -1$. Notice that all assumptions of the above quoted statement are satisfied. The characteristic equation has the form

$$\lambda^2 - 3\lambda - 4 = 0, \text{ and roots } \lambda_1 = 4, \lambda_2 = -1.$$

Therefore, by the quoted theorem there is

$$\lim_{n\to\infty} x_n = 4. \tag{14}$$

However it is impossible because if $x = \{x_n\}_{n=1}^{\infty}$ is any solution of (13) and (14) holds, then left hand side of equality (11) tends to 4, while the right hand side to $\frac{16}{3}$. In the considered example $\lambda_1 = \frac{a}{c}$. Let us observe that in this case our formula (12) gives

$$x_n = \frac{(a-d)C_1(a/c)^n}{cC_1(a/c)^n + cC_2(d/c)^n} = \frac{(a-d)C_1a^n}{cC_1a^n + cC_2d^n}$$

because then $\lambda_2 = \frac{d}{c}$, so for (13) we have

$$x_n = \frac{(4+1)C_1 4^n}{C_1 4^n + C_2(-1)^n} \to 5, \quad \text{for } C_1 \neq 0$$

which coincides with the result obtained by Theorem 2.

The constant solution $x_n = 0$ for all $n \in N$ is related with stationary points which for (E2) (in the case $(\alpha+\delta)^2 + 4\beta > 0$) are defined by

$$\overline{x}_1 = \frac{\alpha-\delta}{2} - \frac{1}{2}\sqrt{(\alpha+\delta)^2+4\beta}, \quad \overline{x}_2 = \frac{\alpha-\delta}{2} + \frac{1}{2}\sqrt{(\alpha+\delta)^2+4\beta}.$$

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF POZNAŃ
Piotrowo 3A
60-965 POZNAŃ, POLAND

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